Polynomial and rational functions

Tuesday, January 12, 2021 2:52 PM

Polynomial functions

$f(z) = \sum_{n=0}^{d} a_n z^n$ By Theorem, $f'(z) = \sum_{n=1}^{d} n a_n z^{n-1}$. Entire function.

Analytic $\forall z \in \mathbb{C}$.

A little bit about polynomials:

Fundamental Theorem of Algebra (will prove it later):

$p(z)$ - polynomial of degree $d \geq 1 \Rightarrow p$ has a root (or zero) z_1,
$p(z_1) = 0$.

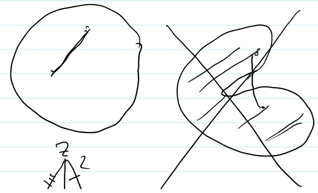
Divide p by $(z - z_1)$: $p(z) = q(z)(z - z_1)$ (remainder is a constant, so it is $p(z_1) = 0$)
q - has degree $d-1$, so also has a root.
so $p(z) = a_d (z - z_1) \dots (z - z_d)$ - by induction.
Some of the roots can be the same! Multiplicity or order of
z_0 as a root: $\# \{ j : z_j = z_0 \}$.
I.e. $p(z) = (z - z_0)^h q(z)$, $q(z_0) \neq 0$. h- order.
Equivalently: $p(z_0) = p'(z_0) = \dots = p^{(h-1)}(z_0) = 0$, $p^{(h)}(z_0) \neq 0$.
Pf. $p'(z) = h (z - z_0)^{h-1} q(z) + (z - z_0)^h q'(z) =$

$$(z - z_0)^{h-1} \underbrace{(h q(z) + (z - z_0) q'(z))}_{r(z)}$$

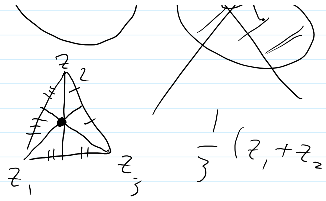
$r(z_0) = h q(z_0) \neq 0$, so has order $h-1$ for p' ∎

Carl Friedrich Gauss

Theorem (Gauss)
Let $z_1, \dots, z_d$ be zeroes of a polynomial p.
Then all the zeroes of $p'(z)$ lie inside the
convex hull of $\{ z_1, \dots, z_d \}$.
Def. Convex hull of $\{ z_1, \dots, z_d \}$ is
$H = \{ w \in \mathbb{C} : \exists m_1, \dots, m_d : m_j \geq 0, \sum_{j=1}^{d} m_j = 1,$
$w = \sum^{d} m_j z_j \}$

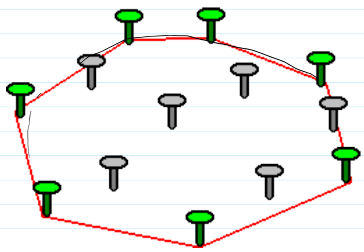


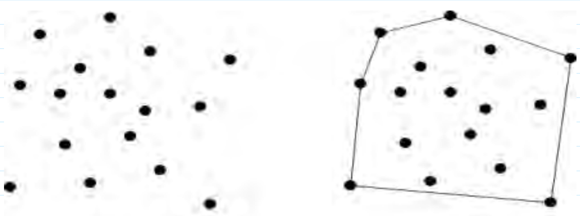
$H = \{ w \in \mathbb{C} : \exists m_1, \dots, m_d : m_j \geq 0, \sum_{j=1}^{d} m_j = 1, w = \sum_{j=1}^{d} m_j z_j \}$.



$\frac{1}{3}(z_1 + z_2 + z_3) = \frac{1}{3} z_1 + \frac{1}{3} z_2 + \frac{1}{3} z_3$

Geometric meaning: the smallest convex set containing $\{z_1, \dots, z_d\}$.





(a) (b)

$(p(z) - c)' = p'(z)$

$p(z) = c$

Corollary. $\forall c \in \mathbb{C}$ the roots of $p'(z)$ lie in the convex hull of the roots $p(z) = c$.

Proof of Theorem

$$\frac{p'(z)}{p(z)} = \sum_{j=1}^{d} \frac{1}{z - z_j} = \sum \frac{\overline{z - z_j}}{|z - z_j|^2}$$

If z_0 - multiple root of p - nothing to prove!

If z_0 is not a root of p, but a root of p' then $\frac{p'(z_0)}{p(z_0)} = 0$.

So $\sum \frac{\overline{z_0 - z_j}}{|z_0 - z_j|^2} = 0$. Or $z_0 = \sum m_j z_j$, where

$$m_j = \frac{|z_0 - z_j|^{-2}}{\sum_{j=1}^{d} |z_0 - z_j|^{-2}}, \quad \sum m_j = 1, \quad m_j \geq 0 \quad \blacksquare$$

Taylor polynomial.

Thm. Let $p(z)$ be a polynomial of degree d, $z_0 \in \mathbb{C}$. Then $p(z) = \sum_{k=0}^{d} \frac{p^{(k)}(z_0)}{k!} (z - z_0)^k$

Proof Induction on d. $d = 0$ $p(z) = C = p(z_0)$.

Step: $\frac{p(z) - p(z_0)}{z - z_0} =: q(z)$ - polynomial.

$p(z) = p(z_0) + (z - z_0) q(z)$,

$p'(z) = q(z) + (z - z_0) q'(z)$

$p''(z) = 2q'(z) + (z - z_0) q''(z)$

$p^{(k)}(z) = k q^{(k-1)}(z) + (z - z_0) q^{(k)}(z)$

$p^{(k)}(z_0) = k q^{(k-1)}(z_0)$.

$$p(z) = p(z_0) + (z - z_0) \sum_{k=0}^{d-1} \frac{q^{(k-1)}(z_0)}{(k-1)!} (z - z_0)^{k-1} = p(z_0) + \sum_{k=1}^{d} \frac{p^{(k)}(z_0)}{k!} (z - z_0)^k$$

induction!

$p(z) = z^2 = (x^2 - y^2) + 2ixy \qquad x = \frac{z + \bar{z}}{}$

induction!

$$p(z) = z^2 = (x^2 - y^2) + 2ixy \qquad x = \frac{z+\bar z}{2}, \quad y = \frac{z - \bar z}{2i}$$

Rational functions

$R(z) = \frac{P(z)}{Q(z)}$, P, Q - polynomials. Assume - no common zeroes (can factor them out).

$$R'(z) = \frac{P'(z)\,Q(z) - Q'(z)\,P(z)}{Q(z)^2}$$ - exists when z is not a pole.

Def. Zeroes of $Q(z)$ are called poles of $R(z)$.
Order (multiplicity) of a pole = order of zero of $Q(z)$.

Example. $R(z) = \frac{(z-i)^2(z+1)}{(z+i)^3}$ has pole $-i$ of order 3,
zero -1 of order 1,
zero i of order 2.

Remark. If z_0 - pole of $R(z)$, then $\lim_{z \to z_0} R(z) = \infty$ (in spherical metric)
So we put $R(z_0) = \infty$

Behavior at ∞: Consider $R_1(z) := R(\frac{1}{z}) = \frac{P(\frac{1}{z})}{Q(\frac{1}{z})}$.

$\frac{1}{z}$ -trick. $R(\infty) := R_1(0)$. $\lim_{z\to\infty} R(z) = \lim_{z \to 0} R_1(z)$

More details: if $P(z) = \sum_{k=0}^{n} a_k z^k$, $Q(z) = \sum_{k=0}^{m} b_k z^k$, $n = \deg P$, $m = \deg Q$.

then $R_1(z) = z^{m-n} \underbrace{\frac{\sum_{k=0}^{n} a_k z^{n-k}}{\sum_{k=0}^{m} b_k z^{m-k}}}_{R_0(z)}$. $a_n \neq 0$, $b_m \neq 0$. $P(\frac{1}{z}) = \sum_{k=0}^{n} a_k z^{-k} = z^{-n} \sum_{k=0}^{n} a_k z^{n-k}$

$Q(\frac{1}{z}) = z^{-m} \sum_{k=0}^{m} b_k z^{m-k}$

$R_0(0) = \frac{a_n}{b_m} \neq 0, \infty$.

So $R(\infty) = R_1(0) = \begin{cases} \frac{a_m}{b_n}, & \deg P = \deg Q \\ 0, & \deg P < \deg Q \\ \infty, & \deg P > \deg Q. \end{cases}$

If $\deg P < \deg Q$, ∞ - zero of $R(z)$, order of 0 is the order of 0 as zero of $R_1(z)$, i.e. $\deg Q - \deg P$

If $\deg P > \deg Q$, ∞ is a pole of $R(z)$ of $\deg P - \deg Q$.

If $\deg P = \deg Q$, ∞ is neither pole nor zero.

$\deg P > \deg Q$: total number of zeroes, counting order =
zeroes of P + # zeroes at ∞ = $\deg P$
total number of poles, ... =
zeroes of Q + # poles at ∞ = $\deg Q + \deg P - \deg Q =$

$\deg P$.

So, the total number of zeroes = the total number of poles = $\max(\deg P, \deg Q)$.

Def. $\max(\deg P, \deg Q)$ is called order of $R(z)$.

Remark. $\forall w \in \mathbb{C}$, the equation $R(z) = w$ has $\deg R$ roots, counting multiplicity, since $R(z) - w$ has the same poles as $R(z)$.

Order $= 1$: Möbius (Linear) maps: $\frac{az+b}{cz+d}$ $\quad ad - bc \neq 0$, $a, b, c, d \in \mathbb{C}$

Polynomials are the rational functions with all the poles at ∞.

Partial fraction decomposition.

Singular part at ∞: If $R(z) = \frac{P(z)}{Q(z)}$ and $\deg P > \deg Q$, let $P(z) = G_\infty(z) Q(z) + S(z)$, $\deg S < \deg Q$,

so $R(z) = G_\infty(z) + \underbrace{\frac{S(z)}{Q(z)}}_{H(z)}$, $\quad H(\infty) = 0$.

$G_\infty(z)$ - polynomial, $\deg G_\infty(z) > 0$, so $G_\infty(\infty) = \infty$.

$G_\infty(z)$ - singular part of R at ∞.

If $\deg P < \deg Q$, let $G_\infty(z) \equiv 0$.

If $\deg P = \deg Q$, let $G_\infty(z) \equiv R(\infty)$ - a constant.

Observe: $\deg G = \max(\deg P - \deg Q, 0)$ - order of ∞ as pole.

Let z_0 be a pole of R. Consider $R_1(\zeta) := R(z_0 + \frac{1}{\zeta})$ - a rational function, $R_1(\infty) = R(z_0) = \infty$, so

$R_1(\zeta) = \frac{P_1(\zeta)}{Q_1(\zeta)}$, $\deg P_1 > \deg Q_1$. Let $G_{z_0}(\zeta)$ - singular part of $R_1(\zeta)$ at ∞, so $R_1(\zeta) = G_{z_0}(\zeta) + H_{z_0}(\zeta)$, $H_{z_0}(\infty) = 0$.

Change back: $R_1(\frac{1}{z - z_0}) = R(z)$, so

$R(z) - G_{z_0}\left(\frac{1}{z - z_0}\right)$ has a zero at z_0 (not a pole).

$G_{z_0}\left(\frac{1}{z - z_0}\right)$ - polynomial of $\frac{1}{z - z_0}$, only has a pole at z_0!

$\deg G_{z_0}$ - order of z_0 as a pole.

Now let $z_1, \ldots, z_n$ be all the poles of R. Then $R(z) - G_\infty(z) - \sum_{k=1}^{n} G_{z_k}\left(\frac{1}{z - z_k}\right)$ has no poles. So it is a constant, equal to zero at ∞. So

$$R(z) = G_\infty(z) + \sum_{k=1}^{n} G_{z_k}\left(\frac{1}{z - z_k}\right)$$ – partial fraction decomposition.