Def. $T:\mathbb{C}\to\mathbb{C}$ $\mathbb{R}$-linear invertible map

angle-preserving if $\forall w,z\in\mathbb{C}$ $|w||z|\langle Tw,Tz\rangle = |Tw||Tz|\langle w,z\rangle$

$\langle w,z\rangle$ - scalar product $= \mathrm{Re}\,w\,\mathrm{Re}\,z + \mathrm{Im}\,w\,\mathrm{Im}\,z = \mathrm{Re}\,z\bar{w} = \mathrm{Re}\,\bar{z}w$

$\left(\frac{\langle w,z\rangle}{|w||z|} = \cos\theta \right.$  $\left.\right)$

Examples $Tz = az$, $Tz = a\bar{z}$.

complex linear — complex anti-linear

Lemma (angle-preserving for linear maps)

The following are equivalent for $\mathbb{R}$-linear map $T:\mathbb{C}\to\mathbb{C}$:

1) T is angle preserving

2) $\exists a\in\mathbb{C}\setminus\{0\}$: $Tz = az\ \forall z\in\mathbb{C}$ $(T = M_a)$
or $Tz = a\bar{z}\ \forall z\in\mathbb{C}$ $(T = M_a\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix})$

3) $\exists s>0 : \langle Tw,Tz\rangle = s\langle w,z\rangle\ \forall z,w\in\mathbb{C}$.

Proof 1) $\Rightarrow$ 2) Let $a = T1$.

Consider $Sz := a^{-1}\cdot Tz$ - angle-preserving.

$S1 = 1$

$\langle Si, S1\rangle = \langle Si, 1\rangle = 0 \Rightarrow Si = ri$ for some $r\in\mathbb{R}$. $r\neq 0$ (invertible $\Rightarrow Si\neq 0$).

$S(1+i) = S1 + Si = 1+ri$ $\quad S(1-i) = 1-ri$

$\langle 1+i, 1-i\rangle = 0 \Rightarrow \langle 1+ri, 1-ri\rangle = \mathrm{Re}\left((1+ri)\overline{(1-ri)}\right) = 1-r^2 = 0 \Rightarrow r=\pm 1$.

$r=1 \Rightarrow Sz = \mathrm{Re}\,z\,S1 + \mathrm{Im}\,z\,Si = z \Rightarrow Tz = aSz = az$

$r=-1 \Rightarrow Sz = \bar{z} \Rightarrow Tz = a\bar{z}$.

2) $\Rightarrow$ 3) $\quad \langle Tz, Tw\rangle = \mathrm{Re}\,Tz\overline{Tw} = |a|^2\,\mathrm{Re}\,z\bar{w}$

3) $\Rightarrow$ 1) $\quad |Tz| = \sqrt{s}|z|$, $|Tw| = \sqrt{s}|w|$. Plug in ■

Conformal maps.

Piecewise smooth arcs.

Real notation Complex notation.

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ $t \in [a,b]$ $z(t)$

<u>Tangent</u>:

$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \neq \vec{0}$ $z'(t) \neq 0.$

Piecewise smooth: $S \subset [a,b]$ - finite

$\forall t \in [a,b] \setminus S,$ $z'(t) \neq 0.$

f - real differentiable map at z_0, $z(t) = z_0$, $z'(t) \neq 0$.

T - differential of f at z_0 ($\frac{|f(z_0+h) - f(z) - T(z)h|}{|h|} \to 0$)

Then $\underline{(f(z(t)))' = T(z'(t))}$ - chain rule.

<u>Tangent map</u> or <u>differential</u>.

<u>Def</u>. f is called angle-preserving at z_0 if its tangent map at z_0 is angle-preserving.

<u>Lemma</u>. The following are equivalent:

1) f is angle-preserving at z_0.
2) Either $\frac{\partial f}{\partial \bar z} = 0$ $\frac{\partial f}{\partial z} \neq 0$ or $\frac{\partial f}{\partial \bar z} \neq 0$, $\frac{\partial f}{\partial z} = 0$
3) T satisfies $\langle Tw, Tz \rangle = s \langle w, z \rangle$ for some $s > 0$.

<u>Proof</u>. This is just a restatement of our previous Lemma ■

<u>Theorem</u>. Let f be continuously real differentiable function in a region D. Then f is angle-preserving if and only if $f \in A(D)$, $f'(z) \neq 0$ $\forall z \in D$ or $\bar f \in A(D)$, $(\bar f)'(z) \neq 0$ $\forall z \in D$.

<u>Proof</u>. By previous Lemma, know "if" part.

Also know: $\forall z \in D$ $\frac{\partial f}{\partial z} \neq 0$, $\frac{\partial f}{\partial \bar z} = 0$; or $\frac{\partial f}{\partial z} = 0$, $\frac{\partial f}{\partial \bar z} \neq 0$.

Consider: $\dfrac{\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar z}}{\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar z}}$ - well defined, can be $+1$ or -1.

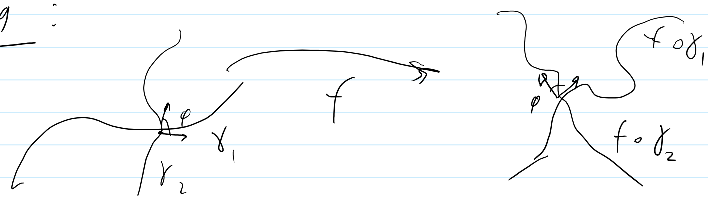
By connectivity of D - it is a constant!

By connectivity of D — it is a constant!

If $= 1 \Rightarrow f \in A(D)$

If $= -1 \Rightarrow \bar{f} \in A(D)$ ∎

Geometric meaning:



If γ_1 and γ_2 intersect at angle φ, then $f \circ \gamma_1$ and $f \circ \gamma_2$ intersect at angle φ.