Complex Analysis

Assignment 5, due March 11

Problem 1 of 5. Let f be an entire function.

(1) Assume that an f satisfies the condition

$$|f(z)| \leq C|z|^d, \quad |z| > R,$$

where C and R are some positive constants. Show that f is a polynomial of degree at most d.
Hint: Use Cauchy Inequlities to estimate $f^{(d+1)}(0)$.

(2) Assume that f has a removable singularity or pole at infinity (i.e. the limit $\lim_{z\to\infty} f(z)$ exists, but might be infinite). Show that f is a polynomial.
Hint: It might be useful to consider the function $f(1/z)$ at 0

Problem 2 of 5. Assume that f is a continuous function on the closed disk $\overline{B(z_0, r)}$ which is analytic on the open disk $B(z_0, r)$. Prove that

$$f(z) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(\xi)}{\xi - z} \, d\xi,$$

where C_r is the positively oriented circle of radius r centered at z_0 and $|z - z_0| < r$.
Be careful: you cannot assume that f is analytic in the closed disk $\overline{B(z_0, r)}$.

Problem 3 of 5. A function $f; \mathbb{R} \mapsto \mathbb{C}$ is called *real-analytic* on $\mathbb{R}$ if for any $x \in \mathbb{R}$ there exists $R_x > 0$ and a sequence of coefficients $(a_n^x)_{n=0}^{\infty}$ such that

$$f(y) = \sum_{n=0}^{\infty} a_n^x (y - x)^n, \text{ if } |y - x| < R_x.$$

Let f be a real analytic function.

(1) Show that there exists a region $\Omega \supset \mathbb{R}$ and a function F, analytic in Ω, such that for any $x \in \mathbb{R}$, $f(x) = F(x)$.

(2) Show that f is infinitely differentiable for any $x \in \mathbb{R}$, and $a_n^x = \frac{f^n(x)}{n!}$.

(3) Show that if (x_n) is a bounded real sequence, and for any n, $f(x_n) = 0$, then $f \equiv 0$.

Problem 4 of 5. Problem 5, page 130 of *Ahlfors.*

Problem 5 of 5. Problem 6, page 130 of *Ahlfors.*