# Classification of Möbius maps

Saturday, January 23, 2021 10:18 AM

Let us start at looking for fixed points (can be $=\infty$ if $c=0$):

$$z=\frac{az+b}{cz+d} \iff \frac{cz^2+(d-a)z-b}{cz+d}=0 \qquad \text{Normalize: } ad-bc=1$$

Case 1: two fixed points: $(d-a)^2 \neq -4bc \overset{ad-bc=1}{\iff} (a+d)^2 \neq 4$.

Case 2: one fixed point: $(d-a)^2 = 4bc \iff (a+d) = \operatorname{Tr}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \pm 2$.
(parabolic).

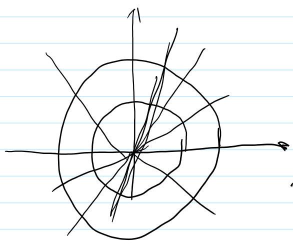
Case 1: Let $z_1, z_2$ be the fixed points.

Start with $z_1 = 0$, $z_2=\infty$.

Then $c=0$, $b=0$. $T(z) = \frac{a}{d}z = a^2 z$
$ad=1$

Case 1a): $|a^2|=1$, $a^2\neq 1$. (not an identity)

$a^2 = e^{i\theta}$ $\quad Tz = e^{i\theta}z$ – rotation.



Rotates the rays $l_\varphi\{\arg z = \varphi\}$ to $l_{\varphi+\theta}$

leaves circles $C_r = \{|z|=r\}$ invariant.

Elliptic.

$\operatorname{Tr} T = a + \frac{1}{a} = e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}} = 2\cos\frac{\theta}{2} \in (-2,2)$
$(\theta\in(0,2\pi))$.

Case 1b) $\quad a^2>0$, $a^2\neq 1$. Hyperbolic.

$Tz = kz$, $k>0$, $k\neq 1$ – dilation.

Leaves the rays $L_\varphi$ invariant.
Shifts $C_r$ to $C_{kr}$. $\qquad \operatorname{Tr} T = \pm\left(\sqrt{k}+\frac{1}{\sqrt{k}}\right) \in (-\infty,-2)\cup(2,\infty)$.

Case 1c $\quad a^2 \notin \mathbb{R}_+ \cup \{|a^2|=1\}$ – loxodromic –
composition of elliptic and hyperbolic with the same fixed points. $\operatorname{Tr} T \notin \mathbb{R}$.

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Arbitrary fixed points $z_1, z_2$

$Sz := \frac{z-z_1}{z-z_2}$ : $\quad Sz_1 = 0$, $Sz_2 = \infty$.

$\tilde{T} = STS^{-1}$ : $\quad \tilde{T}(\infty) = STS^{-1}(\infty) = ST(z_2) = S(z_2) = \infty$
$\tilde{T}(0) = 0$.

$\operatorname{Tr}\tilde{T} = \operatorname{Tr} T$ – by linear algebra.

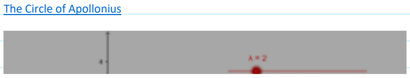
Def $T$ – elliptic/hyperbolic/loxodromic if so is $\tilde{T}$.

$\operatorname{Tr} T \in (-2,2)$ – elliptic

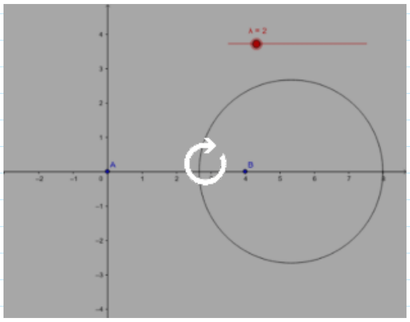
$\operatorname{Tr} T \in (-\infty,-2)\cup(2,\infty)$ $\qquad T = S^{-1}\tilde{T}S$

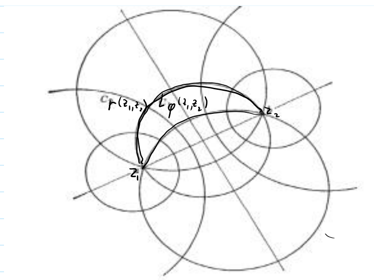
Geometry: $\quad C_r(z_1,z_2) = \{|Sz|=r\} = \left\{\frac{|z-z_1|}{|z-z_2|}=r\right\}$ – circles of Apollonius.

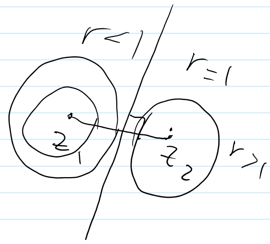
The Circle of Apollonius



[The Circle of Apollonius](#)






Steiner circles

$L_\varphi(z_1,z_2) = \{\arg(Sz)=\varphi\}$ – circular arcs from $z_1$ to $z_2$.

$L_\varphi(z_1,z_2) \perp C_r(z_1,z_2)$ – because $S$ preserves angles.

Reflections: With respect to $C_r(z_1,z_2)$: preserves $L_\varphi(z_1,z_2)$
maps $C_s(z_1,z_2)$ to $C_{r^2/s}(z_1,z_2)$

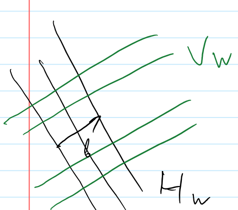
with respect to $L_\varphi(z_1,z_2)$: preserves $C_r(z_1,z_2)$
maps $L_\theta(z_1,z_2)$ to $L_{2\varphi-\theta}(z_1,z_2)$.

Proof. True for $z_1=0$, $z_2=\infty$. $S$ preserves symmetry ■

Case 2 (parabolic) $\operatorname{Tr} T = \pm 2$. $T \neq id$.

$z_0$ – fixed point.

$z_0=\infty$. $Tz = z+b$ $b\neq 0$. (can be written as $T_{\begin{pmatrix}1&b\\0&1\end{pmatrix}}$ or $T_{\begin{pmatrix}-1&-b\\0&-1\end{pmatrix}}$)




$V_w^b = \{w+tb, t\in\mathbb{R}\}$ – vertical horocycles

$H_w^b = \{w+itb, t\in\mathbb{R}\}$ – horizontal

$V_w \perp H_{w'}$.

$TV_w^b = V_w^b$ $TH_w^b = H_{w+b}^b$.

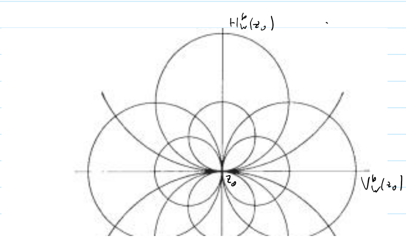
$z_0 \neq \infty$. $Sz := \frac{1}{z-z_0}$.

$\tilde{T} := STS^{-1}$; $\tilde{T}(\infty)=\infty$, no other fixed points

$\operatorname{Tr} T = \operatorname{Tr}\tilde{T} = \pm 2$. $T = S^{-1}\tilde{T}S$

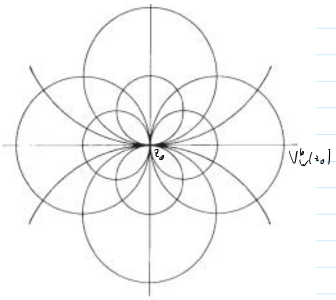
$V_w^b(z_0) = SV_w^b$ – circles through $z_0$, tangent to line $Sb$.

$H_w^b(z_0) = SH_w^b$ – ―― // ―――― $iSb$.




$TV_w^b(z_0) = V_w(z_0)$ $TH_w^b(z_0) = H_{w+b}^b(z_0)$

Reflections are nice.




$T V_w^b(z_0) = V_w(z_0)$ $T H_w^b(z_0) = H_{w+b}^b(z_0)$

Reflections are nice.