Complex Analysis

Assignment 6, due November 16

Problem 1 of 5. Let f be an entire function.

(1) Assume that an f satisfies the condition

$$|f(z)| \le C|z|^d, \quad |z| > R,$$

where C and R are some positive constants. Show that f is a polynomial of degree at most d.
Hint: Use Cauchy Inequlities to estimate $f^{(d+1)}(w)$.

(2) Assume that f has a removable singularity or pole at infinity (i.e. $\lim_{z\to\infty} f(z)$ exists, but might be infinite). Show that f is a polynomial.
Hint: It might be useful to consider the function $f(1/z)$ at 0

Problem 2 of 5. A function $f : \mathbb{R} \mapsto \mathbb{C}$ is called *real-analytic* on $\mathbb{R}$ if for any $x \in \mathbb{R}$ there exists $R_x > 0$ and a sequence of coefficients $(a_n^x)_{n=0}^{\infty}$ such that

$$f(y) = \sum_{n=0}^{\infty} a_n^x (y-x)^n, \text{ if } |y-x| < R_x.$$

Let f be a real analytic function.

(1) Show that there exists a region $\Omega \supset \mathbb{R}$ and a function F, analytic in Ω, such that for any $x \in \mathbb{R}$, $f(x) = F(x)$.

(2) Show that f is infinitely differentiable for any $x \in \mathbb{R}$, and $a_n^x = \frac{f^n(x)}{n!}$.

(3) Show that if (x_n) is a bounded real sequence, and for any n, $f(x_n) = 0$, then $f \equiv 0$.

Problem 3 of 5. Problem 5, page 130 of *Ahlfors*.

Problem 4 of 5. Let f be analytic in the region $\{z : |z| > R\}$. Assume that $\lim_{|z|\to\infty} f(z)$ exists and finite. Let, for $r > R$, $M(r) := \max_{|z|=r} |f(z)|$. Show that $M(r)$ is a decreasing function.
Hint: Consider $f(1/z)$.

Problem 5 of 5. Let P be a polynomial of degree d, $M(r) := \max_{|z|=r} |P(z)|$. Show that for any $0 < r_1 < r_2$, we have

$$\frac{M(r_1)}{r_1^d} \ge \frac{M(r_2)}{r_2^d}.$$

The equality is attained for some $0 < r_1 < r_2$ if and only if $P(z) = cz^d$ for some $c \ne 0$.