Metric spaces

Definition.
Let X be a set. A function $d: X \times X \to \mathbb{R}_+$ is called a **metric** if it satisfies the following three conditions:

1. $\rho(x, y) = 0 \Leftrightarrow x = y$ (positive definitness)
2. $\rho(x, y) = \rho(y, x)$ (symmetry)
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ " (triangle inequality)

A pair (X, ρ), where ρ is a metric on X is called a **metric space**.

Examples.

0. Any set X with $\rho(x, y) = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$ - *discrete metric.*
1. $\mathbb{R}$ with $\rho(x, y) = |x - y|$
2. Any subset of $\mathbb{R}$ with the same metric.
3. *Uniform metric*
   Let S be any set and let $F(S) = \{f: S \to \mathbb{R}_+ : f(S) \text{ is bounded}\}$.
   Define $\rho(f, g) \coloneqq \sup_{x \in S} |f(x) - g(x)|$.
   Particular cases:
   $S = \{1, 2, \ldots, n\}$. Then we get $\mathbb{R}^n$ with the distance $\rho(x, y) = \max|x_i - y_i|$.
   $S = \mathbb{N}$. We get the space of all bounded real sequences. Notation: l_∞ .

Will have many more examples later, as the course proceeds.

Definition.
Let (x_n) be a sequence of elements of X. We say that $\lim_{n \to \infty} x_n = x \in X$ if $\rho(x_n, x) \to 0$.

Definition.
Let (x_n) be a sequence of elements of X. We say that (x_n) is a *Cauchy sequence* if
$\forall \varepsilon > 0 \exists N: m, n > N \Rightarrow \rho(x_n, x_m) < \varepsilon$.

Definition.
Let (x_n) be a sequence of elements of X. We say that x is a *limit point of* (x_n) if
$\forall \varepsilon > 0$ $\{n: \rho(x_n, x) < \varepsilon\}$ is infinite.

Equivalently:
x is a limit point of (x_n) if there exists a subsequence $x_{n_k} \to x$.

Definition.
A metric space (X, ρ) is called *complete* if every Cauchy sequence converges to a limit.

Already know: $\mathbb{R}$ with the usual metric is a complete space.

Theorem.
$F(S)$ with the uniform metric is complete.

Proof.
Let (f_n) be a Cauchy sequence in $F(S)$.
$\forall s \in S$, the sequence of real numbers $f_n(s)$ is a Cauchy sequence (check it!).
Since $\mathbb{R}$ is a complete space, the sequence has a limit. Denote $f(s) \coloneqq \lim f_n(s)$.
Then $\forall s \in S, |f_n(s) - f(s)| = \lim_{m \to \infty} |f_n(s) - f_m(s)| \leq \sup_m \rho(f_n, f_m)$.
Since (f_n) is a Cauchy sequence,
$\forall \varepsilon > 0 \exists N: n > N \Rightarrow \sup_m \rho(f_n, f_m) \leq \varepsilon$.
Re-write it as
$\forall \varepsilon > 0 \exists N: n > N$\Rightarrow $\forall s \in S, |f_n(s) - f(s)| \leq \varepsilon$.
This means that $f \in F(S)$ and $\rho(f_n, f) \to 0$. ■

The space $\mathbb{R}^n$.

Definition.
$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \mathbb{R}}_{n \text{ times}}$, with the metric(will check it later!)

$$|\boldsymbol{x} - \boldsymbol{y}| := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Additional structures.
1. It is an $n-$dimensional vector space.
2. It has an *inner product*

$$\boldsymbol{x} \cdot \boldsymbol{y} := \sum_{i=1}^{n} (x_i y_i)$$

So
$|\boldsymbol{x} - \boldsymbol{y}|^2 = (\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq 0$

Let us now prove an important
Schwarz inequality.
For all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq |\boldsymbol{x}||\boldsymbol{y}|.$
Proof.
Consider
$g(\lambda) := |\boldsymbol{x} + \lambda \boldsymbol{y}|^2 = (\boldsymbol{x} + \lambda \boldsymbol{y}) \cdot (\boldsymbol{x} + \lambda \boldsymbol{y}) = |\boldsymbol{y}|^2 \lambda^2 + 2\boldsymbol{x} \cdot \boldsymbol{y} \lambda + |\boldsymbol{x}|^2.$
$g(\lambda)$ is positive quadratic function of λ. Let us plug in
$\lambda = -\boldsymbol{x} \cdot \boldsymbol{y} / |\boldsymbol{x}|^2 |\boldsymbol{y}|^2$
to get
$0 \leq g(\lambda) = |\boldsymbol{x}|^2 - {(\boldsymbol{x} \cdot \boldsymbol{y})^2}/{|\boldsymbol{y}|^2}$ ■

Now we can check that $|\boldsymbol{x} - \boldsymbol{y}|$ is indeed a metric.
Proof.
Positive definiteness and symmetry are obvious.
Notice that by Schwarz inequality
$|\boldsymbol{x}|^2 + 2\boldsymbol{x} \cdot \boldsymbol{y} + |\boldsymbol{y}|^2 \leq |\boldsymbol{x}|^2 + 2|\boldsymbol{x}||\boldsymbol{y}| + |\boldsymbol{y}|^2$, or
$|\boldsymbol{x} + \boldsymbol{y}|^2 \leq (|\boldsymbol{x}| + |\boldsymbol{y}|)^2.$
Taking the square root, we get
$|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$
which implies triangle inequality.■
Lemma.(Convergence in $\mathbb{R}^n$)
A sequence of vectors (x_n) is a Cauchy sequence iff all n cooordinate sequences are Cauchy sequences.
A sequence of vectors (x_n) converges to a vector $\boldsymbol{x}$ iff all n coordinate sequences converge to the corresponding coordinate of $\boldsymbol{x}$.
Proof.
See the book.■
Corollary.
$\mathbb{R}^n$ is complete.

Topology of metric space

Definition.
Let (X,ρ) be a metric space. An open ball of radius $r>0$ centered at x is defined as $B_r(x) \coloneqq \{y \in X: \rho(x,y) < r\}$.
Definition.
Let (X,ρ) be a metric space, $S \subset X$. Define:
$Int(S) = \{x \in X: \exists r > 0\ B_r(x) \subset S\}$ - the interior of S.
$Ext(S) = \{x \in X: \exists r > 0\ B_r(x) \subset X \setminus S\} = \{x \in X: \exists r > 0\ B_r(x) \cap S = \emptyset\}$ - the exterior of S.
$Bd(S) = X \setminus \big(Int(S) \cup Ext(S)\big) = \{x \in X: \forall r > 0\ B_r(x) \cap X \setminus S \neq \emptyset, B_r(x) \cap S \neq \emptyset\} = \{x \in X: \forall r > 0\ \exists y \notin S, z \in S: |z-x| < r, |y-x| < r\}$ - the boundary of S.

Examples.
1. If X has discrete metric, $\forall S \subset X, Int(S) = S, Ext(S) = X \setminus S, Bd(S) = \emptyset$.
2. If X is the real line with usual metric, $S = \{\frac{1}{n}: n \in \mathbb{N}\}$, then $Int(S) = \emptyset$, $Ext(S) = \mathbb{R} \setminus (S \cup \{0\}), Bd(S) = S \cup \{0\}$.
Remarks.
1. $Int(S) = Ext(X \setminus S), Ext(S) = Int(X \setminus S), Bd(S) = Bd(X \setminus S)$.
2. $S_1 \subset S_2 \Rightarrow Int(S_1) \subset Int(S_2), Ext(S_2) \subset Ext(S_1)$.
Definition.
S is called *open* if $\mathrm{S} = \mathrm{Int(S)}$.
S is called *closed* if $\mathrm{Bd(S)} \subset \mathrm{S}$.
Lemma.
S is open iff $X \setminus S$ is closed.
Proof.
S is open iff $Bd(S) \cap S = \emptyset$ iff $Bd(S) = Bd(X \setminus S) \subset X \setminus S$ iff $X \setminus S$ is closed.■
Lemma.
Union of any number of open sets is open.
Intersection of finitely many open sets is open.
Proof.
Let (O_α) be any collection of open sets.
If $x \in \cup O_\alpha \Rightarrow \exists \alpha: x \in O_\alpha \Rightarrow \exists r > 0: B_r(x) \subset O_\alpha \subset \cup O_\alpha$, so $\cup O_\alpha$ is open.
If $(O_j)_{j=1}^n$ is a finite collection of open sets, then
$x \in \cap O_j \Rightarrow \forall j: x \in O_j \Rightarrow \forall j \exists r_j > 0: B_{r_j}(x) \subset O_j$.
Let $r = \min r_j$. Then
$\forall j\ B_r(x) \subset B_{r_j}(x) \subset O_j \Rightarrow B_r(x) \subset \cap O_j$. So $\cap O_j$ is open. ■
Corollary.
Intersection of any number of closed sets is closed.
Union of finitely many closed sets is closed.
Proof.
We just need to use the identities
$X \setminus (\cup O_\alpha) = \cap (X \setminus O_\alpha)$
$X \setminus \big(\cap O_j\big) = \cup (X \setminus O_j)$■
Examples.
1. $B_r(x)$ is open for all $x \in X, r > 0$.
Proof.
$y \in B_r(x) \Rightarrow \rho(x,y) < r \Rightarrow B_{r-\rho(x,y)}(y) \subset B_r(x)$ by triangle inequality.■
2. $Int(S), Ext(S)$ are open, $Bd(S)$ is closed.
Proof.
$x \in Int(S) \Rightarrow \exists B_{r(x)}(x) \subset S \Rightarrow Int(S) = \cup_{x \in Int(S)} B_{r(x)}(x)$, so it is open as a union of open sets.
$Ext(S) = Int(X \setminus S)$, so it is open.
Finally, $Bd(S) = X \setminus \big(Int(S) \cup Ext(S)\big)$, so it is closed.■

Definition.

The *closure* of a set S is defined as
$Clos(S) := S \cup Bd(S)$.

Theorem. (Alternative characterization of the closure).

$x \in Clos(S)$ iff $\exists x_n \to x, x_n \in S$. ($x$ is a *limit point* of S).

Proof.

Note that $x \in Clos(S)$ iff $\forall r > 0 \exists y \in S: \rho(x,y) < r$.
If $\exists x_n \to x, x_n \in S$, then $\rho(x_n, x) \to 0$, so $\forall r > 0 \exists x_n \in S: \rho(x, x_n) < r$.
Thus $x \in Clos(S)$.
On the other hand, let $x \in Clos(S)$. Fix n, then $\exists y: \rho(x,y) < \frac{1}{n}$. Take $x_n := y$.
Since $\rho(x, x_n) < \frac{1}{n}$, $x_n \to x$. ■

Yet another characterization of closure.

For $S \subset X, x \in X$, define $\text{dist}\,(x,S) := \inf_{s \in S} \rho(x,s)$.
Then $x \in Clos(S)$ iff $\text{dist}\,(x,S) = 0$.

Remark.

$S = Clos(S)$ iff S is closed.

Lemma.

$Clos(S)$ is closed. If C is closed, $S \subset C$, then $Clos(S) \subset C$.

Proof.

Note that
$Bd(S) = X \setminus \big(Int(S) \cup Ext(S)\big)$, so it is closed as a compliment of an open set.
If $S \subset C$, then $Ext(C) \subset Ext(S)$, so $Clos(S) = X \setminus Ext(S) \subset X \setminus Ext(C) = C$■

Remark.

Thus we have another definition of the closed set: it is a set which contains all of its limit points.

Lemma.

Let (X, ρ) be a complete metric space, $S \subset X$. (S, ρ) is a complete metric space iff S is closed in X.

Proof.

Assume that S is closed in X. Let (x_n) be a Cauchy sequence, $x_n \in S$. Since X is complete,
$\exists x \in X, x = \lim x_n$. But S is closed, so $x \in S$.
On the other hand, let (S, ρ) be complete, and let x be a limit point of S, so $x = \lim x_n$ (in X!), $x_n \in S$. Then (x_n) is convergent, so it is Cauchy, so it converges in S. So $x \in S$. Thus S contains all of its limit points, so it is closed.■

Definition.

$S \subset X$ is called *bounded,* if $\exists x \in X, R > 0: S \subset B_R(x)$.

Remark (Hausdorff metric).

For a metric space (X, ρ), let us consider the space $K(X)$ of all nonempty closed bounded subset of X with the following metric:
$\rho_H(A,B) := \max\left(\sup_{a \in A} \text{dist}\,(a,B), \sup_{b \in B} \text{dist}\,(b,A)\right)$.
Check that it is well-defined and a metric!

Continuous functions

Let (X,ρ) and (Y,d) be two metric spaces.

Definition.

Let $x \in X$. Let $f: X \setminus \{x\} \to Y$ be a mapping from $X \setminus \{x\}$ to Y.We say that L is a *limit of f at x*, $L = \lim_{y\to x} f(y)$ if

$\forall\varepsilon > 0 \exists\delta > 0$: $0<\rho(x,y) < \delta \Rightarrow d(f(y),L) < \varepsilon$.

Remark. f does not have to be defined at x!

Example.

$\lim_{x\to 0} {\sin x}/{x} = 1$.

Characterization of the limit in terms of sequences.

$L = \lim_{y\to x} f(y)$, iff for every sequence $x_n \to x, x_n \in X \setminus \{x\}$, we have $\lim f(x_n) = L$.

Proof.

Let $L = \lim_{y\to x} f(y)$, and $x_n \to x, x_n \in X \setminus \{x\}$. Fix $\varepsilon > 0$. Then

$\exists\delta > 0: \rho(x_n,x) < \delta \Rightarrow d(f(x_n),L) < \varepsilon$, and $\exists N: n > N \Rightarrow \rho(x_n,x) < \delta$.

Combining these two assertions we get

$d(f(x_n),L) \to 0$.

Assume that $f(y) \nrightarrow L$ when $y \to x$. It means that

$\exists\varepsilon > 0 \forall\delta > 0, \exists y \in X \setminus \{x\}: \rho(x,y) < \delta, d(f(y),L) \geq \varepsilon$.

In particular, it means that

$\forall n \exists x_n \in X \setminus \{x\}: \rho(x_n,x) < \frac{1}{n}, d(f(x_n),L) \geq \varepsilon$.

Thus $x_n \to x, f(x_n) \nrightarrow L$. ∎

Definition.

$f: X \to Y$ is called *continuous at* $x \in X$ if $f(x) = \lim_{y\to x} f(y)$. Otherwise, f is called dis*continuous at* x.

Sequential definition of continuity

f is continuous at x iff for every sequence $x_n \to x, x_n \in X$, we have $\lim f(x_n) = f(x)$.

Proof.

The same as for the limit. ∎

Topological definition of continuity.

f is continuous at x iff

$\forall\varepsilon > 0 \exists\delta > 0$: $B_\delta(x) \subset f^{-1}\big(B_\varepsilon(f(x))\big)$.

Examples.

1. Identity function is continuous at every point.
2. Every function from a discrete metric space is continuous at every point.
3. The following function on $\mathbb{R}$
   $$f(x) = \begin{cases} 0, x \notin \mathbb{Q} \\ {1}/{q}, x = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N}, GCD(p,q) = 1 \end{cases}$$
   is continuous at every irrational point, and discontinuous at every rational point.
4. $f(x) = \begin{cases} 0, x \notin \mathbb{Q} \\ 1, x \in \mathbb{Q} \end{cases}$
   is discontinuous at every point as a function on $\mathbb{R}$, but continuous at every point as a function on $\mathbb{Q}$.
5. Let X be the usual space $\mathbb{R}^n$ with the standard metric, and Y be the same space with the uniform metric. Then the map $f(\boldsymbol{x}) = \boldsymbol{x}$ is continuous as a function $X \to Y$ and $Y \to X$ - check it!

Definition.

$f: X \to Y$ is called a *continuous function on* X if f is continuous at every point of X.

Topological characterization of continuous functions.

$f: X \to Y$ is a *continuous function on* X iff $\forall U \subset Y$- open, the set $f^{-1}(U)$ is open in X.

Proof.

If f is continuous, and $x \in f^{-1}(U)$, then $\exists \varepsilon > 0\colon\ B_\varepsilon\big(f(x)\big) \subset U$ and $\exists \delta > 0\colon\ B_\delta(x) \subset f^{-1}\Big(B_\varepsilon\big(f(x)\big)\Big) \subset f^{-1}(U)$, so $x \in Int(f^{-1}(U))$. Thus $f^{-1}(U) = Int(f^{-1}(U))$.

On the other hand, $B_\varepsilon\big(f(x)\big)$ is an open set, so $f^{-1}\Big(B_\varepsilon\big(f(x)\big)\Big)$ is an open set which contains x. Thus it also contains some ball centered at x. ■

Compactness

Intuitively: topological generalization of finite sets.

Definition.
A metric space (X, ρ) is called *sequentially compact* if every sequence (x_n) of elements of X has a limit point in X. Equivalently: every sequence has a converging sequence.
Example:
A bounded closed subset of $\mathbb{R}$ is sequentially compact, by Heine-Borel Theorem.
Non-example:
If a subset of a metric space is not closed, this subset can not be sequentially compact: just consider a sequence converging to a point outside of the subset!

Definition.
Let (X, ρ) be a metric space. A subset $Y \subset X$ is called r*-net* if $\forall x \in X$ dist $(x, Y) < r$. A metric space (X, ρ) is called *totally bounded* if $\forall r > 0 \exists$ finite r-net.
Example:
Any bounded subset of R^n.
Non-examples.
1. Any unbounded set.
2. Consider the following subset of l^∞: $X \coloneqq \{(x_n) : \sup x_n \leq 1\}$. X is bounded, but not totally bounded.
   Proof.
   Denote by y^j an element of X which is a sequence with 1 in j-th position, and 0 in all others. Note that $\rho(y^j, y^k) = 1$ if $j \neq k$. Thus X can not have a finite $1/2$-net! ■
3. Infinite space with discrete topology (but any finite space is totally bounded!)

Definition.
Open cover of a metric space (X, ρ) is a collection $(U_\alpha)_{\alpha \in A}$ of open subsets of X, such that $\bigcup_{\alpha \in A} U_\alpha = X$. The space (X, ρ) is called *compact* if every open cover contain a finite sub cover, i.e. if we can cover (X, ρ) by some collection of open sets, finitely many of them will already cover it!
Equivalently: (X, ρ) is compact if any collection of closed sets has non-empty intersection if any *finite* sub collection has non-empty intersection. (For the proof, just pass to the complements).

Example:
Any finite set.
Non-examples.
1. Any unbounded subset of any metric space.
2. Any incomplete space.

Turns out, these three definitions are essentially equivalent.
Theorem.
The following properties of a metric space (X, ρ) are equivalent:
1. (X, ρ) is compact.
2. (X, ρ) is sequentially compact.
3. (X, ρ) is complete and totally bounded.

Proof.
$\mathbf{1. \Rightarrow 2.}$
Assume that (X, ρ) is *not* sequentially compact. Let (x_n) be a sequence without limit points. Then all the sets $F_k \coloneqq \{x_n, n \geq k\}$ are closed, finitely many of them have non-empty intersection, and $\bigcap_k F_k = \emptyset$ -contradiction!
$\mathbf{2. \Rightarrow 3.}$
A limit point of a Cauchy sequence is its limit (check it!), so (X, ρ) is complete if it is sequentially compact.

Assume now that for some $r > 0$ there is no finite r-net. It means that one can inductively construct a sequence (x_n) such that $\rho(x_n, x_m) \geq r$ if $m \neq n$. This sequence does not have a limit point, because for any $y \in X$, $B_{r/2}(x)$ contains onlyone member of the sequence - contradiction.

3. ⇒ 1. (The most interesting part of the proof. It is helpful to compare with the proof of Heine-Borel Theorem).

Let $(U_\alpha)_{\alpha \in A}$ be an open cover without finite sub covers. Call a set **bad** if no finite sub collection of $(U_\alpha)_{\alpha \in A}$ covers it. Thus we assumed that X itself is bad. Notice another property of bad set: if a finite number of other sets covers a bad set, one of them should be bad.

Since there is a finite $1/2$-net, one can find some bad ball $B_{2^{-1}}(x_1)$. Because there is a finite $1/4$-net, one can find some bad ball $B_{2^{-2}}(x_2)$ intersecting the first one. Thus we can inductively construct a sequence of bad balls $B_{2^{-n}}(x_n)$, such that $B_{2^{-n}}(x_n) \cap B_{2^{-n+1}}(x_{n+1}) \neq \emptyset$. Since $\rho(x_n, x_{n+1}) \leq 2^{-(n-1)}$, (x_n) is a Cauchy sequence, so, by completeness of X, it has a limit $x \in X$. $x \in U_\alpha$ for some $\alpha \in A$, since $(U_\alpha)_{\alpha \in A}$ is a cover of X. Since U_α is open, $B_x(\varepsilon) \subset U_\alpha$ for some $\varepsilon > 0$. Now find a large n, such that $\rho(x_n, x) < \varepsilon/2$, and $2^{-n} < \varepsilon/2$. It means that $B_{2^{-n}}(x_n) \subset U_\alpha$, so $B_{2^{-n}}(x_n)$ is covered by **one** set from $(U_\alpha)_{\alpha \in A}$, so it can not be bad - contradiction! ■

Properties of compact sets

1. A subset of $\mathbb{R}^n$ is compact iff it is bounded and closed. (Since totally bounded is the same as bounded in $\mathbb{R}^n$).
2. If X is compact, and $f: X \to Y$ is a continuous map, then $f(X)$ is also compact.
   Proof.
   Let $(U_\alpha)_{\alpha \in A}$ be an open cover of $f(X)$. Then $(f^{-1}(U_\alpha))_{\alpha \in A}$ is an open cover of X. By compactness of X, it has a finite sub cover $f^{-1}(U_{\alpha_1}), f^{-1}(U_{\alpha_2}), \dots, f^{-1}(U_{\alpha_n})$. Then $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$ is a finite open cover of $f(X)$. ∎
3. Let $f: X \to \mathbb{R}$ be a continuous function. Then f is bounded (i.e. $f(X)$ is a bounded set). Moreover, it reaches its maximum and minimum on X: $\exists y \in X, z \in X$, such that for any $x \in X$ we have $f(z) \leq f(x) \leq f(y)$.
   Proof.
   $f(X)$ is compact subset of $\mathbb{R}$, so it is closed and bounded.
   Thus $\sup f(X) \in f(X)$, so $\exists y \in X: \sup f(X) = f(y)$. Similarly, $\exists z \in X: \inf f(X) = f(z)$.
   By the definition of supremum and infimum, for any $x \in X$ we have $f(z) \leq f(x) \leq f(y)$. ∎
4. **Uniform continuity.**
   Definition.
   Let (X, ρ) and (Y, d) be two metric spaces. $f: X \to Y$ is called *uniformly continuous* if $\forall \varepsilon > 0 \exists \delta > 0: x, z \in X,\ \rho(x, z) < \delta \Rightarrow d\big(f(x), f(z)\big) < \varepsilon$.
   Remark.
   It is stronger then usual continuity at every point because δ here depends only on the ε, and not on the point x!
   Non-example.
   $f(x) = x^2$ is continuous at every point of $\mathbb{R}$, but not uniformly continuous!
   Theorem.
   Every continuous function on a compact set is uniformly continuous.
   Proof.
   Let f be a continuous but not uniformly continuous function on compact space (X, ρ). Since f is not uniformly continuous,
   $\exists \varepsilon > 0 \forall n \exists x_n, y_n \in X: \rho(x_n, y_n) < {}^1/_n, d(f(x_n), f(y_n)) \geq \varepsilon.$

   Sequence (x_n) has a subsequence (x_{n_k}) converging to $x \in X$. Since $\rho\big(x_{n_k}, y_{n_k}\big) \to 0$, subsequence (y_{n_k}) also converges to x. By continuity of f at x, $f(x) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k})$. But $d\big(f(x_{n_k}), f(y_{n_k})\big) \geq \varepsilon$, so these two sequences can not have the same limit - contradiction! ∎

Connected metric spaces

Definition.
A metric space (X,ρ) is called *disconnected* if there exist two non empty disjoint open sets $U,V \subset X, U \cap V = \emptyset$: such that $U \cup V = X$.
(X,ρ) is called *connected* otherwise.

The main property.
If $f: X \to Y$ is a continuous function, then $f(X)$ is connected.
Proof.
If $f(X) = U \cup V, U \cap V = \emptyset$, then $X = f^{-1}(U) \cup f^{-1}(V), f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Since X is connected, one of the sets $f^{-1}(U)$ and $f^{-1}(V)$ is empty. Thus either U or V is empty.■
Examples.

1. Any discrete compact space with more than one element is disconnected.
2. $\mathbb{Q}$ is not connected. ($\mathbb{Q} = \{x \in \mathbb{Q}: x < \sqrt{2}\} \cup \{x \in \mathbb{Q}: x > \sqrt{2}\}$).
3. A subset X of real line is said to have *intermediate point property* if $a < b < c, a, c \in X \Rightarrow b \in X$.

Lemma 1.
Nonempty subset of the real line has intermediate point property iff it is a point, an interval, a ray, or the whole real line.
Lemma 2.
Subset of the real line is connected iff it has an intermediate point property.
Corollary.
Nonempty subset of the real line is connected iff it is a point, an interval, a ray, or the whole real line.
Proof of Lemma 1.
Clearly all the sets mentioned in the statement satisfy intermediate point property.
There are four possibilities: X is bounded both above and bellow, X is bounded above but not bellow, X is bounded bellow but not above, X is not bounded above or bellow. I will consider only the first case, others are done the same way.
Let $a = \inf X, b = \sup X$. If $a = b$, then X is just one point.
Let $a \neq b$, and let $a < x < b$. Then, since $x < b$ and $b = \sup X$, $\exists c \in X: x < c < b$.
Same way, $\exists d \in X: a < d < x$. Thus $c, d \in X, c < x < d$. Thus, by intermediate value property, $x \in X$. We just proved that $\forall x: a < x < b \Rightarrow x \in X$. Thus X is an interval (open, closed, semi-open, or semi-closed) with endpoints a and b. ■
Proof of Lemma 2.
First assume that X does not have the intermediate point property, i.e. we can find $a < b < c$: $a, c \in X, b \notin X$. But then both $U := \{x \in X: x > b\}$ and $V := \{x \in X: x < b\}$ are not empty ($a \in U, c \in V$), open, and $U \cup V = X$. Thus X is disconnected.
Assume that X has an intermediate point property, and assume that $X = U \cup V$, $U \cap V = \emptyset$, where U and V are nonempty open sets. Let $a \in U, b \in V$, and let, say, $a < b$ (one of the two numbers has to be larger).
Since X has an intermediate value property, $a \leq x \leq b \Rightarrow x \in X$. This means that $[a,b] \subset X$.
Let $c = \sup\{x \in [a,b] \cap U\} \in X$.
Assume that $c \in U$. Then $c < b$ (since $b \notin U$). Since U is open, $\exists (c-\delta, c+\delta) \subset U$, so $c + \frac{\delta}{2} \in U$ - can not happen because c is an upper bound for U.
Thus $c \in V$. Then $c > a$ (since $a \notin V$). Since V is open, $\exists (c-\delta, c+\delta) \subset V$, so there are no $x \in U: x > c - \delta$ - can not happen because $c = \sup U$. Thus we arrive to a contradiction, which shows that X is connected.■

Theorem (Intermediate Value Theorem).
Let (X,ρ) be a connected metric space, and $f: X \to \mathbb{R}$ be a continuous function. Let

$x, y \in X, b \in \mathbb{R}$, and $f(x) \leq b \leq f(y)$. Then $b = f(z)$ for some $z \in X$.

Proof.

$f(X)$ is a connected subset of $\mathbb{R}$, so it satisfies an intermediate point property. ■

Path-connected spaces

Definition.
Let (X, ρ) be a metric space, $x, y \in X$. A *path* from x to y is a continuous function $\gamma: [a, b] \to X$, such that $\gamma(a) = x, \gamma(b) = y$.
Definition.
(X, ρ) is called *path-connected* if for every two points $x, y \in X$ there exists a path from x to y.
Theorem.
Every path-connected space is connected.
Remark.
The opposite is not true!
Proof.
Let $X = U \cup V$, where U and V are open non-empty nonintersecting sets, and let $x \in U, y \in V$. Let γ be a path joining x and y. Then $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are non-empty nonintersecting open subsets of $[a, b]$, and $\gamma^{-1}(U) \cup \gamma^{-1}(V) = [a, b]$. This contradicts the connectedness of $[a, b]$ ■
Examples.
1. $\mathbb{R}^n$ is path-connected.
2. Any convex subset of $\mathbb{R}^n$ is path-connected.
3. $\mathcal{F}(S)$ is always path-connected, as well as any of its convex subsets.