Complex analysis: limits, continuity, differentiability.

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## Limits and continuity: a short review.

Same as for $\mathbb{R}^2$!

Def. Let $f$ be a function defined on a set $K \subseteq \mathbb{C}$.

$f$ has a limit $A$ as $z \to z_0$ if

$\forall \varepsilon > 0 \quad \exists \delta > 0 : 0 < |z - z_0| < \delta, z \in K \Rightarrow |f(z) - f(z_0)| < \varepsilon.$

Properties. 1) If the limit exists it is unique provided $z_0$ is a limit point of $K$

$(\forall \delta > 0 : \quad B(z_0, \delta) \cap (K \setminus \{z_0\}) \neq \emptyset).$

2) $\lim\limits_{z \to z_0} (f(z) + g(z)) = \lim\limits_{z \to z_0} f(z) + \lim\limits_{z \to z_0} g(z)$ (if both exist)

3) $\lim\limits_{z \to z_0} (f(z) \times g(z)) = \lim\limits_{z \to z_0} f(z) \times \lim\limits_{z \to z_0} g(z)$ (if both exist)

4) $\lim\limits_{z \to z_0} f(z) = A \Leftrightarrow \begin{array}{l} \lim\limits_{z \to z_0} \operatorname{Re} f(z) = \operatorname{Re} A \\ \lim\limits_{z \to z_0} \operatorname{Im} f(z) = \operatorname{Im} A. \end{array}$

5) $\lim\limits_{z \to z_0} f(z) = A \Rightarrow \lim\limits_{z \to z_0} \overline{f(z)} = \overline{A}.$

Proof. 1, 2, 3 - as in real case.

5. $\Leftarrow \quad |f(z) - A| = |\overline{f(z)} - \overline{A}|.$

4 $\Leftarrow$ 5+2, since $\operatorname{Re} f(z) = \dfrac{f(z) + \overline{f(z)}}{2}$

$\operatorname{Im} f(z) = -\frac{i}{2} (f(z) - \overline{f}(z))|.$

**Important property:** $K_1, K_2 \subset K$. Let $\lim\limits_{\substack{z \to z_0 \\ z \in K}} f(z) = A$.

Then $\lim\limits_{\substack{z \to z_0 \\ z \in K_1}} f(z) = \lim\limits_{\substack{z \to z_0 \\ z \in K_2}} f(z) = A$.

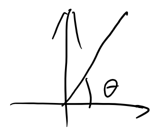
**Proof.** Fix $\varepsilon > 0$, $\exists \delta > 0 \ldots$ $z \in K_1$, $|z - z_0| < \delta \Rightarrow z \in K, |z - z_0| < \delta \Rightarrow |f(z) - A| < \varepsilon$ ∎

**Corollary.** $K_1, K_2 \subset K$, $\lim\limits_{\substack{z \to z_0 \\ z \in K_1}} f(z) \neq \lim\limits_{\substack{z \to z_0 \\ z \in K_2}} f(z) \Rightarrow$

$\lim\limits_{z \to z_0} f(z)$ does not exist.

**Easy and important example:**

$\lim\limits_{h \to 0} \dfrac{\overline{h}}{h}$ does not exist!




On ray $L_\theta$: $h = |h| \operatorname{cis} \theta$

On $L_\theta$: $\dfrac{\overline{h}}{h} = \dfrac{|h| \operatorname{cis}(-\theta)}{|h| \operatorname{cis}(\theta)} = \operatorname{cis}(-2\theta)$ – different on different rays!

**Continuous functions:**

As usual: $f$ is continuous at $z_0$ if $\lim\limits_{\substack{z \to z_0 \\ z \in K}} f(z) = f(z_0)$.

**Remark** All of this can be done at $\infty$, but we need to use **spherical metric**:

$$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} d(f(z), \infty) = 0 \iff \lim_{z \to 0} \frac{1}{f(z)} = 0$$

$$\lim_{z \to \infty} f(z) = A \iff \forall \varepsilon > 0 \ \exists \delta > 0 \ 0 < d(z, \infty) < \delta \Rightarrow |f(z) - A| < \varepsilon$$

$$\iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = A$$

Important (and easy) observation: if $z_0 \neq \infty$ then

$$\lim_{z \to z_0} |z - z_0| = 0 \iff \lim_{z \to z_0} d(z, z_0) = \lim_{z \to z_0} \frac{|z - z_0|}{\sqrt{1+|z|^2}\sqrt{1+|z_0|^2}} = 0.$$

## Analytic functions

Def. $f$ is (complex) differentiable at $z_0$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \text{ exists.}$$

Equivalent definition: $f(z) = f(z_0) + (z - z_0)\varphi(z)$, where $\varphi(z)$ continuous at $z_0$, $\varphi(z_0) = f'(z_0)$.

Proof (of equivalency) $(\Uparrow)$ $\displaystyle\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \varphi(z) = f'(z_0)$

$(\Downarrow)$ Take $\varphi(z) := \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0}, & z \neq z_0 \\ f'(z_0), & z = z_0 \end{cases}$

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Remark. Differentiability at one point is not interesting.

Interesting: differentiability at every point of some $B(z_0, \delta)$ - some neighborhood of $z_0$.

Thm. (the same as in Calculus).

1) If $f'(z)$, $g'(z)$ exist, then

$(f \pm g)'(z) = f'(z) \pm g'(z)$ - exist.

$(fg)'(z) = f'(z)\,g(z) + f(z)\,g'(z)$ - exist

$\left(\frac{f}{g}\right)'(z) = \dfrac{f'(z)\,g(z) - f(z)\,g'(z)}{g^2(z)}$ if $g(z) \neq 0$

2) If $f'(z)$ exist, and $g'$ exist at $f(z)$, then

$(g(f(z)))' = g'(f(z)) \cdot f'(z)$ - exist (Chain Rule).

Proof The same as in Calculus! ∎

Example 0. $f(z) \equiv c$. $f'(z) \equiv 0$.

Example 1 $f(z) = z$, $f'(z_0) = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = 1$.