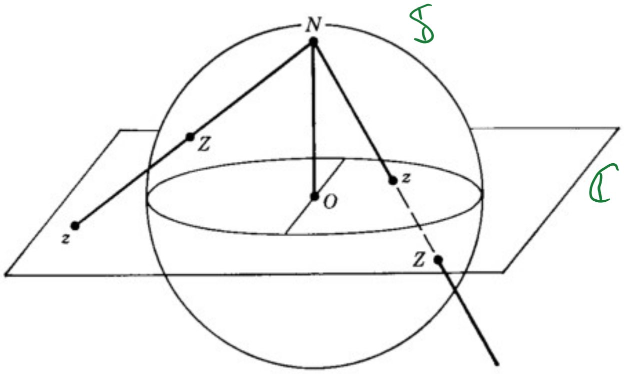
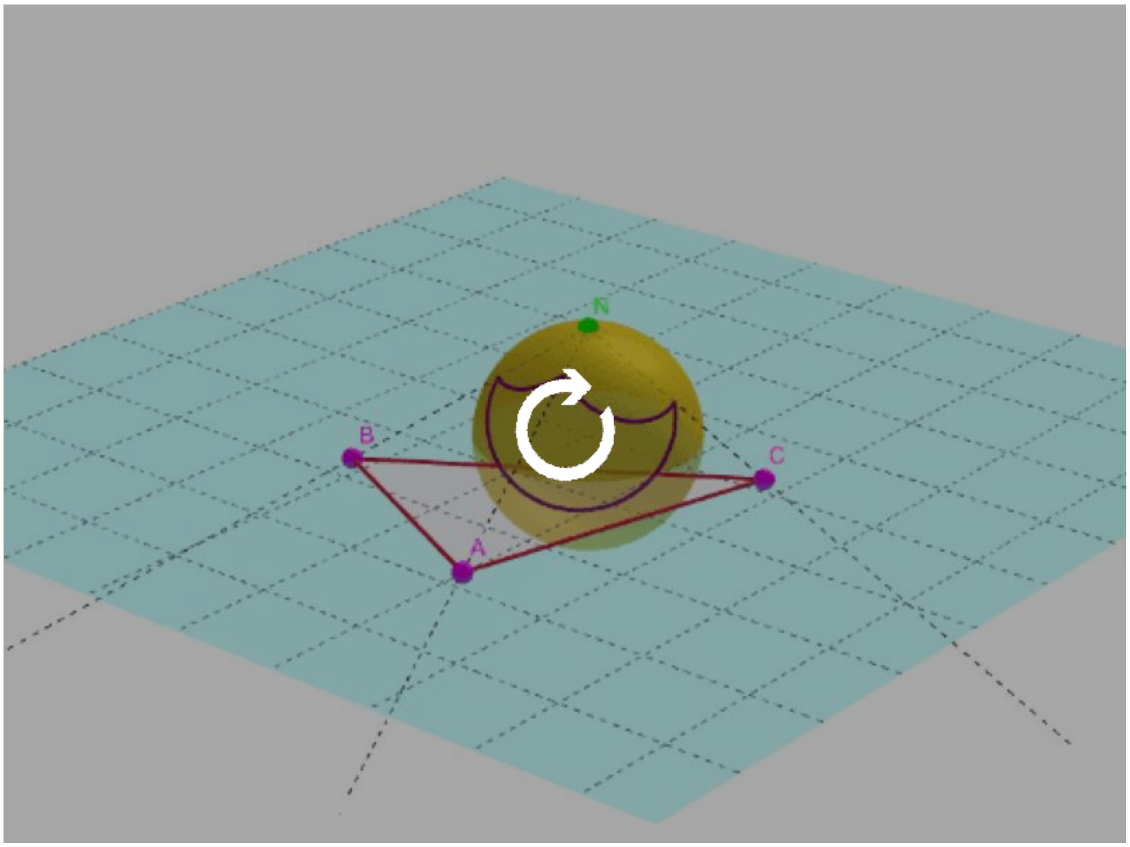
Stereographic projection

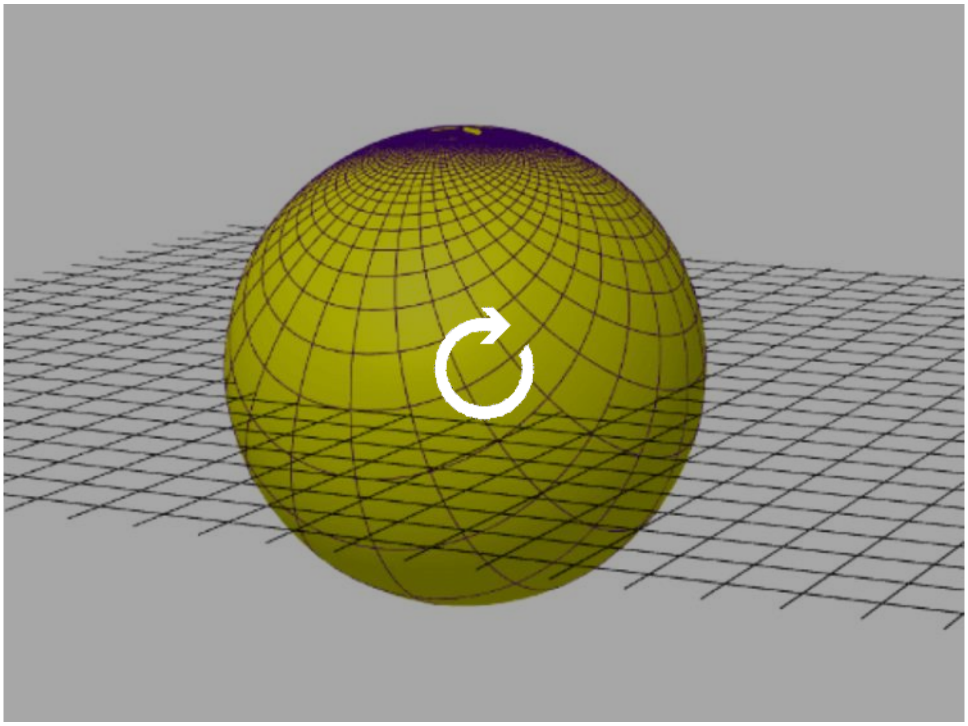
Monday, December 7, 2020 9:02 AM

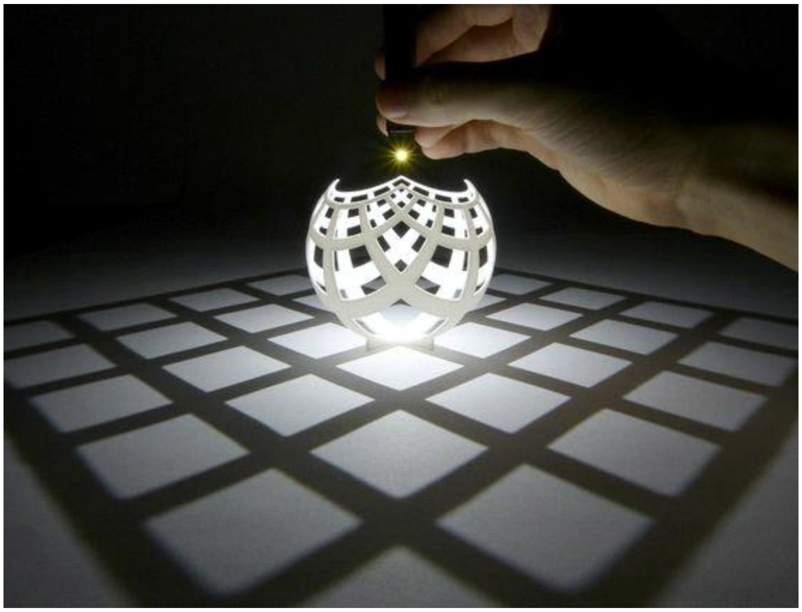


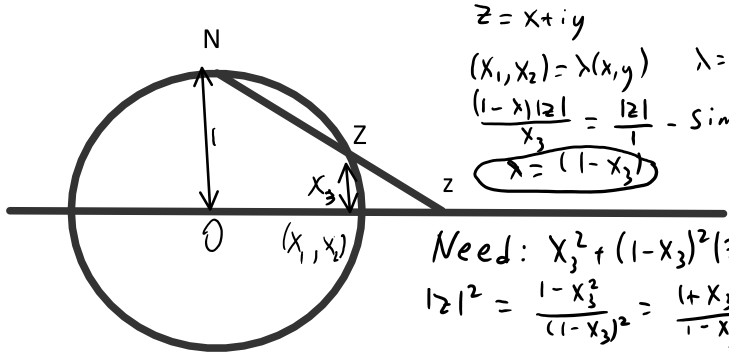
[Stereographic projection]



[Stereographic Projection of Coordinate Grid to Sphere]







$z = x + iy$

$(x_1, x_2) = \lambda(x, y) \qquad \lambda = ?$

$\frac{(1-\lambda)|z|}{x_3} = \frac{|z|}{1}$ - similar triangles!

$\lambda = (1 - x_3)$

Need: $x_3^2 + (1-x_3)^2 |z|^2 = 1$

$$|z|^2 = \frac{1 - x_3^2}{(1-x_3)^2} = \frac{1+x_3}{1-x_3} \Rightarrow \boxed{x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}}$$

$$x_1 = (1 - x_3)\, x = \frac{2x}{|z|^2 + 1} = \boxed{\frac{2 \operatorname{Re} z}{|z|^2 + 1} = x_1}$$

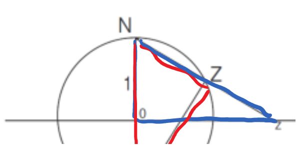
$$\boxed{x_2 = \frac{2 \operatorname{Im} z}{|z|^2 + 1}}$$

Other direction:

$$z = \frac{x_1 + i x_2}{1 - x_3}$$

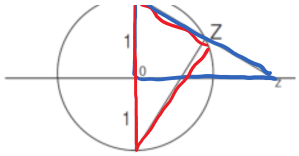
Spherical distance:

$d(z, z') := |Z - Z'|$ - distance on the sphere.



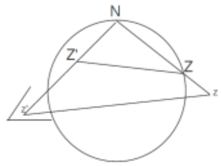
By similarity of triangles:

$$\frac{|N - Z|}{2} = \frac{1}{|N - z|} = \frac{1}{\sqrt{1 + |z|^2}}$$



$$\frac{|N-Z|}{2} = \frac{1}{|N-z|} = \frac{1}{\sqrt{1+|z|^2}}$$

Pythagoras



so

$$\frac{|N-Z|}{|N-z'|} = \frac{2}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}$$

Same reason:

$$\frac{|N-Z'|}{|N-z|} = \frac{2}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}$$

So $\Delta NZZ'$ and $\Delta Nzz'$ are similar!

$$\text{So } \frac{d(z,z')}{|z-z'|} = \frac{|Z-Z'|}{|z-z'|} = \frac{2}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}.$$

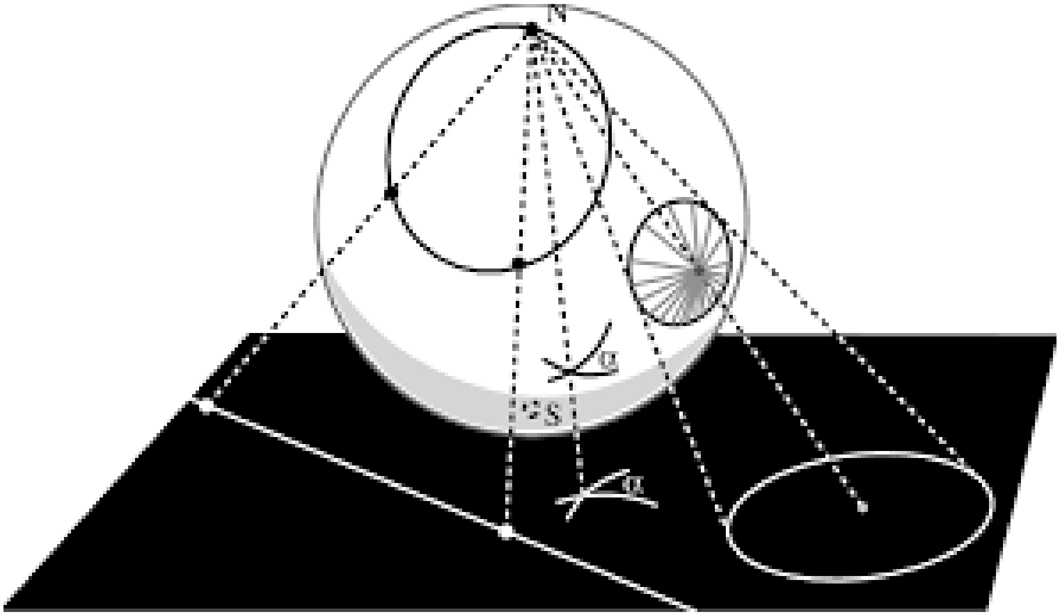
$$d(z,z') = \frac{2|z-z'|}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}$$

$$|Z-N| = d(z,\infty) = \lim_{z'\to\infty} d(z,z') = \frac{2}{\sqrt{1+|z|^2}}.$$

Bonus (+1 pt): $\tilde{d}(z,z') = \text{dist}_{\text{spherical}}(Z,Z') = ?$ (in terms of z, z').

<u>Circles and straight lines on $\mathbb{C}$ are mapped to circles.</u>

<u>Proof.</u> For straight line: the image is the intersection of the sphere S with the plane through the line and N: <u>a circle through N</u>!



Circle: computation:

$$(x-a)^2+(y-b)^2=r^2$$

substitute by formula: $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1-x_3}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$a x_1 + b x_2 + \frac{1+r^2-a^2-b^2}{2} x_3 = \frac{a^2+b^2-r^2+1}{2}$$ - equation of a plane!

So, again, the image is plane intersected with $\mathbb{S}$!

And any plane intersecting $\mathbb{S}$ and not through N is of this form!

Can be done geometrically: see

[Proof (stereographic projection proof that circles on the sphere project to circles in the plane)]

Some transformations:

On $\mathbb{C}$	On $\mathbb{S}$
$z \to \bar{z}$	$(x_1, x_2, x_3) \to (x_1, -x_2, x_3)$
$z \to \frac{1}{\bar{z}}$	$(x_1, x_2, x_3) \to (x_1, x_2, -x_3)$
$z \to \frac{1}{z}$	$(x_1, x_2, x_3) \to (x_1, -x_2, -x_3)$

All of them preserve $d(z, z')$!

Does $z \to z+1$ preserve $d(z, z')$? No!