GRT Learning Seminar

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On the geometry of the Feigin-Odesskii Poisson structures on $\mathbb{C}P^n$

(Mykola Matviichuk)

X - manifold (real/complex)

$\{,\}: \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ Poisson bracket

if
- Lie bracket
- $\{f, gh\} = \{f,g\}h + g\{f,h\}$

Equivalently: One can define a Poisson structure by specifying $\pi \in \Gamma(\Lambda^2 T_X)$ + integrable $[\pi,\pi]_S = 0$ ← Schouten bracket

$[\ ,\ ]_S : \Lambda^k T \times \Lambda^l T \longrightarrow \Lambda^{k+l-1} T$

- continuation of Lie bracket of vector fields
- satisfy graded Jacobi
- satisfy graded Leibniz w.r.t. $\wedge$

i.e. $[v \wedge w, y] = [v,y] \wedge w \pm v \wedge [w,y]$

$\pi \in \Gamma(\Lambda^2 T) \longleftrightarrow \{f,g\} := \pi(df \wedge dg)$

$[\pi,\pi]_s = 0 \longleftrightarrow$ Jacobi

Examples

0) $\pi \equiv 0$

1) (X,ω)-symplectic $\Rightarrow$ Poisson

$\{f,g\} := \omega(H_f \wedge H_g)$ ← Hamiltonian vector fields.

Say, $X = \mathbb{C}^{2n} = \langle x_1, y_1, \dots, x_{2n}, y_{2n} \rangle$

$\omega = \sum dx_i \wedge dy_i$

$\{f,g\} = \sum \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}$

In general, $\Pi \in \Gamma(\Lambda^2 T) \Rightarrow \Pi_{\#} : T^* \longrightarrow T$

$$\Pi_{\#}(df) = \Pi(df, -) \in \Gamma((T^*)^*) = \Gamma(T)$$

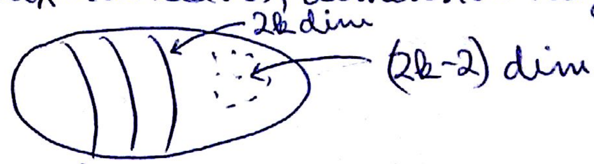
$\mathrm{Im}(\Pi_{\#})$ – at each $x \in X$, is a subspace $T_x X$

i.e. $\mathrm{Im}(\Pi_{\#})$ defines $\underbrace{\text{a}}_{\text{singular}}$ distribution on X.

singular (i.e. the rank can jump)

$[\Pi, \Pi]_S = 0 \Rightarrow \mathrm{Im}(\Pi_{\#})$ integrable

$\rightsquigarrow$ can foliate X with leaves, dimension might jump



$\Pi|_{leaf}$ – comes from a sympl. form.

Feigin – Odesskii Poisson structures:

① $\mathbb{C}P^2$

$$\Pi \in \Gamma(\Lambda^2 T, \mathbb{C}P^2) = \Gamma(\mathcal{O}(3)).$$

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^3 \to Q \to 0$$

$0 \to \mathcal{O} \to (\mathcal{O}(1))^{\oplus 3} \to T\mathbb{C}P^2 \to 0$ Euler short exact sequence

$$\det(\mathcal{O}(1))^{\oplus 3} = \cancel{\det \mathcal{O}} \otimes \det(T\mathbb{C}P^2) = \Lambda^2 T\mathbb{C}P^2$$

$\det(\mathcal{O}(1))^{\oplus 3} = \mathcal{O}(3)$, $\det \mathcal{O} = \mathcal{O}$

Moral: To define a bivector, need a section of $\mathcal{O}(3)$.

Fix $\Pi \in \Gamma(\mathcal{O}(3))$ i.e. a degree 3 homogeneous poly. in x_0, x_1, x_2.

$[\Pi, \Pi]_S \in \Gamma(\Lambda^3 T\, \mathbb{C}P^2) \rightsquigarrow$ has to $\equiv 0$

$E = \{\Pi = 0\}$ – degree 3 curve in $\mathbb{C}P^2$, generically smooth

By genus formula, $g_E = \frac{(3-1)(3-2)}{2} = 1$.

$\Rightarrow E$ – elliptic curve.

One can show: $\mathbb{C}P^2 \setminus E$ is a symplectic leaf.
$\forall x \in E$ - 0-dim'l symplectic leaf.

2 $\mathbb{C}P^3$

$\pi \in \Gamma(\Lambda^2 T\mathbb{C}P^3) \neq \Gamma(\mathcal{O}(4))$.

Need two polynomials to cut out a curve.

Fix two (generic) $f, g \in \Gamma(\mathcal{O}(2))$

"Wronskian" = "$fdg - gdf$" $\in \Gamma(\mathcal{O}(2) \otimes \mathcal{O}(2) \otimes T^*)$

"$df \in \Gamma(\mathcal{O}(2) \otimes T^*)$" need jet bundles.

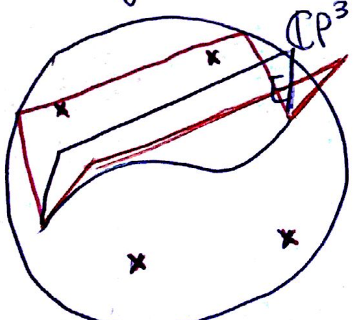
$\mathcal{O}(2) \otimes \mathcal{O}(2) \otimes T^* = \mathcal{O}(4) \otimes T^* = \Lambda^3 T\mathbb{C}P^3 \otimes T^* = \Lambda^2 T\mathbb{C}P^3$

$\rightsquigarrow \pi = fdg - gdf$ - Poisson bivector.

What is $\{f = g = 0\}$?

- curve
- degree 4

$\Big\} \Rightarrow$ This has to be elliptic curve E.



$\forall x \in E$ forms a 0-dim'l sympl. leaf.
$\exists$ 4 more sympl. leafs of dim = 0. (x)
2-dim'l sympl. leafs:
$\{\lambda f + \mu g = 0\} \setminus E \quad [\lambda : \mu] \in \mathbb{C}P^1$
"pencil of surfaces"

$$\Delta_f \rightarrow \begin{array}{cc} & 4 \\ 4 & \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \end{array} \leftarrow \Delta_g$$

$\det(\lambda \Delta_f + \mu \Delta_g)$

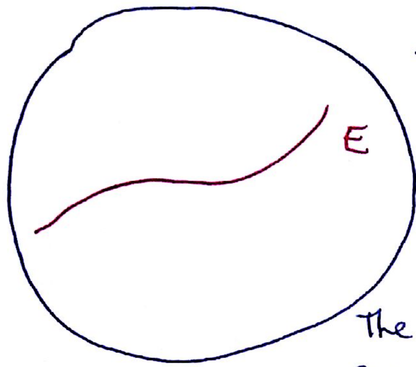
For generic $[\lambda : \mu] \rightsquigarrow \mathbb{P}^1 \times \mathbb{P}^1 \setminus E$

$\exists$ 4 values of $[\lambda : \mu] \rightsquigarrow$ cone over $\mathbb{P}^1$.

For higher $\mathbb{C}P^n$'s:

Fix $E \hookrightarrow \mathbb{C}P^n$

$V \in \Gamma(TE)$ - v.f. on E $\rightsquigarrow$ Feigin–Odeskii Poisson structure on $\mathbb{C}P^n$



n even:

0-dim'l sympl. leaves for n even: E

2-dim'l sympl. leaves: $\mathrm{Sec}_1 E = \overline{\{\text{all secants through } E\}}$ "secant" variety

The 3-dim'l manifold (singular) $\mathrm{Sec}_1 E$ is foliated by 2-dim'l symplectic leaves away from E.

4-dim'l sympl. leafs: $\mathrm{Sec}_2 E = \overline{\{\text{all planes through 3 points in } E\}}$

$\dim \mathrm{Sec}_2 E = 5$

$\mathrm{Sec}_2 E \setminus \mathrm{Sec}_1 E$ – smooth, foliated by 4-dim'l sympl. leafs.

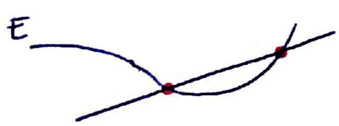
6-dim'l sympl. leafs ... so on

$\mathrm{Sec}_d E$ – singular

$\mathrm{Sec}_1 E$: $\quad$ resolved by $\longleftrightarrow$

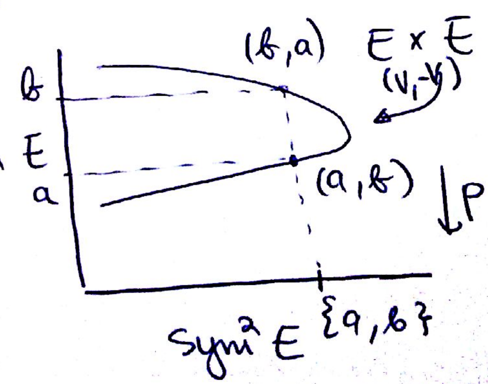
$$\begin{array}{ccc} \mathbb{P}' & \longrightarrow & \mathbb{P} \\ & & \downarrow \\ & & \mathrm{Sym}^2 E \end{array}$$

$\uparrow$ smooth

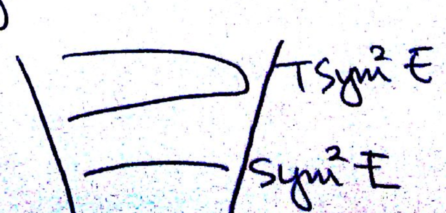


Idea for smoothness:

$\mathrm{Sym}^n \mathbb{C}P^1 = \{\{x_1, \dots, x_n\} \subseteq \mathbb{C}P^1\} \xleftrightarrow{1:1} s \in \mathbb{P}(\Gamma(\mathcal{O}(n))) = \mathbb{P}(\mathbb{C}^{n+1}) = \mathbb{C}P^n$



$Tp: T(E \times E) \longrightarrow T(\mathrm{Sym}^2 E)$ gives $E \times E \hookrightarrow \mathrm{tot}\,(T(\mathrm{Sym}^2 E))$



coHiggs field on V:

$\phi \in \Gamma(\mathrm{End}(V) \otimes TX)$ + integrability

Spectral correspondence

$\dashrightarrow$ spectral variety + line bundle on it.

+ Polishchuk's result

Thm Poisson structures on $\mathrm{tot}(P)$ which project onto the 0 Poisson str. on S

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & P \\ & & \downarrow \\ & & S \end{array}$$

$\Updownarrow$ 1:1

co-Higgs fields on V - rk. 2 v. bundle s.t. $P = \mathbb{P}\left(\begin{smallmatrix} V \\ \downarrow \\ S \end{smallmatrix}\right)$