Fractional linear (Möbius) transformations.

Monday, December 7, 2020 9:05 AM

<u>Def.</u> Let $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ $\left(= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$

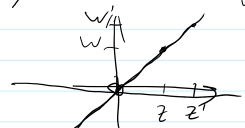
$z \to \dfrac{az+b}{cz+d}$ is called <u>fractional-linear</u> map (or <u>Möbius</u> map)

Rational map of order 1. $\left(\dfrac{az+b}{cz+d}\right)' = \dfrac{ad-bc}{(cz+d)^2}$ - exists when $z \neq -\frac{d}{c}, \infty$

Complex projective line.

Consider $P\mathbb{C} := \mathbb{C}^2 \setminus \{0\} = \left\{ \begin{pmatrix} z \\ w \end{pmatrix}, z, w \in \mathbb{C}; |z|^2 + |w|^2 \neq 0 \right\} / \sim$

$\begin{pmatrix} z \\ w \end{pmatrix} \sim \begin{pmatrix} z' \\ w' \end{pmatrix}$ if $zw' = z'w$ - complex "line" through $\vec{0}$ and $\begin{pmatrix} z \\ w \end{pmatrix}$.



Natural map: $\hat{\mathbb{C}} \leftrightarrow P\mathbb{C}$ $\quad \lambda \neq \infty \Rightarrow \left[\begin{pmatrix} \lambda \\ 1 \end{pmatrix}\right]$

$\lambda = \infty \Rightarrow \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ - an invertible matrix $\mathbb{C}^2 \to \mathbb{C}^2$, preserves lines through origin.

$\left[A\begin{pmatrix} z \\ w \end{pmatrix}\right] = \left[\begin{pmatrix} az+bw \\ cz+dw \end{pmatrix}\right]$ - a map of $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, given by $\frac{az+b}{cz+d}$. Denoted by T_A.

<u>Some notations:</u> $GL(2, \mathbb{C}) = \{ 2\times 2$ matrices, $\det \neq 0\}$.

$SL(2, \mathbb{C}) = \{ 2\times 2$ matrices, $\det = 1\}$.

$A \in GL(2, \mathbb{C}) \Leftrightarrow \frac{1}{\sqrt{\det A}} A \in SL(2, \mathbb{C})$.

Observe: $T_{AB} z = \left[AB\begin{pmatrix} z \\ 1 \end{pmatrix}\right] = \left[A\left(B\begin{pmatrix} z \\ 1 \end{pmatrix}\right)\right] = T_A(T_B z)$. So $T_A^{-1} = T_{A^{-1}}$.

So the map $A \mapsto T_A$ - group homomorphism
$\ker T = \{\pm I\}$ in $SL(2, \mathbb{C})$
$\ker T = \{\lambda I, \lambda \in \mathbb{C} \setminus \{0\}\}$ in $GL(2, \mathbb{C})$
$PSL(2, \mathbb{C}) := SL(2, \mathbb{C}) / \{\pm I\} = GL(2, \mathbb{C}) / \{\lambda I\}$

If you are familiar with groups.

<u>Theorem.</u> Any Möbius map maps lines and circles to lines and circles.

<u>Proof.</u> If $c = 0$, $z \to \frac{a}{d} z + \frac{b}{d}$ $(ad - bc = ad \neq 0 \Rightarrow d \neq 0)$ composition of multiplication by $\frac{a}{d}$ and shift by $\frac{b}{d}$ $\Rightarrow$ preserves circles and lines.

If $c \neq 0$: $\dfrac{az+b}{cz+d} = \dfrac{a}{c} + \dfrac{bc - ad}{c^2 (z + \frac{d}{c})}$ - composition of shift by $\frac{d}{c}$,

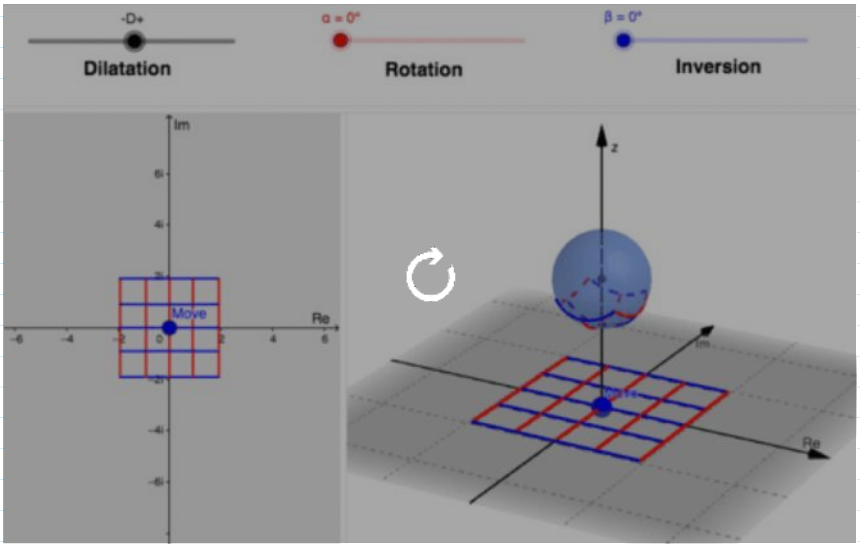
$z \to \frac{1}{z}$ (inversion), multiplication by $\dfrac{bc-ad}{c^2}$, and another shift by $\frac{a}{c}$.

$z \to \frac{1}{z}$ preserves circles and lines, because so does stereographic

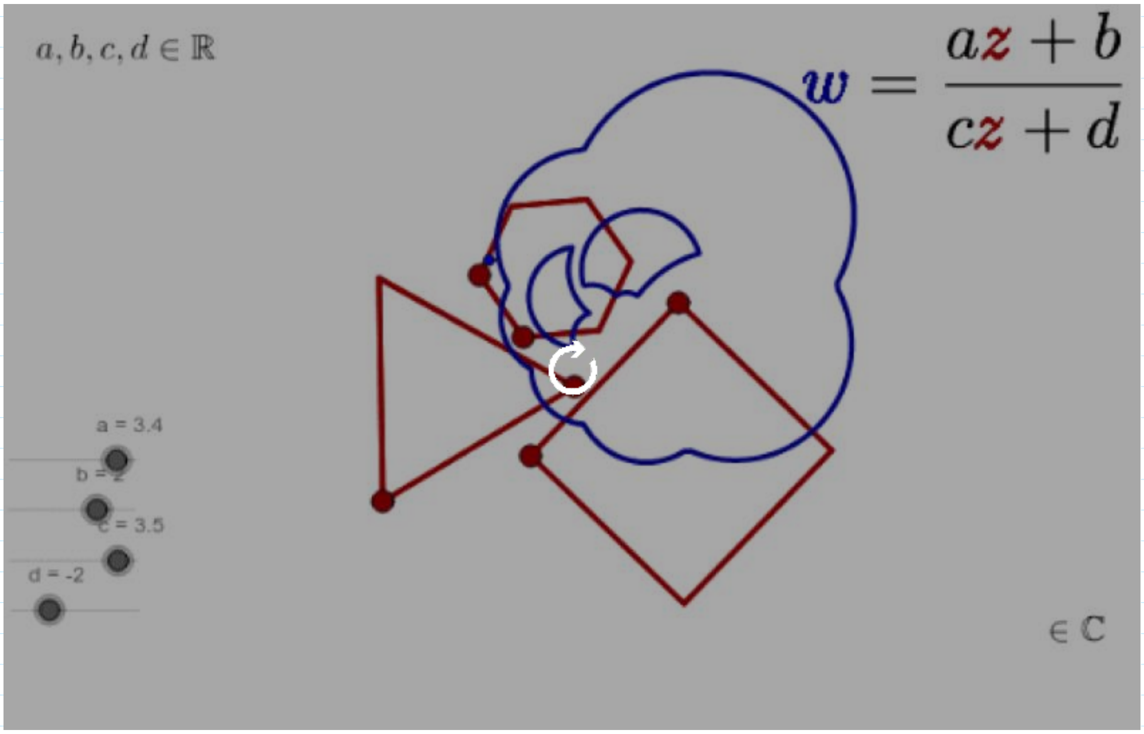
projection P and $R: (x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3)$
$(z \to \frac{1}{z} = P \circ R \circ P^{-1})$. So the whole composition also preserves them ∎

What we just observed: any Möbius map is a composition of some dilations ($z \to rz$, $r>0$), rotations ($z \to e^{i\theta}z$), translations ($z \to z+a$), and inversions ($z \to \frac{1}{z}$).

[Möbius transformation]



[Mobius transformations]:animation.


$a, b, c, d \in \mathbb{R}$
$w = \dfrac{az + b}{cz + d}$
a = 3.4
c = 3.5
d = -2
$\in \mathbb{C}$