Complex Analysis

Assignment 8, due November 30

Problem 1 of 5. Let f be a non-constant function analytic in a region Ω, and the sequence of the functions (f_n), analytic in Ω, converge to f locally uniformly. Let $f(z_0) = 0$. Show that there exists a sequence $z_n \to z_0$ such that for any n, $f_n(z_n) = 0$.

Problem 2 of 5. Problem 3, parts (b), (d), (f), (g), (h); page 161 of *Ahlfors*.

Problem 3 of 5. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be an analytic function in $\mathbb{C} \setminus \{0\}$. Assume that

$$\forall N \in \mathbb{N}\, \exists m > N, n > N \ : a_n \neq 0, a_{-m} \neq 0.$$

Let $M(r) := \max_{|z|=r} |f(z)|$. Show that for any $k \in \mathbb{N}$

$$\lim_{r \to 0} \frac{1}{r^k M(r)} = \lim_{r \to \infty} \frac{r^k}{M(r)} = 0.$$

Problem 4 of 5. Problem 1, page 186 of *Ahlfors*.

Problem 5 of 5. Use the cotangent trick to compute

$$\sum_{n=1}^{\infty} \frac{1}{n^6 + n^2}.$$

Be careful at $z = 0$!