Let $K \subset \mathbb{C}$ (or $\hat{\mathbb{C}}$ - extended complex plane), $K^c = \mathbb{C}\setminus K$ (or $\hat{\mathbb{C}}\setminus K$).

Def. K is called a neighborhood of $z \in K$ if $\exists \delta > 0: B(z,\delta) \subset K$.

Def Interior of K: $\operatorname{Int} K = \{z : \exists \delta > 0 : B(z,\delta) \subset K\}$.

Exterior of K: $\operatorname{Ext} K = \{z : \exists \delta > 0 : B(z,\delta) \cap K = \emptyset\} = \operatorname{Int}(K^c)$

Boundary of K: $\partial K := (\operatorname{Int} K \cup \operatorname{Ext} K)^c = \partial K^c = \{z : \forall \delta > 0 \; B(z,\delta)\cap K \neq \emptyset, \; B(z,\delta)\cap K^c \neq \emptyset\}$

(Accumulation)
Limit points of K: $L(K) := \{z : \exists z_n \in K,\ z_n \neq z,\ z_n \to z\} \subset K \cup \partial K$.


$z_n \in K$
$z_n \neq z$

Isolated points of K: $I(K) := K \setminus L(K)$. $\exists \delta > 0 \; B(z,\delta) \cap K = \{z\}$.

$\operatorname{Clos} K = K \cup \partial K = (\operatorname{Ext} K)^c = K \cup L(K)$. $(z \in \partial K \setminus K \Rightarrow z \in L(K))$

Example. $\operatorname{Int} B(z,\delta) = \operatorname{Int} \overline{B}(z,\delta) = B(z,\delta)$ $\quad \partial B(z,\delta) = \{w : |w-z| = \delta\}$

$\operatorname{Clos} B(z,\delta) = \operatorname{Clos} \overline{B}(z,\delta) = \overline{B}(z,\delta)$ $\quad \overset{\parallel}{\partial \overline{B}(z,\delta)}$ $\quad I(B(z,\delta)) = I(\overline{B}(z,\delta)) = \emptyset$

Def. K is called open if $K = \operatorname{Int} K$.
K is called closed if $\partial K \subset K \Leftrightarrow K^c$ is open $\Leftrightarrow L(K) \subset K$.

Properties. 1) $(U_\alpha)_{\alpha \in I}$ - family of open sets. $\bigcup_{\alpha\in I} U_\alpha$ - open,
$\bigcap_{\alpha \in I} U_\alpha$ - open if I is finite

2) $(F_\alpha)_{\alpha \in I}$ - closed sets. $\bigcap F_\alpha$ - closed, $\bigcup F_\alpha$ - closed if I is finite.

3) U - open, $U \subset K \Rightarrow U \subset \operatorname{Int}(K)$.

4) F - closed, $K \subset F \Rightarrow \operatorname{Clos}(K) \subset F$.

Def. $K \subset \mathbb{C}$ (or $\hat{\mathbb{C}}$, or $\mathbb{R}$, or...) is called connected if the following holds: $K \subset U_1 \cup U_2$, U_1, U_2 - open, $U_1 \cap U_2 = \emptyset$. then $K \subset U_1$ or $K \subset U_2$.

Remark. If K is open, $U_1 \cap K$, $U_2 \cap K$ - open. So equivalent:
$K = K_1 \cup K_2$, K_1, K_2 - open $\Rightarrow K_1 = \emptyset$ or $K_2 = \emptyset$.

If K is closed, $U_1^c \cap K$ - closed, $U_2^c \cap K$ - closed,
$U_1^c \cap K \subset U_2$, $U_2^c \cap K \subset U_1$.

So $K = K_1 \cup K_2$, K_1, K_2 - closed $\Rightarrow K_1 = \emptyset$ or $K_2 = \emptyset$.

Theorem $(K_\alpha)_{\alpha \in I}$ - a family of connected sets, $\bigcap K_\alpha \neq \emptyset \Rightarrow \bigcup K_\alpha$ - connected.

Proof. Let $z_0 \in \bigcap K_\alpha$. Let $\bigcup K_\alpha \subseteq U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$.
$z_0 \in U_1 \Rightarrow \forall \alpha : K_\alpha \subseteq U_1 \cup U_2$, $U_1 \cap K_\alpha \neq \emptyset \Rightarrow U_2 \cap K_\alpha = \emptyset \Rightarrow K_\alpha \subset U_1$.
So $\bigcup K_\alpha \subset U_1$ ∎

Theorem. $K \subset \mathbb{C}$ can be uniquely decomposed
$K = \bigcup K_\alpha$, K_α - connected, non-empty. Connected component.

Proof. For $z \in K$, let $K_z := \bigcup_{\substack{F \subset K \\ z \in F \\ F\text{- connected}}} F$. By previous Thm, K_z - connected.

$z \neq z' \Rightarrow$ either $K_z \cap K_{z'} = \emptyset$ or $K_z \cap K_{z'} \neq \emptyset \Rightarrow K_z \cup K_{z'}$ - Connected $\Rightarrow$
$K_z = K_{z'}$ (maximal connected set containing z) ∎

Theorem (Generalized intermediate value Theorem).
Let K be connected, f - continuous on K. Then $f(K)$ is connected.

Proof. Left as exercise (use: if $U \subset f(K)$ - open $\Rightarrow$ $f^{-1}(U)$ - open). ■

Def. A continuous $\gamma : [a,b] \to \mathbb{C}$ is called <u>an arc</u> or <u>a path</u> from $\gamma(a)$ to $\gamma(b)$

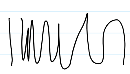
Def. $K \subset \mathbb{C}$ is called <u>path-connected</u> if $\forall\, z_1, z_2 \in K$
$\exists\, \gamma : [a,b] \to K$ - an arc in K from z_1 to z_2. (γ - continuous function).
Easier to check: $B(z,\delta)$, $\overline{B}(z,\delta)$ are path-connected.

Thm. Let $U \subset \mathbb{C}$ - open. Then U is connected if and only if it is path-connected.

Remark. For any $K \subset \mathbb{C}$, path-connected $\Rightarrow$ connected.

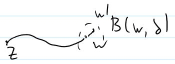
Proof of remark. Let $K \subset V_1 \cup V_2$, V_1, V_2 - open, $V_1 \cap V_2 = \emptyset$,
$z_1 \in V_1$, $z_2 \in V_2$. Let $\gamma : [a,b] \to K$, $\gamma(a) = z_1$, $\gamma(b) = z_2$,
$t = \sup\{s : \gamma(s) \in V_1\}$ — non-empty ($\gamma(a) \in V_1$).
$\gamma(t) \in V_1 \underset{t<b}{\Rightarrow} \exists\, B(\gamma(t), \varepsilon) \subset V_1 \Rightarrow$ by continuity $\exists\, \delta > 0$:
$\gamma(t-\delta, t+\delta) \subset B(\gamma(t), \varepsilon)$.
So $\gamma(t + \frac{\varepsilon}{2}) \in V_1$ — contradiction.
So $\gamma(t) \in V_2$, so $t > a$, so, same reasoning, $\forall\, t-\delta < s \le t$, $\gamma(s) \in V_2$ - not supremum.
Contradiction. ■

Example: $K = \{-1 \le \operatorname{Im} z \le 1, \operatorname{Re} z = 0\} \cup \{\operatorname{Im} z = \sin \frac{1}{\operatorname{Re} z}, 0 < \operatorname{Re} z \le 1\}$



Connected but not path-connected.

Proof of Theorem (open connected $\Rightarrow$ path-connected)
Let $z \in U$. $U_z := \{w \in U : \exists$ a path in U from z to $w\}$.
$z \ne z' \Rightarrow$ either $U_z \cap U_{z'} = \emptyset$ or $U_z \cap U_{z'} \ne \emptyset$ $\exists\, w \in U_z \cap U_{z'}$,
path γ from z to w, path γ' from z' to w.
so $\exists$ path from z to z', and $U_z = U_{z'}$.
So $U = \bigcup U_z$. Let us show that U_z is open. Indeed



$w \in U_z$, $\exists\, B(w,\delta) \subset U \Rightarrow \forall w' \in B(w,\delta)$ $\exists$ path from z to $w' \Rightarrow$ $B(w,\delta) \subset U_z \Rightarrow w \in \operatorname{Int} U_z$.
So if $U_z \ne U$ then $U_z' := \bigcup_{U_{z'} \ne U_z} U_{z'}$ - open. $U_z \cup U_z' = U$ - contradiction! ■

Remark. Even more is true (with the same proof): if U is open, connected, then $\forall\, z, w \in D$ $\exists\, \gamma$ - a path consisting of intervals parallel to one of the axes, joining z to w.

Theorem. $U \subset \mathbb{C}$, open $\Rightarrow$ each of its connected components

Theorem. $V \subset \mathbb{C}$, open $\Rightarrow$ each of its connected components is open.

Proof. $V = \bigcup V_\alpha$, $w \in V_\alpha \Rightarrow \exists B(w,\delta) \subset V$. $B(w,\delta)$ - connected, $w \in B(w,\delta) \cap V_\alpha \neq \emptyset \Rightarrow V_\alpha \cup B(w,\delta)$ - connected $\Rightarrow B(w,\delta) \subset V_\alpha$ ∎

Def (Important). A region is an open connected subset of $\mathbb{C}$.

We will talk about functions analytic in regions.

Def $\mathcal{A}(D)$ - functions analytic in a region D.

Theorem. $f \in \mathcal{A}(D)$, $f' \equiv 0 \Rightarrow f \equiv$ const in D.

Proof. For $z \in D$, let $V_z := \{w \in D : f(w) = f(z)\}$.

Then V_z is open: $w \in V_z$, $B(w,\delta) \subset V \Rightarrow$
$\forall w' \in B(w,\delta)$: $f(w') = f(w) = f(z)$ (by Theorem for disk) $\Rightarrow$
$B(w,\delta) \subset V_z$.

$V = \bigcup V_z$, $f(z) \neq f(z')$ for some $z, z' \Rightarrow V_z \cap V_{z'} = \emptyset$.
Contradiction ∎

$f : K \to \mathbb{C}$

Def. f is analytic on $K \subset \mathbb{C}$ if $\exists V$-open, $K \subset V$, $F : V \to \mathbb{C}$; $F \in \mathcal{A}(V)$, $F|_K = f$.

Notation: $\mathcal{A}(K)$.

Concept of compactness.

Let K be a closed bounded ($\exists R : K \subset B(0,R)$) subset of $\mathbb{C}$.

1) $K \subset \bigcup_{\alpha \in I} V_\alpha$, V_α - open $\Rightarrow \exists$ $U_1, U_2, \ldots U_n$: $K \subset U_1 \cup \ldots \cup U_n$.
(compactness)

2) $(z_n) \subset K \Rightarrow \exists z_{n_k}$ - subsequence, $\lim_{k\to\infty} z_{n_k} = z \in K$
(sequential compactness).

3) $f : K \to \mathbb{R}$ - continuous. Then $f(K)$ is closed, bounded.
In particular, $\max_{z \in K} f(z)$, $\min_{z \in K} f(z)$ are finite, achieved at some points of K

4) $f : K \to \mathbb{C}$ - continuous $\Rightarrow f(K)$ is bounded and closed.

5) $f : K \to \mathbb{R}$ or $f : K \to \mathbb{C}$ - continuous. Then f is uniformly continuous
$\forall \varepsilon > 0 \; \exists \delta > 0$: $|z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \varepsilon$.

No proof.

Any closed subset of $\hat{\mathbb{C}}$ is compact.

Closed $K \subset \mathbb{C}$. K is closed in $\hat{\mathbb{C}} \Leftrightarrow K$ is bounded.

Any closed subset of $\mathbb{C}$ is compact.

Closed $K \subset \mathbb{C}$. K is closed in $\hat{\mathbb{C}} \Longleftrightarrow K$ is bounded.

K - unbounded $\Rightarrow \infty \in \mathrm{Clos}_{\hat{\mathbb{C}}} K$

$\infty \notin K$.

Theorem (K_n) - compact, $K_n \supset K_{n+1} \neq \emptyset \Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Proof. If $\bigcap = \emptyset$, then $V_n = K_n^c$ - open, $K_1 \subset \bigcup V_n$, no finite subcover $(V_1 \cup \dots \cup V_n = V_n = K_n^c)$ ∎

Remark. The same true for any closed $K \subset \hat{\mathbb{C}}$ in spherical metric.

An application of compactness.

Def Let $S \subset \mathbb{C}$, $f_n, f : S \to \mathbb{C}$. We say that f_n converges to f locally uniformly if $\forall \varepsilon > 0 \ \forall z \in S \ \exists \delta(\varepsilon, z) > 0, N(\varepsilon, z)$: $n > N$, $w \in B(z, \delta) \cap S \Rightarrow |f_n(w) - f(w)| < \varepsilon$

Since δ and N depend on ε and z, it is a very weak assumption.

Theorem. If S is compact, $f_n \to f$ - locally uniformly

Then $f_n \rightrightarrows f$ - converges uniformly.

Proof. Fix $\varepsilon > 0$. Then $S \subset \bigcup_{z \in S} B(z, \delta(\varepsilon, z))$. Since S - compact,

$S \subset B(z_1, \delta_1) \cup \dots \cup B(z_k, \delta_k)$. Take $N = \max_{1 \le j \le k} (N(z_j, \varepsilon))$.

Then $w \in S \Rightarrow \exists j : w \in B(z_j, \delta_j)$

$n > N \Rightarrow n > N(z_j, \varepsilon) \Rightarrow |f_n(w) - f(w)| < \varepsilon$ ∎