Theorem (Schwarz Lemma). Let \( f \in \mathcal{A}(D) \), \( |f(z)| \leq 1 \) \( \forall z \in D \), \( f(0) = 0 \). Then \( \forall z \in D \) \( |f(z)| \leq |z| \), and \( |f'(0)| \leq 1 \).
If for some \( z \in D \) \( |f(z)| = 1 \) or \( |f'(0)| = 1 \), then \( f(z) = e^{i\theta}z \). (\( f \) is a rotation by \( \theta \)).

Proof. Let \( \varphi(z) = \left( \frac{f(z)}{z}, \frac{f'(0)}{2} \right) \). Then \( \varphi \in \mathcal{A}(D \setminus \{0\}) \), \( \lim_{|z| \to 0} \varphi(z) = \lim_{|z| \to 0} \frac{f(z)}{z} = f'(0) \), so \( \varphi \in \mathcal{A}(D) \).
Take \( r \in D \). Then, by maximum principle, \( \forall z \in B(r) \):
\[
|\varphi(z)| = \max_{|z|=r} |\varphi(z)| = \max_{|z|=r} \left| \frac{f(z)}{z} \right| \leq \frac{1}{r}.
\]
So \( \forall z \in B(1) \) we have \( |\varphi(z)| \leq 1 \) \( \Rightarrow |f(z)| \leq 1 \).
If for some \( z \), \( |\varphi(z)| = 1 \) (which is \( |f(z)| = |z| i \neq 2z0 \))
then \( |\varphi| \) reaches maximum at \( z \), so
\[
\varphi(z) = \text{const.} \quad (\text{const} \in D) \quad \text{const} = e^{i\theta}.
\]
\[
f(z) = \frac{z}{e^{i\theta}}.
\]

An invariant form of Schwarz Lemma.

Theorem (Schwarz-Pick).
Let \( f \in \mathcal{A}(D) \), \( f : D \to D \) (i.e. \( \forall z \in D \) \( |f(z)| < 1 \)).
Then \( \forall z_1, z_2 \in D \)
\[
|f(z_1) - f(z_2)| \leq |z_1 - z_2|.
\]
\[
\left| \frac{f(z_2) - f(z_1)}{1 - \frac{z_2}{z_1}} \right| \leq \left| \frac{z_2 - z_1}{1 - \frac{z_2}{z_1}} \right| \quad \text{and} \quad \left| f(z) \right| \leq \frac{1}{1 - |z|^2}
\]

If the equality is reached for some \( z_1, z_2 \in \mathbb{D} \) or for some \( z \in \mathbb{C} \), then \( f \) is a Möbius transformation \( \mathbb{D} \to \mathbb{D} \).

**Proof.** For \( w \in \mathbb{D} \), denote \( S_w(z) = \frac{z - w}{1 - \overline{w}z} \). Then \( S_{f(z)}(z) \) is a Möbius map \( \mathbb{S}^1 \to \mathbb{S}^1 \).

Consider the map \( g(z) = S_{f(z)}(z) \). Then \( g(0) = S_{f(0)}(0) = S_{f(0)}(z_1) = \{ \} \) and \( g : \mathbb{D} \to \mathbb{D} \)

So, by Schwarz lemma. (Since each map \( \partial \mathbb{D} \))

\[ |S_{f(z)}(z) + S_{f(z)}(w)| \leq |z| \quad \text{for } z \in \mathbb{D} \text{ and } w \in \mathbb{D} \]

Then \( S_{f(z)}(z) + S_{f(z)}(w) = 2z \). Hence, \( S_{f(z)}(z) = S_{f(z)}(w) = \frac{f(z) - f(w)}{1 - f(z)f(w)} \).

So it implies the inequality

\[ \left| \frac{f(z) - f(w)}{1 - \frac{z}{w}} \right| \leq \left| \frac{z - w}{1 - \frac{z}{w}} \right| \]

Let \( z_1 \to z_1 \) to get the second inequality.

Finally, equality is reached in any of the inequalities \( \Rightarrow \)

\[ |g(z)| = |z| \text{ for some } \tau \text{ or } \left| g'(0) \right| = \left| \right| \text{ for some } \tau \]

\[ f = S_{f(0)}^{-1} \circ S_w \circ S_{f(z)} \to \text{ Möbius } \]

**Def.** \( \rho (z_1, z_2) := \left| \frac{z_1 - z_2}{1 - \frac{z_1}{z_2}} \right| = \sqrt{\left| \frac{z_1 - z_2}{1 - \frac{z_1}{z_2}} \right|} \)

Quasi-hyperbolic metric.

Möbius maps fixing circle preserve:

1) Cross-ratio.
2) Points symmetric with respect to the unit circle.

So they preserve \( \rho \) if \( \tau e^{i \theta} \to \frac{\tau}{e^{i \theta}} \) then \( \rho (\tau z, \tau w) = \rho (z, w) \).

**Why is \( \rho \) metric?**

\[ \rho (z_1, z_2) = 0 \quad \text{iff} \quad z_1 = z_2 \] Obvious.

\[ \rho (z_1, z_2) = \rho (z_2, z_1) \]

\[ \rho (z_1, z_2) + \rho (z_2, z_3) \geq \rho (z_1, z_3) \]

Möbius is invariant, so map \( z_1 \to 0, z_2 \to 1, z_3 \to r > 0 \)

Then \( \rho (z_1, z_2) = r \), \( \rho (z_2, z_3) = \frac{r - 1}{1 - r} \), \( \rho (z_1, z_3) = \frac{r - 1}{1 - r} \)

For fixed \( r \), the image of the circle \( \{ |z| = 1 \} \) under \( S(z) = \frac{r z}{1 - rz} \) a circle symmetric with \( \mathbb{R} \), \( S(1z) = \frac{r - 1}{1 - r(1)} \), \( S(-1z) = \frac{r - 1}{1 - r(1)} \), \( S(1z) = \frac{r + 1}{1 + r} \), \( S(-1z) = \frac{r + 1}{1 + r} \), \( \rho (z_1, z_2) < r \).
Theorem. Let \( f \in A(D) \), \( f : D \to D \) be bijection. Then \( f \) is a Möbius map.

Proof. \( \forall z_1, z_2 \in D \).

\[
\rho(f(z_1), f(z_2)) \leq \rho(z_1, z_2)
\]

But \( f^{-1} : D \to D \), analytic.

So \( \rho(z_1, z_2) = \rho(f^{-1}(f(z_1)), f^{-1}(f(z_2))) \leq \rho(f(z_1), f(z_2)) \).

So \( \rho(z_1, z_2) = \rho(f(z_1), f(z_2)) \Rightarrow f \) is Möbius.

Corollary. \( \rho \) is invariant under all conformal bijections of \( D \) to itself.

Hyperbolic metric.

How to measure the length of curve?

\[
l(Y) = \int_{Y} \frac{1}{1-|z|^2} \, dt
\]

Know: Under Möbius, \( \frac{|f'(z)|}{1-|f(z)|^2} = \frac{1}{1-|z|^2} \). By Schwarz-Pick.

For any \( f \in A(D), f : D \to D \)

\[
\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}.
\]

Definition: Hyperbolic length of a path:

Let \( \gamma \) be a piecewise differentiable arc, parametrized by \( \gamma(t), t \in \mathbb{C}, \mathbb{E} \).

\[
l_{H}(\gamma) = \int_{\gamma} \frac{1}{1-|z|^2} \, dt = \int_{\gamma} \frac{|d\gamma|}{1-|z|^2}
\]

in \( D \)

Restatement of Schwartz-Pick:

For any curve \( \gamma \in D \) and any \( f : D \to D \) analytic

\[
l_{H}(\gamma) < l_{H}(\gamma). \quad \text{If the equality is reached for one curve, then } f \text{ is Möbius. If } f \text{ is Möbius, then } \forall \gamma : l_{H}(\gamma) = l_{H}(\gamma)
\]

Proof. \[
\frac{|f'(z)||z'(t)|}{1-|f(z)|^2} \leq \frac{12 |z|^2}{1-|z|^2}. \quad \text{Equality for one point(\( \infty \))}
\]

Equality for all points(\( \infty \)) if and only if \( f \) is Möbius.
Henri Poincaré

Any two points can be joined by a straight line. (This line is unique given that the points are distinct)

1. Any straight line segment can be extended indefinitely in a straight line.
2. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
3. All right angles are congruent.

The shortest Y, the arc of circle orthogonal to \( |z_1| = 1 \), joining \( z_1 \) and \( z_2 \).

Proof. Every thing (LHS, RHS, circles orthogonal to \( |z_1| = 1 \)) are Möbius invariant.
So we can map \( z_1 \) to 0, \( z_2 \) to a positive number \( r > 0 \).

Consider any path \( s \) from 0 to \( r \), \( s(t) = x(t) + y(t)i \)

\[
\int_0^1 \frac{|s'(t)|}{1 - |s(t)|^2} dt \geq \int_0^1 \frac{|x'(t)|}{1 - x(t)^2} dt \geq \int_0^1 \frac{x'(t)}{1 - x(t)^2} dt \geq \log \frac{1 + x(1)}{1 - x(1)} \bigg|_{t=0} \log 1 - 1 \]

with equality reached exactly when \( y \equiv 0 \) (\( y \equiv 0 \)) and

\( x'(t) \parallel x(t) \), i.e. when \( s = [0, r] \), travelled once.

Hyperbolic geometry:

Points: points in \( \mathbb{D} \)
Lines: circular arcs or intervals orthogonal to \( |z| = 1 \).

Poincaré disk model of hyperbolic geometry:

Satisfies all Euclidean Axioms except for paralles:

1. Any two points can be joined by a straight line. (This line is unique given that the points are distinct)
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. Through a point not on a given straight line, one and only one line can be drawn that never meets the given line.

Spherical geometry. Can be defined the same way on $\mathbb{S}^2$:

$$d_\mathbb{S}(x,y) = \sqrt{2(1 - \cos d)}$$
$$d_\mathbb{S}(z_1,z_2) = \inf \, d_\mathbb{S}(z)$$

Also satisfies all Euclidean Axioms except for parallels: lines are great circles, so there are no parallels!

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Euclidean</th>
<th>Spherical</th>
<th>Hyperbolic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infinitesimal length</td>
<td>$</td>
<td>dz</td>
<td>$</td>
</tr>
<tr>
<td>Oriented isometries</td>
<td>$e^{\theta z} + b$</td>
<td>rotations</td>
<td>conformal self-maps</td>
</tr>
<tr>
<td>Curvature</td>
<td>0</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>Geodesics</td>
<td>lines</td>
<td>great circles</td>
<td>circles $\perp$ unit circle</td>
</tr>
<tr>
<td>Angles of triangle</td>
<td>$\pi$</td>
<td>$&gt; \pi$</td>
<td>$&lt; \pi$</td>
</tr>
</tbody>
</table>

$z^2 - x^2 - y^2 = 0$