Generalizing the Multivariable Alexander Polynomial

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Advances in Quantum and Low-Dimensional Topology, University of Iowa, March 11, 2016

The Multivariable Alexander Polynomial:

- multivariable version of the classical Alexander polynomial
- invariant for long, regular, virtual knots and links
- $\mathrm{vMVA}(L) \in \mathbb{Z}[t_i, t_i^{-1}]$

GOAL: Study its generalizations to tangles.

First generalization (J. Archibald '06)

$v\mathcal{T}_n =$ regular v-tangles with ends labelled by $|X^{\mathrm{in}}| = |X^{\mathrm{out}}| = n$

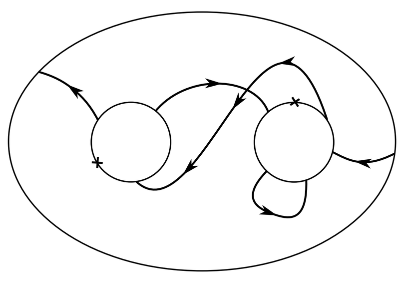
$\mathrm{AHD}(X^{\mathrm{in}}, X^{\mathrm{out}}) = \Lambda^n(X^{\mathrm{out}}) \otimes \Lambda^n(X^{\mathrm{in}} \cup X^{\mathrm{out}})$

Tangle invariant, which is a **circuit algebra morphism**:

$$\mathrm{tMVA} : (v\mathcal{T}_n, \text{gluing}) \longrightarrow (\mathrm{AHD}(X^{\mathrm{in}}, X^{\mathrm{out}}), \text{interior mult.})$$

Oriented Circuit Algebra

An **OCA** is a collection $\mathcal{V}$ of objects indexed by pairs $(n, m) \in \mathbb{N}^2$ and morphisms $\mathcal{F}$ indexed by *circuit diagrams*:



$: V_{1,2} \otimes V_{3,2} \longrightarrow V_{1,1}$

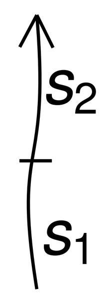
The Alexander matrix

Given a diagram D_T for a tangle T:

- $\{I, X^{\mathrm{in}}, X^{\mathrm{out}}\}$ label the internal, incoming, and outgoing arcs
- $\{t_j\}$=variables associated to the strands

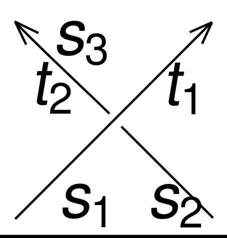
The **Alexander matrix** $M(D_T)$ is built via the rules:

$$M(D_T) = \begin{array}{c} \\ I \\ X^{\mathrm{out}} \end{array} \begin{array}{ccc} I & X^{\mathrm{out}} & X^{\mathrm{in}} \\ \hline \end{array}$$



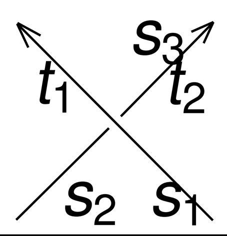
$\mapsto$

	s_1	s_2
s_2	-1	1



$\mapsto$

	s_1	s_2	s_3
s_3	$1 - t_2$	-1	t_1



$\mapsto$

	s_1	s_2	s_3
s_3	$t_2 - 1$	$-t_1$	1

Definition of tMVA

For any regular, virtual tangle T with m strands, (n open):

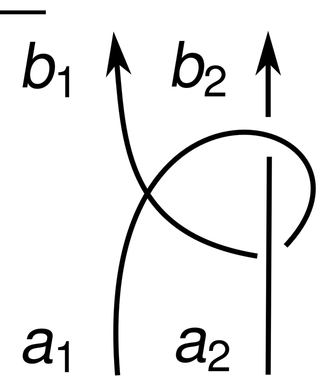
$$\mathrm{tMVA}(T) = \prod_{k=1}^{m} t_k^{-\mu(k)/2} w \otimes \left[\sum_{i_1 < \ldots < i_n} M(D_T)^{i_1 < \ldots < i_n} s_{i_1} \wedge \ldots \wedge s_{i_n} \right]$$

where: $(w \in \Lambda^n(X^{\mathrm{out}})) =$ a choice of ordering of X^{out}

$M(D_T)^{i_1 < \ldots < i_n} =$ minor with columns I and $\{s_{i_1}, \ldots, s_{i_n}\} \subset X^{\mathrm{out}} \cup X^{\mathrm{in}}$

$\mu(k) = \#$ times strand k over-crosses

Example



$\longrightarrow$

	b_1	b_2	a_1	a_2
b_1	t_2	0	-1	$1 - t_1$
b_2	0	t_1	$1 - t_2$	-1

$$\mathrm{tMVA}(T) = (t_1 t_2)^{-1/2} b_1 \wedge b_2 \otimes [(t_1 t_2) b_1 \wedge b_2 + t_2(1 - t_2) b_1 \wedge a_1$$
$$- t_2 b_1 \wedge a_2 + t_1 b_2 \wedge a_1 - t_1(1 - t_1) b_2 \wedge a_2 + (t_1 + t_2 - t_1 t_2) a_1 \wedge a_2]$$

Properties of tMVA

- It is a circuit algebra morphism.
- Satisfies "Overcrossings Commute", so is a w-tangle invariant.
- For u-tangles, can get R1 invariance: $\mathrm{tMVA}' = \prod_k t_k^{rot(k)/2} \mathrm{tMVA}$
- Can recover vMVA:
  $$\frac{\mathrm{tMVA}\left(\xrightarrow{a} \boxed{T} \xrightarrow{b}\right)}{t_a - 1} = \mathrm{vMVA}\left(\xrightarrow{a} \boxed{T} \xrightarrow{b}\right)(b \otimes b - a \otimes a)$$
- Gives easy verification of many local vMVA relations (Conway's second and third identity, Murakami's fifth axiom, doubled delta move).

Handout browser courtesy of Dror Bar-Natan:

`http://drorbn.net/index.php?title=HandoutBrowser.js`

Hodge operators

For a fixed $w \in \Lambda^n(X^{\mathrm{out}})$:

$$\Lambda^n(X^{\mathrm{in}} \cup X^{\mathrm{out}}) \cong \bigoplus_{k=0}^{n} \Lambda^k(X^{\mathrm{in}}) \otimes \Lambda^{n-k}(X^{\mathrm{out}})$$
$$\xrightarrow{*_w} \bigoplus_{k=0}^{n} \Lambda^k(X^{\mathrm{in}}) \otimes \Lambda^k(X^{\mathrm{out}}) \cong \bigoplus_{k=0}^{n} \mathrm{Hom}(\Lambda^k(X^{\mathrm{in}}), \Lambda^k(X^{\mathrm{out}}))$$

where $*_w(\alpha) = (-1)^{*_w}\gamma \ \Leftrightarrow\ (-1)^{*_w}\alpha \wedge \gamma = w$.

Theorem

For any tangle T, let $\lambda =$deg 0, $\phi =$deg 1 component of the image. If $\lambda \neq 0$, the image of T in $\bigoplus_{k=0}^{n} \mathrm{Hom}(\Lambda^k(X^{in}), \Lambda^k(X^{out}))$ is precisely $\lambda \cdot \Lambda(\phi/\lambda)$.

Reduction of tMVA

For a **pure** tangle T, identify $X^{\mathrm{in}} \cong X^{\mathrm{out}} = X$ using the var's $\{t_i\}$.

$(A \in M_{X \times X}(\mathbb{R}(t_i)))$=matrix of deg 1 coefficients under $*_w$.

For $0 \neq \lambda \in \mathbb{R}(t_i)$, tMVA reduces to:

$$rMVA(T) := \prod_{k=1}^{m} t_k^{-\mu(k)/2}(\lambda, A) \in R_X = \mathbb{R}(\sqrt{t_i}) \times M_{X \times X}(\mathbb{R}(\sqrt{t_i}))$$

The induced operations on R_X are:

- $$\begin{array}{c|ccc} \lambda & a & b & X^{\mathrm{in}} \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ X^{\mathrm{out}} & \phi & \psi & \Xi \end{array} \xrightarrow[t_a, t_b \to t_c]{m_c^{ab}} \begin{array}{c|cc} \lambda + \beta & c & X^{\mathrm{in}} \\ \hline c & \gamma + \frac{\beta\gamma - \alpha\delta}{\lambda} & \epsilon + \frac{\beta\epsilon - \delta\theta}{\lambda} \\ X^{\mathrm{out}} & \phi + \frac{\beta\phi - \alpha\psi}{\lambda} & \Xi + \frac{\beta\Xi - \psi\theta}{\lambda} \end{array}$$
- $$\left(\begin{array}{c|c} \lambda_1 & X_1^{\mathrm{in}} \\ \hline X_1^{\mathrm{out}} & A_1 \end{array}, \begin{array}{c|c} \lambda_2 & X_2^{\mathrm{in}} \\ \hline X_2^{\mathrm{out}} & A_2 \end{array} \right) \xrightarrow{*} \begin{array}{c|cc} \lambda_1\lambda_2 & X_1^{\mathrm{in}} & X_2^{\mathrm{in}} \\ \hline X_1^{\mathrm{out}} & \lambda_2 A_1 & 0 \\ X_2^{\mathrm{out}} & 0 & \lambda_1 A_2 \end{array}$$

Metamonoids

A **meta-monoid** is a collection of sets $\{G_X\}_{X=\text{a finite set}}$ together with maps between them:

$$m_z^{xy} : G_{\{x,y\} \cup X} \longrightarrow G_{\{z\} \cup X} \quad \text{"multiplication"}$$
$$* : G_X \times G_Y \longrightarrow G_{X \cup Y} \quad \text{"union"}$$

satisfying:

1. "Monoid axioms": $m_z^{xy} \circ m_x^{uv} = m_z^{ux} \circ m_x^{vy}$
2. A list of "set manipulation" axioms.

E.g. Pure X-labelled v-tangles form a metamonoid.

Remark: rMVA is a metamonoid morphism and recovers the Gassner (and Burau) representations on pure braids.

Second generalization (Bar-Natan '12)

$\zeta : \{\text{ribbon } S^1\text{s and } S^2\text{s in } \mathbb{R}^4\} \xrightarrow{\text{invariant}} \{\text{free-Lie and cyclic words}\}$.

It reduces to a metamonoid morphism:

$z : \{X\text{-labelled, pure w-tangles}\} \longrightarrow \Gamma_X = \mathbb{R}(t_i) \times M_{X \times X}(\mathbb{R}(t_i))$,

where Γ_X has similar m_z^{xy} and $*$ to R_X.

Main result:

Theorem ("rMVA and z are essentially the same.")

The map $\Gamma_X \longrightarrow R_X$ taking $(\lambda, A) \longrightarrow (\lambda, -\lambda A)$ is a metamonoid morphism.

It induces a partial trace operation on Γ_X.

$$\begin{array}{c|cc} \lambda & c & X \\ \hline c & \alpha & \theta \\ X & \psi & \Xi \end{array} \xrightarrow{\mathrm{tr}_c} \begin{array}{c|c} \lambda(1-\alpha) & X \\ \hline X & \Xi + \frac{\psi\theta}{1-\alpha} \end{array}$$

Future directions:

OCA vs. MM, extend tr_c to ζ, $\lambda = 0$?, categorification?