Power series: complex power series, uniform convergence

Monday, December 7, 2020 9:04 AM

Crush course in complex series and sequences:

Nothing new.

Some details: $z_n \to z \iff \begin{cases} \operatorname{Re} z_n \to \operatorname{Re} z \\ \operatorname{Im} z_n \to \operatorname{Im} z \end{cases}$ $\quad |z_n - z| \to 0$

$\sum_{n=1}^{\infty} z_n = a \iff \begin{cases} \sum \operatorname{Re} z_n = \operatorname{Re} a \\ \sum \operatorname{Im} z_n = \operatorname{Im} a \end{cases}$

Cauchy test: $\sum z_n$ converges $\iff \forall \varepsilon > 0\ \exists N : m > n > N : \left| \sum_{k=n}^{m} z_k \right| < \varepsilon.$

Remark $\sum_{n=1}^{\infty} z_n$ - converges $\Rightarrow |z_n| \to 0$ $\quad \not\Leftarrow$

Def. $\sum_{n=1}^{\infty} z_n$ converges absolutely $\iff \sum_{n=1}^{\infty} |z_n| < \infty$

Converging series, important. $\sum_{n=1}^{\infty} z_n$ converges $\iff \sum_{n=k}^{\infty} z_n \to 0$

$A = \left\{ \frac{(-1)^n}{n},\ n \in \mathbb{N} \right\}$ $\quad -1, -\frac{1}{3}, \dots, \frac{1}{2}$

lower bounds $\quad$ upper bounds

Supremum and infinum: $A \subset \mathbb{R}$

$a \in \mathbb{R}$ - upper bound for A if $x \in A \Rightarrow x \le a$

$a = \sup A$ if $\forall a'$ - u.b., $a' \ge a$ - least u.b. $\quad$ no u.b. $\sup A = +\infty$

$a \in \mathbb{R}$ - lower bound for A if $x \in A \Rightarrow x \ge a$

$a = \inf A$ if $\forall a'$ - l.b., $a' \le a$ - greatest l.b. $\quad$ no l.b. $\inf A = -\infty$

Lim sup and lim inf. (a_n) - a real sequence.

$R_n := \{ a_k, k \ge n \}$ $\quad u_n := \sup R_n \quad u_n \downarrow$

$\quad \ell_n := \inf R_n \quad \ell_n \uparrow$

$\overline{\lim}$ $\quad \limsup a_n := \lim u_n = \inf u_n$

$\underline{\lim}$ $\quad \liminf a_n := \lim \ell_n = \sup \ell_n$

$\overline{\lim}\, a_n = \sup \{ b : \# \{ n : a_n > b \} \text{ is infinite} \} = \inf \{ c : \# \{ n : a_n > c \} \text{ - finite} \}.$

Example. $a_n = (-1)^n + \frac{\sin n}{n}$ $\quad \overline{\lim}\, a_n = 1, \ \underline{\lim}\, a_n = -1.$

? $b = 1$ $\quad \# \{ n : a_n \ge 1 \}$ - infinite. There infinitely many even n: $\sin n > 0$. ?

$b > 1$ $\quad \exists N : \frac{1}{N} < b - 1 \Rightarrow \left(n > N \Rightarrow a_n = (-1)^n + \frac{\sin n}{n} \le 1 + \frac{1}{n} < b \right.$

$\Rightarrow \left(\{ n : a_n \ge b \} \subset \{ 1, \dots, N \} \right)$ finite.

Property. $u = \overline{\lim}\, a_n$ is the unique number such that: 1) $\forall \varepsilon > 0 : \{ n : a_n > u + \varepsilon \}$ is finite

2) $\forall \varepsilon > 0 : \{ n : a_n > u - \varepsilon \}$ - infinite.

$b < 1$ $\quad \exists N : \frac{1}{N} < 1 - b \Rightarrow (n = 2k, n > N \Rightarrow$

$\Leftarrow a_n = 1 - \frac{\sin n}{n} \ge 1 - \frac{1}{n} > b)$

$\# \{ n : a_n > b \}$ - infinite.

Another characterization: $\overline{\lim}\, a_n = \sup \{ b : \# \{ n : a_n > b \} \text{ - infinite} \} = \inf \{ b : \# \{ n : a_n > b \} \text{ - finite} \}.$

$\underline{\lim}\, a_n = \inf \{ b : \# \{ n : a_n < b \} \text{ - infinite} \} = \sup \{ b : \# \{ n : a_n < b \} \text{ - finite} \}$

$\lim_{n \to \infty} a_n$ - exists if and only if $\overline{\lim}_{n \to \infty} a_n = \underline{\lim}_{n \to \infty} a_n.$

Bonus problem (+1 pt):

Let $a_n > 0$ be an unbounded sequence.

Prove that $\overline{\lim} \left(1 + \frac{1}{a_n} \right)^{a_n} = e.$

$\overline{\lim}\, a_n \log \left(1 + \frac{1}{a_n} \right) = 1.$

Uniform vs pointwise $\quad f_n, f : K \to \mathbb{C}, \quad K \subset \mathbb{C}.$

$f_n \to f$ pointwise if $\forall z \in K\ \forall \varepsilon > 0\ \exists N(z, \varepsilon) : |f_n(z) - f(z)| < \varepsilon \iff \forall z \in K\ |f_n(z) - f(z)| \to 0$ $\quad n > N(z,\varepsilon)$

$f_n \rightrightarrows f$ - uniformly if $\forall \varepsilon > 0\ \exists N(\varepsilon) : n > N\ |f_n(z) - f(z)| < \varepsilon \iff \sup_{z \in K} |f_n(z) - f(z)| \to 0.$

Thm. Uniform limit of a sequence of continuous functions is continuous. $\exists \varepsilon_n \to 0: \forall z \in K \; |f_n(z) - f(z)| < \varepsilon_n$

Same proof as in real case.

Example: On $[0,1]$, $x^n \to \begin{cases} 0, & x<1 \\ 1, & x=1 \end{cases}$

M-test for uniform convergence of series.

Thm. $K \subset \mathbb{C}$. $\sum_{n=1}^{\infty} M_n < \infty$, $M_n \geq 0$.

Then if $\forall z \in K$, $\forall n$: $|f_n(z)| \leq M_n$, then $\sum f_n(z)$ converges uniformly on K.

Proof. $\sum M_n$ - converges $\Rightarrow$ $\forall \varepsilon > 0 \; \exists N: m > n > N \; \sum_{k=n}^{m} M_k < \varepsilon \Rightarrow$

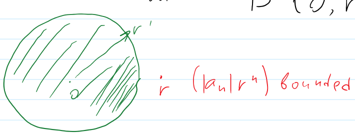
$\forall \varepsilon > 0 \; \exists N: m > n > N \; \sum_{k=n}^{m} |f_k(z)| < \varepsilon \Rightarrow \sum_{k=1}^{\infty} f_k(z) =: f(z)$ exists $\forall z \in K$.

$\left| f(z) - \sum_{k=1}^{n} f_k(z) \right| < \sum_{k=n+1}^{\infty} M_k \to 0$, so uniform $\blacksquare$

Def. Let $(a_n)_{n \geq 0}$ be a sequence of complex numbers. $\sum_{n=0}^{\infty} a_n z^n$ is a (formal) power series.

$s_n(z) = \sum_{k=0}^{n} a_k z^k$ - nth partial sum, a polynomial.

Lemma. Let for some $r \geq 0$, $\{|a_n| r^n\}$ is bounded. Then for any z with $|z| < r$, $\sum a_n z^n$ converges. Moreover, if $r' < r$, then the series converges uniformly on $\overline{B}(0, r')$.



Remark. Not on $\overline{B}(0,r)$ or even $B(0,r)$! Consider $\sum z^n$, $r=1$

Proof. When $|z| \leq r'$, $|a_n z^n| \leq |a_n (r')^n| = |a_n r^n| \left(\frac{r'}{r}\right)^n < C \left(\frac{r'}{r}\right)^n$

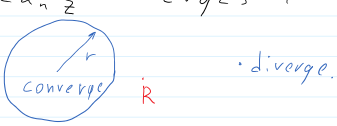
$\sum C \left(\frac{r'}{r}\right)^n < \infty$, so we can use M-test $\blacksquare$

Note: $\{|a_n| r^n\}$ - not bounded $\Rightarrow$ for z with $|z| = r$, $\sum a_n z^n$ diverge ($|a_n z^n| = |a_n| r^n \not\to 0$).

Theorem. For any power series $\sum a_n z^n$ let $R := \sup\{r : \text{the sequence } (|a_n| r^n) \text{ is bounded}\}$. Then

1) $\sum a_n z^n$ converges uniformly on $\overline{B}(0,r)$ for any $r < R$.

2) $\sum a_n z^n$ diverges for any $|z| > R$.



Def R is called radius of convergence.

Proof. 1) $r < R \Rightarrow \exists \; r < r' < R \Rightarrow r' \in \{t : |a_n| t^n \text{ is bounded}\} \Rightarrow$

$(|a_n|(r')^n)$ - bounded $\underset{\text{Lemma}}{\Longrightarrow}$ 1)

2) $|z| > R \Rightarrow |z| \notin \{t : (|a_n| t^n) \text{ is bounded}\} \Rightarrow (|a_n||z|^n)$ - unbounded $\Rightarrow$ 2)

<u>Cauchy-Hadamard formula for radius of convergence.</u>

$R = \dfrac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$ (Convention for this formula: $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$.)

Jacques Salomon Hadamard

<u>Proof.</u>

$r < \dfrac{1}{\overline{\lim} \sqrt[n]{|a_n|}} \Rightarrow \underset{\overline{\lim}\sqrt[n]{r^n |a_n|}}{\underbrace{\overline{\lim}\, r \sqrt[n]{|a_n|}}} < 1 \Rightarrow \exists a < 1, N : n > N \Rightarrow r^n |a_n| < a^n < 1.$ bounded

So $(r^n |a_n|)$ is bounded.

So $r \le R \Rightarrow R \geq \dfrac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$

If $(r^n |a_n|)$ - bounded, $r^n |a_n| \le M \Rightarrow r \sqrt[n]{|a_n|} \le M^{\frac{1}{n}} \Rightarrow$

$\overline{\lim}\, r \sqrt[n]{|a_n|} \le \lim_{n \to \infty} M^{\frac{1}{n}} = 1$. So $r \le \dfrac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$.

So $\dfrac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$ is an upper bound for $\{r : (|a_n| r^n) \text{ bounded}\}$ ∎

$\sum a_n z^n$ has radius of convergence $R = \dfrac{1}{\overline{\lim}_{n \to \infty} \sqrt[n]{|a_n|}}$.

converges uniformly in any disk $\{|z| \le r\}$, $r < R$.

$|z| < R$. continuous in $\{|z| < R\}$.

<u>Lemma</u> Let $|z| \le r$, $|w| \le r$. Then

$|z^n - w^n| \le n |z - w| r^{n-1}$.

<u>Proof</u> $(z^n - w^n) = (z - w)(z^{n-1} + z^{n-2} w + \dots + w^{n-1})$

So $|z^n - w^n| \le |z - w| \sum_{k=0}^{n-1} \underbrace{|w^k z^{n-k-1}|}_{\le r^{n-1}} \le n r^{n-1} |z - w|$ ∎

<u>Theorem.</u> Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with the radius of convergence R.
Then $f(z)$ is analytic on $B(0, R)$ and
$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ (termwise derivative).

<u>Proof.</u> Let $f_1(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$.
First, observe that the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is R.

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Proof. Let R_1 be the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$. Then, since $|n a_n| \geq |a_n|$, $R_1 \leq R$.

On the other hand, if $r < R$, then take $r_0: r < r_0 < R$

$$\lim_{n\to\infty} n|a_n|r^n = \lim_{n\to\infty} n|a_n| r_0^n \cdot \frac{r^n}{r_0^n} = \lim_{n\to\infty} |a_n| r_0^n \cdot \lim_{n\to\infty} n\left(\frac{r}{r_0}\right)^n = 0 \cdot 0$$

So $r < R_1$. So we have: $r < R \Rightarrow r < R_1 \Rightarrow R_1 \geq R \Rightarrow R_1 = R$.

$\lim n q^n = 0$ if $q < 1$.

Proof $q = \frac{1}{1+\delta}$, $\delta > 0$

$(1+\delta)^n > \frac{n(n-1)}{2} \delta^2$.

So $n q^n < \frac{2n}{n(n-1)\delta^2} = \frac{2}{\delta^2(n-1)} \to 0$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{k} a_n z^n + \sum_{n=k+1}^{\infty} a_n z^n = S_k(z) + R_k(z).$$

$f_1(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$ $\quad S_n'(z)$ - nth partial sum for f_1.

Let $|z|, |z_0| < r < R$

$$\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0) = \underbrace{\left(\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)\right)}_{\text{I}} + \underbrace{\left(S_n'(z_0) - f_1(z_0)\right)}_{\text{II}} + \underbrace{\frac{R_n(z)-R_n(z_0)}{z-z_0}}_{\text{III}}$$

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| = \left|\sum_{k=n+1}^{\infty} \frac{a_k(z^k - z_0^k)}{z-z_0}\right| \overset{\text{Lemma}}{\leq} \sum_{k=n+1}^{\infty} |a_k| k r^{k-1}$$

Now: fix $\varepsilon > 0$:

Find n: $\left|S_n'(z_0) - f_1(z_0)\right| < \frac{\varepsilon}{3}$ (Partial sum!)

$\sum_{k=n+1}^{\infty} k|a_k| r^{k-1} < \frac{\varepsilon}{3}$ $\quad (r < R = R_1)$

Find $\delta > 0$: $|z - z_0| < \delta \Rightarrow \left|\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)\right| < \frac{\varepsilon}{3}$

[using differentiability of S_n.

Thus $|z - z_0| < \delta \Rightarrow$

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0)\right| \leq |\text{I}| + |\text{II}| + |\text{III}| < \varepsilon.$$

So $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f_1(z_0)$ ∎

Corollary (Taylor) $\sum a_n z^n$ is infinitely differentiable for $|z| < R$,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}$$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

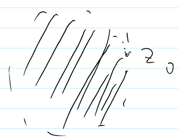
Brook Taylor

Proof. Induction. Plug in $z=0$. ∎

What happens for $|z|=R$?

If $\sum |a_n| R^n$ converges (the series converges absolutely), then $\sum a_n z^n$ converges uniformly in $\overline{B(0,R)}$, by M-test. So it is continuous in $\overline{B(0,R)}$.

Example: $\sum \frac{z^n}{n^2}$ $R=1$.



$B(0,R) \cup \{z_0\}$.

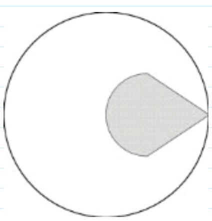
But not always the case: $\sum \frac{z^n}{n}$ - converges for $z \neq 1$, $|z|=1$. (Why?).

$\sum z^n$ - converges only for $|z|<1$.

What happens if $\sum a_n z_0^n$ converges for some z_0 with $|z_0|=R$?

Thm (Stolz). Let $f(z)=\sum a_n z^n$ has the radius of convergence R. Let $|z_0|=R$, and $\sum a_n z_0^n$ converges. Then the series $\sum a_n z^n$ converges uniformly inside the Stolz angle

$S_k := \{ z : |z-z_0| \leq k(R-|z|) \}$. In particular, inside this angle $\lim\limits_{\substack{z\to z_0 \\ z\in S_k}} f(z) = f(z_0)$ - non-tangential convergence.



Otto Stolz

Proof. First, consider $\underbrace{z=z_0\zeta}$, $\underbrace{\sum a_n z^n} \to \underbrace{\sum a_n z_0^n \zeta^n}$.

We can take $z_0=1$. Subtract $f(1)$ from a_0 to get

$f(1) = \sum\limits_{n=0}^{\infty} a_n = 0.$

Abel's trick:

$t_n := \sum\limits_{k=0}^{n} a_k$. $\quad a_n = t_n - t_{n-1}$

$t_{-1} := 0$ $\quad t_n \to 0$

$s_n(z) = \sum\limits_{k=0}^{n} a_k z^k = \sum\limits_{k=0}^{n} (t_k - t_{k-1}) z^k = \sum\limits_{k=0}^{n-1} t_k (z^k - z^{k+1}) + t_n z^n =$

$(1-z)\sum\limits^{n-1}$ ...

$s_n(z) = \sum_{k=0} a_k z^k = \sum_{k=0}^{n} (t_k - t_{k-1}) z^k = \sum_{k=0}^{n-1} t_k (z^k - z^{k+1}) + t_n z^n =$

$(1-z) \sum_{k=0}^{n-1} t_k z^k + t_n z^n$. $\qquad t_n z^n \to 0$, so

$f(z) = (1-z) \sum_{k=0}^{\infty} t_k z^k$

and

$f(z) - s_n(z) = (1-z) \sum_{k=n}^{\infty} t_k z^k - t_n z^n$

$|f(z) - s_n(z)| \le |1-z| \sum_{k=n}^{\infty} |t_k| |z|^k + |t_n| \quad (|z| \le 1)$

Choose N: $k \ge N \Rightarrow |t_k| < \varepsilon$

$\boxed{\text{In } S_K: \ |1-z| \le K(1-|z|)}$

so for $n \ge N$: $\sum_{k=n}^{\infty} |t_k| |z|^k \le \varepsilon \frac{1}{1-|z|}$

So $|f_n(z) - s_n(z)| \le \varepsilon K + \varepsilon$.