$B_{w_0} = \langle F_{\beta_1}^{(a_1)} \cdots F_{\beta_K}^{(a_K)} \mid \beta_i \in \Lambda_+$ canonical basis.

B_{w_0} is independent of choice w_0

$\mathcal{L} := \mathrm{span}_{\mathbb{Z}[q]} B_{w_0}$

$= \mathrm{span}_{\mathbb{Z}[q]} B.$

If V_λ is a $U_q^- \mathfrak{g}$-module, then $B \cap I_\lambda$ is a basis of I_λ.

$V_\lambda = U_q^- \mathfrak{g} / I_\lambda$

$\Rightarrow B \setminus I_\lambda$ spans V_λ

<u>Fact</u> $I_\lambda = \sum_{i \in I} U_q^- \mathfrak{g}\, F_i^{c_i+1}$ where c_i are coefficients in $\lambda = \sum c_i w_i$.

Coideal subalgebras, categorification and duality

Iva

<u>Ref</u> (Ehrig - Stroppel)

0) <u>Motivation</u>. $\mathfrak{g}$ type A :

Brundan - Kleshchev '09

rep th. of Hecke alg $\longleftrightarrow$ Lusztig's canonical basis

grading on cyclotomic Hecke alg's (also from parabolic cat $\mathcal{O}$) $\rightsquigarrow$ Kazhdan-Lusztig polynomials

using: cyclotomic quotients of quiver Hecke alg [Khovanov - Lauda '08, '09]

$\mathfrak{g}$: other types : $\mathfrak{g} \xrightarrow[\text{duality}]{\text{Schur-Weyl}}$ Brauer alg.

In this paper : $\mathfrak{g}$ = type D.

$\mathcal{O}^{\mathfrak{p}}(\mathfrak{g}) \xleftrightarrow{\text{relate}}$ Brauer algebras and degeneration VW-algebras. (Combinatorics)

Main goals : 1) Connect VW-alg with cat $\mathcal{O}$ and KL combinatorics

2). obtain canonical basis for rep's of coideal subalg in quantum groups

3). Categorify skew Howe duality.

I) Preliminaries.

① $\mathfrak{g} = \mathfrak{so}_{2n}$ Fix $4 \leq n \in \mathbb{N}$. $I^+ = \{1, 2, \cdots, n\}$

$I = I^+ \cup -I^+$

$V = \mathbb{C}\{v_i \mid i \in I\}$.

$\mathfrak{gl}(V)$ $J = (J_{k,l} = \delta_{k,-l})$ $k, l \in I$

$$\begin{array}{c|c} & n \cdots 1 \; -1 \cdots -n \\ \hline \begin{array}{c} n \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -n \end{array} & \begin{bmatrix} 0 & \begin{matrix} & 1 \\ \cdot & \end{matrix} \\ \begin{matrix} & 1 \\ 1 & \end{matrix} & 0 \end{bmatrix} \end{array} = J.$$

$\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$

$= \{A \in \mathfrak{gl}(V) \mid JA + A^{tr}J = 0\}$

$= \{A \in \mathfrak{gl}(V) \mid A_{ij} = -A_{-j,-i}\}$

$= \{X \in \mathfrak{gl}(V) \mid \langle Xv, w\rangle + \langle v, Xw\rangle = 0\}$

$\langle -,- \rangle$ induced by J

$\mathfrak{h} = \{A \in so_{2n}(\mathbb{C}) \mid \text{diagonal}\}$.

$\mathfrak{h}^* = \mathbb{C}\{\varepsilon_i \mid i \in I^+\}$ s.t. $wt(v_i) = \begin{cases} \varepsilon_i, & i \in I^+ \\ -\varepsilon_i, & i \in I^- \end{cases}$

simple roots $\alpha_0 = \varepsilon_1 + \varepsilon_2$

$\alpha_i = \varepsilon_{i+1} - \varepsilon_i \quad 1 \leq i \leq n-1$

root.

$\alpha \in R_n = R(so_{2n}) = \{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\}$

$\downarrow$

X_α root vector w/ weight α.

$\forall i \in I^+$: $X_i \in \mathfrak{h}$ dual to ε_i. $so_{2n}(\mathbb{C}) = \mathbb{C}\{X_\gamma \mid \gamma \in R_n \cup I^+\}$

$\mathfrak{n}^+ = \{X_{\varepsilon_i \pm \varepsilon_j} \mid i > j\}$ Borel: $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$.

$\mathfrak{n}^- = \{X_{-(\varepsilon_i \pm \varepsilon_j)} \mid i > j\}$.

$V = L(\varepsilon_n)$ natural rep.

$\mathfrak{l} = \mathbb{C}\{X_{\pm(\varepsilon_i - \varepsilon_j)} \mid i > j\} \oplus \mathfrak{h} \cong gl_n(\mathbb{C})$ Levi subalg

Standard max. parabolic $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$ (of type A corresp to $\alpha_1, \cdots, \alpha_{n-1}$)

$\forall i \in I$: $v_i^* = v_{-i}$ the element dual to v_i under $\langle -, - \rangle$. $X_\gamma^* =$ dual to X_γ under the Killing form of $\mathfrak{g} = so_{2n}$

2) $\mathfrak{g}$-endomorphisms.

$V = L(\varepsilon_n)$.

M = $\mathfrak{g}$-mod. Linear endomorphisms $V \otimes V \to V \otimes V$

$\tau : v \otimes w \longmapsto \langle v, w \rangle \sum_{i \in I} v_i \otimes v_i^*$

$\sigma : v \otimes w \longmapsto w \otimes v$.

$d \geq 0$

They induce maps in $\mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d})$

$S_i = \mathrm{Id}_M \otimes \mathrm{Id}_V^{\otimes(i-1)} \otimes \sigma \otimes \mathrm{Id}_V^{\otimes(d-i-1)}$

$e_i = \mathrm{Id}_M \otimes \mathrm{Id}_V^{\otimes(i-1)} \otimes \tau \otimes \mathrm{Id}_V^{\otimes(d-i-1)}$ $\quad 1 \leq i \leq d-1$

Casimir $C = \sum_{\gamma \in R_n \cup I^+} X_\gamma X_\gamma^* \subset \mathcal{U}\mathfrak{g}$

pseudo-Casimir $\Omega = \sum_{\gamma \in R_n \cup I^+} X_\gamma \otimes X_\gamma^* \subset \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$;

(Relation : $\Omega = \frac{1}{2}(\underbrace{\Delta}_{\text{coproduct of } \mathcal{U}(\mathfrak{g}).}(C) - C \otimes 1 - 1 \otimes C)$)

$\Omega_{ij} = \sum_{\gamma \in R_n \cup I^+} 1 \otimes \cdots \otimes \underset{i}{X_\gamma} \otimes \cdots \otimes \underset{j}{X_\gamma^*} \otimes \cdots \otimes 1 \in \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d})$

$y_i = \sum_{0 \leq k < i} \left(\Omega_{ki} + \frac{2n-1}{2} \mathrm{Id}\right) \in \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d})$, $i \in I$

M = h, wt $\mathfrak{g}$-mod

($\exists\ 0 \neq m \in M$ $\quad \mathfrak{n}^+ \cdot m = 0$. $\quad \mathfrak{h} m \subseteq \mathbb{C} m$. $\quad \Rightarrow \mathrm{End}_{\mathfrak{g}}(M) = \mathbb{C}$)

$\mathcal{U}(\mathfrak{n}^-) \cdot m = M$

* $\lambda \in \mathfrak{h}^* \longrightarrow$ 1-dim $\mathbb{C}_\lambda$

Verma: $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ h.w.

$\Downarrow$

$L(\lambda)$ irred.

II) VW-algebras. (replacement for degenerate affine Hecke alg's in type A)

Def $d \in \mathbb{N}$

A VW-alg./affine Nazarov algebra with Ξ = set of parameters $\omega_k \in \mathbb{C}$, $k \geq 0$, is:

$\mathbb{W}_d = \mathbb{W}_d(\Xi)$

- generators s_i, e_i, y_j $\quad 1 \leq i \leq d-1$, $1 \leq j \leq d$

- relations

$\left(\begin{array}{c} 1 \leq a, b \leq d-1 \\ 1 \leq c < d-1 \\ 1 \leq i, j \leq d \end{array} \right)$

① $s_a^2 = 1$

② i) $s_a s_b = s_b s_a \quad |a-b| > 1$

ii) $s_c s_{c+1} s_c = s_{c+1} s_c s_{c+1}$

iii) $s_a y_i = y_i s_a \quad i \notin \{a, a+1\}$.

③ $e_a^2 = \omega_0 e_a$

④ $e_1 y_1^k e_1 = \omega_k e_1 \quad k \in \mathbb{N}$.

⑤ i) $s_a e_b = e_b s_a$, $e_a e_b = e_b e_a \quad |a-b| > 1$.

ii) $e_a y_i = y_i e_a \quad i \notin \{a, a+1\}$

$y_i y_j = y_j y_i$

⑥ ⑦ ⑧

* all relations are symmetric / come from symm. pairs.

$\Rightarrow \mathbb{W}_d(\Xi) \cong \mathbb{W}_d(\Xi)^{opp}$ canonically.

<u>Lemma</u> M: h-wt mod $\quad \mathfrak{g} = \mathfrak{so}_{2n}$

$\forall k \in \mathbb{N} \; \exists \; a_k(M) \in \mathbb{C}$ s.t. $e_1 y_1^k e_1 = a_k(M) e_1 \in \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d})$

$a_0(M) = 2n$.

<u>Thm</u> M : h. wt module

$\Xi_M = \{ a_k(M) \mid k \geq 0 \}$

$\Rightarrow \exists$ well-defined right action $M \otimes V^{\otimes d} \circlearrowleft \mathbb{W}_d(\Xi_M)$

$M \otimes V^{\otimes d} \ni$

$p \cdot s_i = s_i(p), \qquad p \cdot w_k = a_k(M) p.$

$p \cdot e_i = e_i(p), \qquad p \cdot y_j = y_j(p), \qquad 1 \leq i \leq d-1, \; 1 \leq j \leq d$

In particular, we have hom of algebras

$$\Psi_M : \mathbb{W}_d(\Xi_M) \longrightarrow \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d}).$$

(can be modified for types B & C)

II) Parabolic setting

$\mathcal{O}' = \bigoplus_{\lambda \text{ wt}} \mathcal{O}_{\chi_\lambda} \qquad (\chi_\lambda$: central character$)$

integral blocks for cat $\mathcal{O}$ of $\mathfrak{g}$.

$\cup$

$\mathcal{O}^{\mathfrak{p}}(n)$ = subcat of $\mathfrak{p}$-finite modules.

weight $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathfrak{h}^*$

integral in $\mathbb{Z}\{\varepsilon_i\}$ or $(\mathbb{Z}+\frac{1}{2})\{\varepsilon_i\}$.

$\Rightarrow$ supported on integers or half-integers.

Parabolic Verma module

$$M^{\mathfrak{p}}(\lambda) = \text{max quotient of } M(\lambda) \text{ which is locally } \mathfrak{p} \text{ finite.}$$

$$= \begin{cases} 0 & \lambda \text{ not } \mathfrak{p}\text{-dominant} \\ U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E(\lambda), & \text{where } E(\lambda) \text{ some f-d. } \mathfrak{l}\text{-mod with h.w. } \lambda. \end{cases}$$

$\mathfrak{p}$-dominant wts:

$$\Lambda = \{ \lambda \in \mathfrak{h}^* \mid \lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i, \quad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \}$$

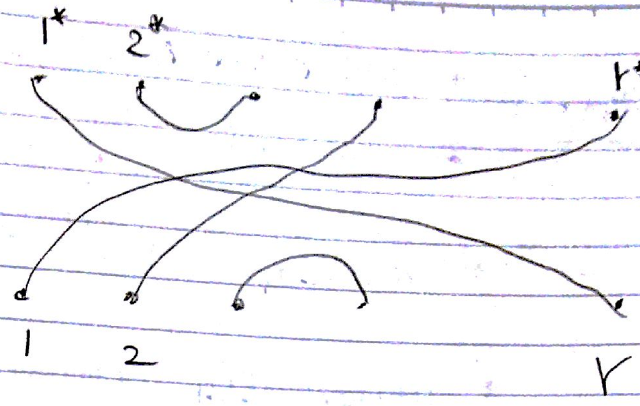
$$= \underset{\text{supp. } \mathbb{Z}}{\Lambda^1} \cup \underset{\mathbb{Z}+\frac{1}{2}}{\Lambda^0}$$

Brauer alg. $\quad r \in \mathbb{Z}_{\geq 0}$.

Brauer diagram on $2r$ vertices is a partition b of the set $\{1, 2, \cdots, r, 1^*, 2^*, \cdots, r^*\}$ into r subsets of cardinality 2.

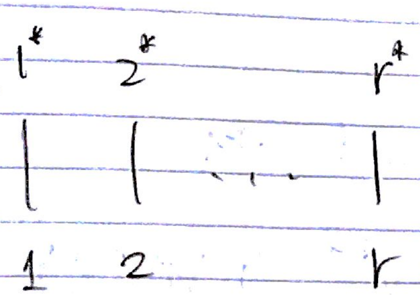
$B[r]$ = the set of all Brauer diagrams.

Graphically:



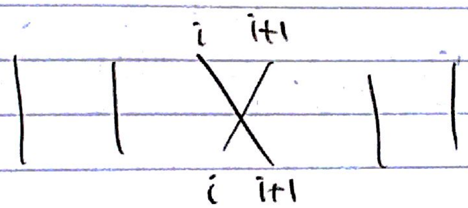
- each vertex connects to another one
- "=" if they connect the same vertices

"unit" 1



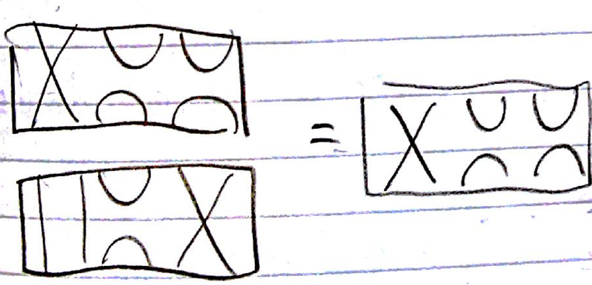
$\tilde{e}_i$

"$\tilde{s}_i$"



Concatenation: $b \circ b'$: - identify i^* in b with i in b'

- remove any loops



$c(b, b')$ = # loops from $b \circ b'$

Def A Brauer algebra $Br_d(\delta)$ for $d \in \mathbb{Z}_{\geq 0}$, $\delta \in \mathbb{C}$ is the $\mathbb{C}$-alg with
- basis $b \in B[d]$
- mult $b \cdot b' := \delta^{c(b,b')}\, b \circ b'$.

Properties
* associative w/ 1
* generated by $\tilde{s}_i$, $\tilde{e}_i$, $1 \le i \le d-1$
* $\dim Br_d(\delta) = d!! = 1 \cdot 3 \cdot 5 \cdots d$
* generically semisimple for $\delta \geq 0$

$(\delta \neq 0 \;\&\; \delta \geq d-1)$ or $(\delta = 0, d = 1, 3, 5, \dots)$

Remark $M = L(0)$ trivial rep.

$\Rightarrow$ the action $M \otimes V^{\otimes d} \circlearrowleft \mathbb{W}_d(\Xi)$

factors through

$$Br_d(2n) \overset{\text{canon.}}{\simeq} \mathbb{W}_d(\Xi) \Big/ \left(y_1 - \tfrac{2n-1}{2}\right)$$

III) Cyclotomic quotients and admissibility

Def The parameters ω_a, $a \geq 0$ are admissible if

$$\omega_{2a+1} + \frac{1}{2}\omega_{2a} - \frac{1}{2}\sum_{b=1}^{2a} (-1)^{b-1} \omega_{b-1}\omega_{2a-b+1} = 0.$$

Note: Admissible $\Rightarrow$ nice basis for $\mathbb{W}_d(\Xi)$

$b \in Br_d(\delta) \rightsquigarrow$ in terms of $\tilde{e}_i$, $\tilde{s}_i$

$\rightsquigarrow$ corr. element $B \in \mathbb{W}_d(\Xi)$.

For $\gamma, \eta \in \mathbb{Z}^r_{\geq 0}$, $b \in Br_d(\delta)$.

$$y_1^{\gamma_1} y_2^{\gamma_2} \cdots y_d^{\gamma_d} \, B \, y_1^{\eta_1} y_2^{\eta_2} \cdots y_d^{\eta_d}$$

regular if $\gamma_i \neq 0 \Rightarrow$ $i =$ left endpoint of horizontal arc in b.

$\gamma_i = 0 \Rightarrow$ $i^* =$ " "

If the parameters Ξ are admissible, this is a basis for $\mathbb{W}_d(\Xi)$.

<u>Def</u> Cyclotomic W-alg. of level ℓ with parameters $u = (u_1, u_2, \cdots, u_\ell) \in \mathbb{C}^\ell$:

$$\mathbb{W}_d(\Xi, u) := \mathbb{W}_d(\Xi) \Big/ \prod_{i=1}^{\ell} (y_1 - u_i).$$

<u>Special case</u> $M = M^b(\underline{\delta})$ $\quad \delta \in \mathbb{Z} \rightsquigarrow \underline{\delta} = \delta \omega_0 = \delta \sum_{i=1}^{n} \varepsilon_i$

<u>Proposition</u>

$\exists$ isom of $\mathfrak{g}$-mods

$$M(\underline{\delta}) \otimes V \cong M(\underline{\delta} - \varepsilon_1) \oplus M(\underline{\delta} + \varepsilon_n)$$ eigenspace decomp for y_1

with eigenvalues $\alpha = \frac{1}{2}(1 - \delta)$

$$\beta = \frac{1}{2}(\delta + 2n - 1)$$

<u>Thm</u> For $n \geq 2d$, $\delta \in \mathbb{Z}$, get an isom

$$\Psi^{d,n}_{M(\underline{\delta})} : \mathbb{W}_d(\Xi_{M(\underline{\delta})}; \alpha, \beta) \xrightarrow{\sim} \mathrm{End}_{\mathfrak{g}}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})$$