Let (a_n) be a bounded sequence of real numbers. Let us define sets

$$R_n := \{a_k \,:\, k \geq n\}\,.$$

Let $s_n := \sup R_n$, $m_n := \inf R_n$. Since $R_{n+1} \subset R_n$, we have $s_n \geq s_{n+1}$, $m_n \leq m_{n+1}$. Thus both sequences (s_n) and (m_n) are monotone and bounded (Why?), so they are both convergent to their infimum and supremum correspondingly.

Definition. Limit superior of (a_n) is defined as

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} s_n = \inf s_n.$$

Limit inferior of (a_n) is defined as

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} m_n = \sup m_n.$$

Example. Let $a_n = (-1)^n$. Then, for any n, $s_n = 1$, $m_n = -1$ and

$$\limsup a_n = 1, \qquad \liminf a_n = -1.$$

Proposition. *Let (a_n) be a bounded sequence of real numbers. Then*

$$\lim_{n\to\infty} a_n = a$$

if and only if

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$$

Proof. Let $\lim_{n\to\infty} a_n = a$. Fix $\varepsilon > 0$. For some N, if $n \geq N$ then $|a_n - a| < \varepsilon/2$. Then, for such n, $R_n \subset V_{\varepsilon/2}(a)$. Therefore,

$$a - \varepsilon < m_n = \inf R_n \leq \sup R_n = m_n < a + \varepsilon.$$

So for $n \geq N$ we have $|m_n - a| < \varepsilon$ and $|s_n - a| < \varepsilon$. So

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n = a$$

and

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} m_n = a.$$

On the other hand, $m_n \geq a_n \geq s_n$ (since $a_n \in R_n$). So, by the Squeezed sequence lemma, if

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} s_n = a = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} m_n,$$

then

$$\lim_{n\to\infty} a_n = a.$$

□