GRT Learning Seminar

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Coideal subalgebras

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Ref: Letzter [Coideal subalgebras and quantum symmetric pairs]
[L]

Idea

corr. quantum object

* Hopf alg: $\Bbbk G$, $U(L)$, $U_q(\mathfrak{g})$ $\quad \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$
* study interesting subalg. of some interesting Hopf alg./quantum gps.

I) General setting/results:

1) Hopf algebras:

H Hopf alg./$\Bbbk$ is a vector space over $\Bbbk$ with:

$\mu : H \otimes H \longrightarrow H$ multiplication
$\imath : \Bbbk \longrightarrow H$ unit
] coalg. homomorphisms

$\Delta : H \longrightarrow H \otimes H$ comultiplication
$\varepsilon : H \longrightarrow \Bbbk$ counit
] alg. homomorphisms

$\sigma : H \longrightarrow H$ antipode

such that:

$(H, \mu, \imath)$ is an algebra.

(H, Δ, ε) is a coalgebra

$$\begin{tikzcd} H \otimes H \arrow[r, "\mathrm{id} \otimes \sigma", "\sigma \otimes \mathrm{id}"'] & H \otimes H \arrow[d, "\mu"] \\ H \arrow[u, "\Delta"] \arrow[r, "\imath \circ \varepsilon"'] \arrow[ur, phantom, "\circlearrowright"] & H \end{tikzcd}$$

2) Coideal (subalg.) and general results.

- A subspace $I \subseteq H$ is a <u>left coideal</u> if $\Delta(I) \subseteq H \otimes I$ (similarly, right coideal if $\Delta(I) \subseteq I \otimes H$)

i.e. $I \subseteq H$ is a left (resp. right) H-comodule.

General results $\quad$ e.g. $\langle x_i, y_i, t_i^{\pm 1} | \ldots \rangle$

(A) Assume $\Bbbk G \subseteq H$ (G a group), $H = \Bbbk G \oplus Y$

$P: H \longrightarrow \Bbbk G$ projection. Assume moreover that

$(P \otimes id) \circ \Delta : H \longrightarrow \Bbbk G \otimes H$ makes H a left $\Bbbk G$-comodule.

Then any left coideal of H is a left $\Bbbk G$-module.

$$\left[H = \bigoplus_{g \in G} H_g \rightsquigarrow I = \bigoplus_{g \in G} I \cap H_g \right]$$

(B) $H \circlearrowleft$ adjoint action

$$(\operatorname{ad} a) b = \sum a_{(1)} b \, \sigma(a_{(2)}) \quad \text{where } \Delta(a) = \sum a_{(1)} \otimes a_{(2)}.$$

I left coideal in H, M Hopf subalg.

Then, $(\operatorname{ad} M) I$ is an $(\operatorname{ad} M)$-invariant left coideal of H.

$\Delta((\operatorname{ad} a) b) \in H \otimes (\operatorname{ad} M) I$.

II) More specific results

1) Group algebra $\Bbbk G$:

anti alg. hom

- $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, $\sigma(g) = g^{-1}$
- If I coideal subalg., then $I \cap G$ is a semigroup and I is spanned by $I \cap G$ as a vector space.

2) Universal enveloping algebra $U(L)$.

- $\Delta(x) = 1 \otimes x + x \otimes 1$, $\varepsilon(x) = 0$, $\sigma(x) = -x$
- [Mo] The set of coideal subalg. of $U(L)$ is precisely $\{ U(L') \mid L' \text{ subalg. of } L \}$.

3) Quantized enveloping alg. $U_q(\mathfrak{g})$

- $\mathfrak{g}$ complex semisimple Lie alg. with root system Δ, simple roots $\Pi = \{\alpha_i\}$

$(-,-)$ standard form on $\mathfrak{h}^*_{\mathbb{R}}$

- $U_q(\mathfrak{g})$ is the unital assoc. alg. over $\mathbb{C}(q)$ generated by $x_i, y_i, t_i^{\pm 1}$ $(1 \le i \le n)$ $\leftarrow q^{h_i}$
with standard rel'ns.

(1) $x_i y_j - y_j x_i = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ $\qquad q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$

(2) The t_i's generate a free abelian gp. T of rank n.

(3) $t_i x_j t_i^{-1} = q^{(\alpha_i, \alpha_j)} x_j$, $t_i y_j t_i^{-1} = q^{-(\alpha_i, \alpha_j)} y_j$.

(4) Quantum Serre relations

$$\Delta(t) = t \otimes t \ (t \in T), \ \varepsilon(t) = 1, \ \sigma(t) = t^{-1}$$

$$\Delta(x_i) = x_i \otimes 1 + t_i \otimes x_i, \ \varepsilon(x_i) = 0, \ \sigma(x_i) = -t_i^{-1} x_i$$

$$\Delta(y_i) = y_i \otimes t_i^{-1} + 1 \otimes y_i, \ \varepsilon(y_i) = 0, \ \sigma(y_i) = -y_i t_i$$

Note: $U_q(\mathfrak{g})$ specializes to $U(\mathfrak{g})$ as $q \to 1$.

$A = \mathbb{C}[q^{\pm 1}]$, $\hat{U} = \langle x_i, y_i, t_i^{\pm 1}, \dfrac{t_i - 1}{q_i - 1} \rangle A$

$\hat{U} \otimes_A \mathbb{C} \cong U(\mathfrak{g})$.

$S \subseteq U = U_q(\mathfrak{g})$, S specializes to $\bar{S}$ in $U(\mathfrak{g})$.

$S \cap \hat{U} \rightsquigarrow \bar{S} \subseteq U(\mathfrak{g})$

"quantum analogs"

(A) Some specific coideal subalg. of $U = U_q(\mathfrak{g})$.

(a) The locally finite part of U.

$F(U) = \{ v \in U \mid \dim (\operatorname{ad} U) v < \infty \}$.

This is a left coideal subalg. of U.

Note: Related to quantum Harish-Chandra modules.

[classical: $(\mathfrak{g}, \mathfrak{m})$ $\quad V =$ sum of simple $\mathfrak{m}$-modules]

(b) Nilpotent & parabolic subalg.

- $G^- = \langle y_i t_i \rangle_{i=1}^n$, $U^+ = \langle x_i \rangle_{i=1}^n$

Easily check that G^-, U^+ are left coideal.

$$\rightsquigarrow U = G^- \otimes \overset{\Bbbk T}{\overset{\parallel}{U^\circ}} \otimes U^+$$

- Can study coideal subalg. of G^-, U^+ and generators (ad T-modules)

G^- quantum analog of $\mathcal{U}(n^-)$

(B) More generally:

$$G^+ = \langle x_i t_i^{-1} \rangle_{i=1}^n, \quad U^- = \langle y_i \rangle_{i=1}^n$$

$$U \cong U^- \otimes (\Bbbk T) \otimes G^+$$

$$\Rightarrow U = \bigoplus_{t \in T} U^- G^+ t$$

- I is a left coideal of U, then $I = \bigoplus_{t \in T} (I \cap U^- G^+ t)$
- I left coideal subalg. s.t. $I = \sum_{\lambda, \mu, t} (I \cap U_\lambda^- G_\mu^+ t)$

and $I \cap T$ is a group. Then $I \cap G^-, I \cap U^+, I \cap U^\circ$ are (ad T)-submodules and left coideal subalg. of I.

We have iso. of vector spaces:

$$I \cong (I \cap G^-) \otimes (I \cap U^\circ) \otimes (I \cap U^+)$$

given by multiplication

<u>Note</u>: One can generalize this using some filtrations on U. (technical).

4) <u>Quantized function algebra</u>: $R_q[G]$

- $U = U_q(\mathfrak{g}) \rightsquigarrow U^*$

"Hopf dual"

$U^{dual} = \{ \xi \in U^* \mid \xi(I) = 0$ for some 2-sided ideal I of finite codim$\}$

• V is a left U-mod, define $c^V_{\xi,v} \in U^{dual}$

$v \in V$ $\quad c^V_{\xi,v}(a) = \xi(av)$ "matrix coeff."
$\xi \in V^*$

$$C^V = \mathrm{span}_{\mathbb{k}}\{c^V_{\xi,v} \mid \xi \in V^*, v \in V\}$$

<u>Define</u>

$$R_q[G] = \text{spanned by } C^{V(\mu)} \quad (\mu \in P^+)$$

$$\overset{\text{Peter-Weyl}}{\cong} \bigoplus_{\mu \in P^+} V(\mu) \otimes V(\mu)^*$$

<u>Note</u>: 1) $R_q[G]$ specializes to $R[G]$ as $q \to 1$.
Hopf subalg. of U^{dual}

2) Peter-Weyl

• I left coideal subalg. of U.

M = left U-module

$$M^I_\ell = \{m \in M \mid a.m = \varepsilon(a)m \ \forall a \in I\}$$

"left I-invariants of M"

<u>Theorem</u> $\forall I$ left coideal subalg. of U,
$R_q[G]^I_\ell$ is a left coideal subalg. of $R_q[G]$.

I is quantum analog of $U(\mathfrak{a})$.

$\mathfrak{a} \subseteq \mathfrak{g} \rightsquigarrow H \subseteq G$

$\rightsquigarrow R_q[G]^I_\ell$ corresponds to $R[G/H] = R[G]^H$

$$R[G]^H \subset R[G]$$

One reason might be interested: quantum homogeneous spaces.

$$G \times G/H \to G/H$$