Complex Analysis

Assignment 6, due March 18

Problem 1 of 5. Problems 1 and 2, page 133.

Problem 2 of 5. Let f be analytic in the region $\{z : |z| > R\}$. Assume that $\lim_{|z|\to\infty} f(z)$ exists and finite. Let, for $r > R$, $M(r) := \max_{|z|=r} |f(z)|$. Show that $M(r)$ is a decreasing function.
Hint: Consider $f(1/z)$.

Problem 3 of 5. Let P be a polynomial of degree d, $M(r) := \max_{|z|=r} |P(z)|$. Show that for any $0 < r_1 < r_2$, we have

$$\frac{M(r_1)}{r_1^d} \geq \frac{M(r_2)}{r_2^d}.$$

The equality is attained for some $0 < r_1 < r_2$ if and only if $P(z) = cz^d$ for some $c \neq 0$.

Problem 4 of 5.

(1) Let f and g be two analytic maps of the unit disk. Assume that g is conformal (analytic and injective), that $f(\mathbb{D}) \subset g(\mathbb{D})$, and that $f(0) = g(0)$. Show that for any $r \leq 1$, $f(r\mathbb{D}) \subset g(r\mathbb{D})$ and that $|f'(0)| \leq |g'(0)|$.
Hint: Consider the function $h := g^{-1} \circ f$.

(2) Let h be an analytic and bounded by 1 function in the unit disk. Show that for all z, $0 < |z| < 1$, we have

$$|f(z) - f(0)| \leq |z| \frac{1 - |f(0)|^2}{1 - |f(0)||z|}.$$

For which f can the equality be attained?
Hint: Use the previous part with the function $g(z) = (z + f(0))/(1 + \overline{f(0)}z)$.

Problem 5 of 5.

(1) A region Ω is called *star-shaped* if there exists a point $z_0 \in \Omega$ such that for any other point $z \in \Omega$, the segment $[z_0, z]$ is a subset of Ω. Prove that Ω is simply connected.

(2) A region Ω is called *almost convex* if if for any point $z \notin \Omega$ one can find a ray $l \subset \Omega^c$ such that $z \in l$. Prove that Ω is simply connected.