1) $\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} u_n := \sup\{a_k,\ k\ge n\}$ ($\exists$ because $u_n \downarrow$)

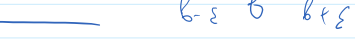
2) $a = \overline{\lim} a_n$ if $\forall \varepsilon > 0$ $\{n: a_n > a+\varepsilon\}$ - finite

$\{n: a_n > a-\varepsilon\}$ - infinite.

3) $L(a_n) = \{b: b = \lim_{k\to\infty} a_{n_k}\} = \{b: \forall \varepsilon > 0.\ \{n: a+\varepsilon > a_n > b-\varepsilon\}$ infinite $\}$

limit set

$\overline{\lim} a_n = \sup L(a_n) = \max L(a_n)$.

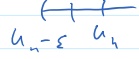


1) $\Rightarrow$ 2) $\forall \varepsilon$: $a = \lim u_n$

$n \ge N \Rightarrow u_n \notin \{n: a_n > a+\varepsilon\}$ - finite. $\exists N: u_N < a+\varepsilon$ ($u_n \to a$)

$a_n \le u_N$ $n \ge N$

$u_n \in \{n: a_n > a-\varepsilon\}$ - infinite. $\forall n: u_n \ge a$ $\forall n\ \exists k \ge n: a_k > u_n - \varepsilon \Rightarrow a_k \ge a-\varepsilon$



2) $\Rightarrow$ 3) 1. $a = \lim a_{n_k}$.

2. $b = \lim a_{n_k} \Rightarrow b \le a$.

Proof of 1.

$\forall \varepsilon > 0$ $\{n: a_n > a-\varepsilon\}$ - infinite, $\{n: a_n > a+\varepsilon\}$ - finite $\Rightarrow$

$\{n: a+\varepsilon > a_n > a-\varepsilon\}$ - infinite.

$\varepsilon = \frac{1}{k}$ $\exists n_k > n_{k-1}: a_{n_k} \in (a-\varepsilon, a+\varepsilon)$

$a_{n_k} \to a \quad (|a_{n_k} - a| < \frac{1}{k})$.

Proof of 2.

$b > a$ $\quad \{n : a_n > a + \underbrace{\frac{b-a}{2}}_{>0}\}$ - finite

$\{n : a_n > b - \frac{b-a}{2}\}$ - finite.

No subsequence!

3) $\Rightarrow$ 1) $\quad u_{n_k} \geq a_{n_k}$ $\quad \lim a_{n_k} \leq \lim u_{n_k} = \lim u_n$

$\forall k \; \exists n_k > k \quad a_{n_k} > b_k - \frac{1}{k}$. $u_k > \lim a_{n_k} > u_k - \frac{1}{k}$

$\lim$ $\quad$ $\lim$

Polya "How to solve a problem."

A reference for bonus problem