Introduction: coming attractions

Friday, November 13, 2020 9:32 AM

Fundamental theorem of Algebra

On the invention of Complex numbers:
Look at a cubic equation of the form $t^3 + pt + q = 0$
Famous Cardano's formula(1545!):

$$t = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Only one root? There are always three complex cubic roots!
$t^3 - 3t = 0.\ p = -3,\ q = 0.$ Nice real roots: 0, $\pm\sqrt{3}$
Formula:
t=$\sqrt[3]{\sqrt{-1}} + \sqrt[3]{\sqrt{-1}}$.
What is wrong? We need to take all possible values of cubic root of i!

Every polynomial with complex coefficients has a root.
In fact, if it has degree d, it has at least d roots, counting multiplicity.

Complex differentiation vs real differentiation

Real case:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Function can be differentiable only once: $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x \leq 0 \end{cases}$

- Even infinitely differentiable functions can have nothing to do with their Taylor series: $\boxed{f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases} \quad f^{(k)}(0) = 0 \ \forall k}$

$f(x) = \sum a_k (x - x_0)^k$ - real analytic functions.

Differentiable $\neq$ Infinitely Differentiable $\neq$ Analytic

Complex case:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

- analytic functions.

- Any function differentiable at a neighborhood of a point is infinitely differentiable.

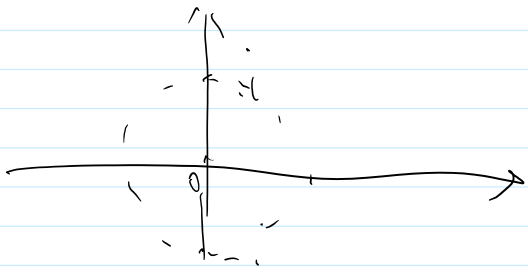
- Differentiable functions are automatically equal to the sum of their Taylor series:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad a_k = \frac{f^{(k)}(z_0)}{k!}$$

Differentiable = Infinitely Differentiable = Analytic

$\frac{1}{1-x} = \sum_{n=0} x^n$, $|x|<1$ $x=1$ - singularity

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, $|x|<1$ No real singularities, but $z=\pm i$ - complex singularity



Complex Antiderivative

Real case:

Any continuous function has an antiderivative: $F'(x) = f(x)$

$F(x) := \int_0^x f(t)\,dt$ - FTC!

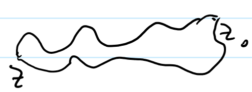
x $x+h$

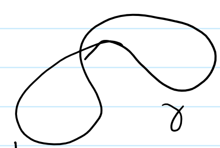
Complex case:

Only differentiable functions have antiderivatives.

$F(z) := \oint_{z_0}^{z} f(w)\,dw$ - line integral.

But over which path?





$\oint_\gamma f(z)dz = 0$

$|f(z+h) - f(z) - T(h)| \to 0$

$|\underbrace{f(z+h) - f(z)}_{\lambda = f'(z)} - \lambda h| \to 0$

Conformal maps and Riemann Theorem

Special important class: conformal maps: injective analytic functions.

$\varphi: \Omega_1 \to \Omega_2$ - angle and orientation preserving map.

Theorem (Riemann) If Ω is a simply-connected domain ("a domain without holes") $\Omega \neq \mathbb{C}$. Then $\exists\, \varphi: \mathbb{D} \to \Omega$ - conformal bijection!