Complex plane: algebraic and geometric properties.

Monday, December 7, 2020 9:01 AM

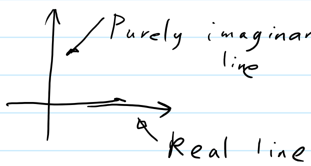
Complex numbers as vectors. Addition and multiplication

$z = \underbrace{a+ib} \rightsquigarrow \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}$ $\quad i \rightsquigarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad 1 \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Notation: $a = \operatorname{Re} z$
$b = \operatorname{Im} z$

$\mathbb{R}^2$ with addition and multiplication.

$\mathbb{R}$ identified with $\left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} : a \in \mathbb{R} \right\}$.



Addition: usual vector addition: $(a+ib)+(c+id) = (a+c)+i(b+d)$

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

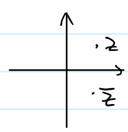
Multiplication: $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac-bd \\ ad+bc \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Notation: $\mathbb{C} := \mathbb{R}^2$ with addition and multiplication.

Absolute value and conjugate

$z = a+ib \qquad \bar{z} = a-ib$ - conjugate.



$\operatorname{Re} z = \frac{z+\bar{z}}{2}, \quad \operatorname{Im} z = \frac{z-\bar{z}}{2i} = -\frac{i}{2}(z-\bar{z})$.

An example: Line $ax+by=c$, $a^2+b^2 \neq 0$ $\qquad x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ $\qquad \overset{\mu}{\frac{a-ib}{2}} z + \frac{a+ib}{2}\bar{z} = c$

$\boxed{\mu z + \bar{\mu}\bar{z} = c \quad \mu \neq 0}$ $c \in \mathbb{R}$

Properties 1) $\overline{z+w} = \bar{z} + \bar{w}$
2) $\overline{zw} = \bar{z}\,\bar{w}$ $\Big\}$ just computation

Absolute value:

$|z|^2 = x^2+y^2 = (x+iy)(x-iy) = z\bar{z}$

Properties: 1. $|z+w| \le |z|+|w|$

2. $|zw| = |z||w|$.

Proof. 1. The usual triangle inequality.

2. $|zw|^2 = zw\overline{zw} = z\bar{z}\cdot w\bar{w} = |z|^2|w|^2$.

Notations : $B(z,\delta) = \{w : |z-w|<\delta\}$ - open

$\bar{B}(z,\delta) = \{w : |z-w|\le\delta\}$ - closed

balls centered at z, radius δ.

Complex numbers form a field.

(P1)	(Associative law for addition)	$a + (b + c) = (a + b) + c$.
(P2)	(Existence of an additive identity)	$a + 0 = 0 + a = a$.
(P3)	(Existence of additive inverses)	$a + (-a) = (-a) + a = 0$.
(P4)	(Commutative law for addition)	$a + b = b + a$.
(P5)	(Associative law for multiplication)	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
(P6)	(Existence of a multiplicative identity)	$a \cdot 1 = 1 \cdot a = a$; $1 \neq 0$.
(P7)	(Existence of multiplicative inverses)	$a \cdot a^{-1} = a^{-1} \cdot a = 1$, for $a \neq 0$.
(P8)	(Commutative law for multiplication)	$a \cdot b = b \cdot a$.
(P9)	(Distributive law)	$a \cdot (b + c) = a \cdot b + a \cdot c$.

$\frac{1}{z} = \frac{\bar{z}}{|z|^2} \Longleftarrow z\bar{z} = |z|^2$

$\frac{w}{z} = \frac{w\bar{z}}{|z|^2}$

Matrix form of a Complex Number.

Fix $z = a+ib$. Map $w = x+iy \to z(x+iy)$ in vector form:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax-by \\ bx+ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$M_z := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ $\qquad M_z \cdot w = zw$

Moreover: $M_z + M_w = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} a+x & -b-y \\ b+y & a+x \end{pmatrix} = M_{z+w}$

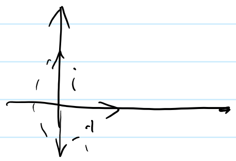
$M_z \cdot M_w = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} ax-by & -bx-ay \\ bx+ay & ax-by \end{pmatrix} = M_{zw}$.

Let $\mathcal{M} := \left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a,b \in \mathbb{R}\right\}$.

Then $\varphi: \mathbb{C} \to \mathcal{M}$, $\varphi(z) := M_z$ – field isomorphism (bijection preserving $+$ and $\times$).

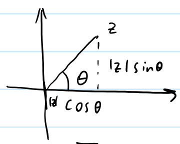
What is $M_{\bar z}$? M_z^T $\quad M_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $M_z^T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = M_{\bar z}$.

Remark. $z \longrightarrow \bar z$ – linear map. What is the matrix?



$$\begin{array}{cc} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \bar 1 \quad \bar i \end{array}$$

Polar form of a Complex Number



z in polar form: $(|z|, \theta)$ θ-angle with $\mathbb{R}_+$.

Real case: $(r, \theta) \longmapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

Complex notation: $z = |z|(\cos\theta + i\sin\theta)$.

Temporary notation: $\operatorname{cis}\theta := \cos\theta + i\sin\theta$ $\quad$ ($e^{i\theta}$ – later)

$|\operatorname{cis}\theta| = 1$.

z – absolute value $|z|$

$\theta = \arg z$ – not unique. $\arg z = \{\theta + 2\pi k, k \in \mathbb{Z}\} \in \mathbb{R}/_{\sim}$ $\quad x \sim y \Leftrightarrow x - y = 2\pi k, k \in \mathbb{Z}$.

Principal value of argument: $-\pi < \operatorname{Arg} z \le \pi$

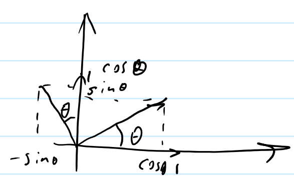
$\arg \bar z = -\arg z = \{\theta : -\theta \in \arg z\}$.

$2\arg z$ – well defined.

$\frac{\arg z}{2}$ – is not!

Rotation as a multiplication.

Rotation of $\mathbb{R}^2$ – linear map.



Matrix: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = M_{\operatorname{cis}\theta}$

$\operatorname{cis}\theta \cdot w = M_{\operatorname{cis}\theta} w$ – w rotated by θ. $\quad \boxed{\arg \operatorname{cis}\theta\, w = \arg w + \theta}$

$z = |z| \operatorname{cis}(\arg z)$. $\quad zw = |z| \operatorname{cis}(\arg z)\, w$. $\quad zw$ – w rotated by $\arg z$ dilated by $|z|$.

$\forall \theta_1, \theta_2 \in \arg z$, $\operatorname{cis}\theta_1 = \operatorname{cis}\theta_2$.

Theorem. 1) $\arg zw = \arg z + \arg w$, $|zw| = |z||w|$.

2) $\arg \frac{z}{w} = \arg z - \arg w$, $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

($\arg z + \arg w$ – take $\theta_1 \in \arg z$, $\theta_2 \in \arg w$ $\quad \arg z + \arg w = \{\theta_1 + \theta_2 + 2\pi k\}$ does not depend on θ_1, θ_2! Prove it!)

Proof. 1) just done

2) $\arg \frac{1}{w} = \arg \frac{\bar w}{|w|^2} = -\arg w$, $(\arg w + \arg \bar w = \arg 1)$ ∎

N.B. $\operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w$ – $\boxed{\text{not always!}}$ Example?

$z = w = -i$ $\quad \operatorname{Arg} z = \operatorname{Arg} w = -\frac{\pi}{2}$

$z = w = -i \qquad \operatorname{Arg} z = \operatorname{Arg} w = -\frac{\pi}{2}$

$zw = -1 \qquad \operatorname{Arg} -1 = \pi \neq -\frac{\pi}{2} + \left(-\frac{\pi}{2}\right).$

Trigonometry done right: deMoivre formula

$$\operatorname{cis}(\theta_1+\theta_2) = \operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 \qquad \boxed{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)}$$

de Moivre formula

$$\cos(\theta_1+\theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

$$\sin(\theta_1+\theta_2) = \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2$$

$$\cos n\theta = \frac{1}{2}\left(\operatorname{cis}^n\theta + \overline{\operatorname{cis}^n\theta}\right) = \frac{1}{2}\left((\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\right) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C_n^{2k} \sin^{2k}\theta \cos^{n-2k}\theta$$

$$\sin n\theta = -\frac{i}{2}\left((\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n\right) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k C_n^{2k+1} \sin^{2k+1}\theta \cos^{n-1-2k}\theta$$

Powers and Roots

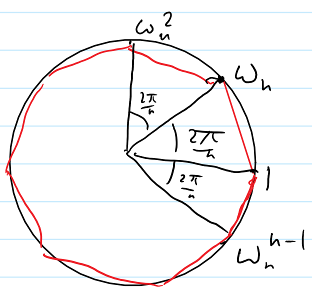
Integer powers: $z^n = |z|^n \operatorname{cis}(n \arg z).\ n \in \mathbb{N}, n \in \mathbb{Z}$

$$i^{239} = 1 \cdot \operatorname{cis}\left(239 \cdot \frac{\pi}{2}\right) = \operatorname{cis}\frac{3\pi}{2} = -i.$$

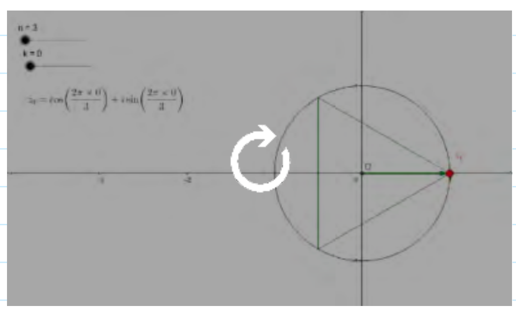
$$(1+i)^{239} = (\sqrt{2})^{239} \operatorname{cis}\left(239\frac{\pi}{4}\right) = \sqrt{2} \cdot 2^{119} \cdot \operatorname{cis}\frac{7}{4}\pi = 2^{119}(1-i).$$

Roots of 1. $z^n = 1.\ \begin{cases} n \operatorname{Arg} z \in \{2\pi k,\ k \in \mathbb{Z}\}. \\ |z| = 1 \end{cases}$

$\omega_n := \operatorname{cis}\left(\frac{2\pi}{n}\right).$ Solutions: $1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}.$ $\quad \arg \omega_n^{n-1} = \left\{\frac{2\pi(n-1)}{n} + 2\pi k\right\}$



Roots of Unity



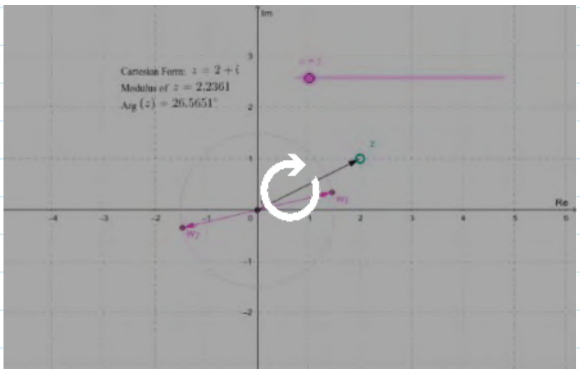
nth roots of $w \neq 0$: $z^n = w$.

z_1, z_2 – two roots. Then $\left(\frac{z_1}{z_2}\right)^n = \frac{w}{w} = 1$. So $\frac{z_1}{z_2}$ – root of 1!

So: n roots. $z_0, z_0\omega_n, \dots, z_0\omega_n^{n-1}$

$|z_0| = |w|^{1/n}$. $n \operatorname{Arg} z_0 \in \arg w$.

nth Roots of Complex Numbers



$w = |w|(\cos\varphi + i\sin\varphi)$

Can take

$z_0 = |w|^{1/n}\left(\cos\frac{\varphi}{n} + i\sin\frac{\varphi}{n}\right)$