Analytic functions; Cauchy-Riemann equations

Monday, December 7, 2020 9:03 AM

## Limits and continuity: a short review.

Same as for $\mathbb{R}^2$!

Def. Let $f$ be a function defined on a set $K \subseteq \mathbb{C}$.

$f$ has a limit $A$ as $z \to z_0$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 : 0 < |z - z_0| < \delta, \ z \in K \Rightarrow |f(z) - f(z_0)| < \varepsilon.$$

Properties. 1) If the limit exists it is unique provided $z_0$ is a limit point of $K$

$$(\forall \delta > 0 : \ B(z_0, \delta) \cap (K \setminus \{z_0\}) \neq \emptyset).$$

2) $\lim\limits_{z \to z_0} (f(z) + g(z)) = \lim\limits_{z \to z_0} f(z) + \lim\limits_{z \to z_0} g(z)$ (if both exist)

3) $\lim\limits_{z \to z_0} (f(z) \times g(z)) = \lim\limits_{z \to z_0} f(z) \times \lim\limits_{z \to z_0} g(z)$ (if both exist)

4) $\lim\limits_{z \to z_0} f(z) = A \Leftrightarrow \lim\limits_{z \to z_0} \operatorname{Re} f(z) = \operatorname{Re} A$, $\lim\limits_{z \to z_0} \operatorname{Im} f(z) = \operatorname{Im} A$.

5) $\lim\limits_{z \to z_0} f(z) = A \Rightarrow \lim\limits_{z \to z_0} \overline{f(z)} = \overline{A}$.

Proof. 1, 2, 3 - as in real case.

5. $\Leftarrow$ $|f(z) - A| = |\overline{f(z)} - \overline{A}|$.

4 $\Leftarrow$ 5+2, since $\operatorname{Re} f(z) = \dfrac{f(z) + \overline{f(z)}}{2}$

$\operatorname{Im} f(z) = -\frac{i}{2} (f(z) - \overline{f}(z))$.

Important property: $K_1, K_2 \subset K$. Let $\lim\limits_{\substack{z \to z_0 \\ z \in K}} f(z) = A$.

Then $\lim\limits_{\substack{z \to z_0 \\ z \in K_1}} f(z) = \lim\limits_{\substack{z \to z_0 \\ z \in K_2}} f(z) = A$.

Pr. Fix $\varepsilon > 0$, $\exists \delta > 0$...

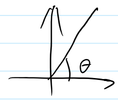
Proof. $z \in K_1$, $|z - z_0| < \delta \Rightarrow z \in K$, $|z - z_0| < \delta \Rightarrow |f(z) - A| < \varepsilon$ ■

Corollary. $K_1, K_2 \subset K$, $\lim\limits_{\substack{z \to z_0 \\ z \in K_1}} f(z) \neq \lim\limits_{\substack{z \to z_0 \\ z \in K_2}} f(z) \Rightarrow$

$\lim\limits_{z \to z_0} f(z)$ does not exist.

## Easy and important example:

$\lim\limits_{h \to 0} \dfrac{\bar{h}}{h}$ does not exist!



On ray $L_\theta$: $h = |h| \operatorname{cis} \theta$

On $L_\theta$: $\dfrac{\bar{h}}{h} = \dfrac{|h| \operatorname{cis}(-\theta)}{|h| \operatorname{cis}(\theta)} = \operatorname{cis}(-2\theta)$ – different on different rays!

## Continuous functions:

As usual: $f$ is continuous at $z_0$ if $\lim\limits_{\substack{z \to z_0 \\ z \in K}} f(z) = f(z_0)$.

Remark All of this can be done at $\infty$, but we need to use spherical metric:

$$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} d(f(z), \infty) = 0 \iff \lim_{z \to 0} \frac{1}{f(z)} = 0$$

$$\lim_{z \to \infty} f(z) = A \iff \forall \varepsilon > 0\ \exists \delta > 0 : d(z, \infty) < \delta \Rightarrow |f(z) - A| < \varepsilon$$

$$\iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = A$$

Important (and easy) observation: if $z_0 \neq \infty$ then

$$\lim_{z \to z_0} |z - z_0| = 0 \iff \lim_{z \to z_0} d(z, z_0) = \lim_{z \to z_0} \frac{|z - z_0|}{\sqrt{1+|z|^2}\sqrt{1+|z_0|^2}} = 0.$$

## Analytic functions

Def. $f$ is (complex) differentiable at $z_0$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \text{ exists.}$$

Equivalent definition: $f(z) = f(z_0) + (z - z_0)\varphi(z)$, where $\varphi(z)$ continuous at $z_0$, $\varphi(z_0) = f'(z_0)$.

Proof (of equivalency) ($\Uparrow$) $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} = \lim\limits_{z \to z_0} \varphi(z) = f'(z)$

($\Downarrow$) Take $\varphi(z) := \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0}, & z \neq z_0 \\ f'(z_0), & z = z_0 \end{cases}$.

---

Remark. Differentiability at one point is not interesting.
Interesting: differentiability at every point of some $B(z_0, \delta)$ - some neighborhood of $z_0$.

Thm. (the same as in Calculus).

1) If $f'(z)$, $g'(z)$ exist, then

$(f \pm g)'(z) = f'(z) \pm g'(z)$ - exist.

$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$ - exist

$\left(\frac{f}{g}\right)'(z) = \dfrac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}$ if $g(z) \neq 0$

2) If $f'(z)$ exist, and $g'$ exist at $f(z)$, then

$(g(f(z)))' = g'(f(z)) \cdot f'(z)$ - exist (Chain Rule).

Proof The same as in Calculus! ∎

Example 0. $f(z) \equiv c$. $f'(z) \equiv 0$.

Example 1 $f(z) = z$, $f'(z_0) = \lim\limits_{z \to z_0} \dfrac{z - z_0}{z - z_0} = 1$.

Example 5. Non-differentiable: $f(z) = \bar{z}$.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0} = \lim_{h \to 0} \frac{\bar{h}}{h}$$ - does not exist!

---

Real and Complex differentiability.

<u>Complex</u>: $\lim\limits_{h\to 0} \dfrac{|f(z+h)-f(z)-f'(z)h|}{|h|} = 0$

<u>Real</u>: $\lim\limits_{h\to 0} \dfrac{|f(z+h)-f(z)-T(h)|}{h} = 0$  $f = u+iv$

$T(h): \mathbb{R}^2 \to \mathbb{R}^2$ - linear map. $h = x+iy = \begin{pmatrix} x \\ y \end{pmatrix}$

From calculus, $T(h) = \dfrac{\partial f}{\partial x}x + \dfrac{\partial f}{\partial y}y = \left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right)x + \left(\dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}\right)y$ - in complex form

$T(h) = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ - in real form.

$x = \dfrac{h+\bar h}{2}$, $y = \dfrac{h-\bar h}{2i}$.

$T(h) = \dfrac{\partial f}{\partial x}\dfrac{h+\bar h}{2} - \dfrac{i}{2}\dfrac{\partial f}{\partial y}(h-\bar h) = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y}\right)h + \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right)\bar h$.

<u>Notation</u>: $\dfrac{\partial f}{\partial z} := \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y}\right)$  $\dfrac{\partial f}{\partial \bar z} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right)$

When is real differentiable function complex differentiable?

$\lim\limits_{h\to 0}\dfrac{f(z+h)-f(z)}{h} = \lim\limits_{h\to 0}\dfrac{T(h)}{h} = \lim\limits_{h\to 0}\dfrac{\partial f}{\partial z}\cdot\dfrac{h}{h} + \lim\limits_{h\to 0}\dfrac{\partial f}{\partial \bar z}\dfrac{\bar h}{h}$

So, since $\lim\limits_{h\to 0}\dfrac{\bar h}{h}$ - does not exist, we get.

<u>Theorem</u> (Cauchy-Riemann) Let $f$ be a real-differentiable function at $z_0$. It is complex differentiable if and only if $\dfrac{\partial f}{\partial \bar z}(z_0) = 0$.

Augustin-Louis Cauchy

Bernhard Riemann

<u>Remark</u>: $f'(z) = \dfrac{\partial f}{\partial z}$ in this case.

Other form: $\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} + i\left(\dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}\right) = 0$. <u>Or</u>

$\Big(\partial u - \partial v\Big)$

Other form: $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = 0$. Or

$$\begin{pmatrix} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{pmatrix} \text{ — Cauchy–Riemann equations.}$$

Matrix of $T(h)$: $\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} = M_{f'(z)}$,

$$\boxed{f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}}$$

**Remark** Need to assume real-differentiability apriori.

$\exists f$: $\frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar z}$ – exist everywhere, $\frac{\partial f}{\partial \bar z} \equiv 0$, yet $f$ is not everywhere analytic! $\left( f(z) = \begin{cases} e^{-\frac{1}{z^4}}, & z \neq 0 \\ 0, & z = 0 \end{cases} \right)$

**Thm (Looman–Menchoff)** If $f = u + iv$ is continuous $\forall z \in B(z_0, r)$, all the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist $\forall z \in B(z_0, r)$, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Then $f$ is analytic in $B(z_0, r)$.

**Theorem** Let $f$ be analytic in $B(z_0, \delta)$, $f'(z) \equiv 0$ $\forall z \in B(z_0, \delta)$. Then $f(z) \equiv$ const.

**Proof.** $f'(z) = 0 \Rightarrow \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \equiv 0 \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \equiv 0 \overset{\text{Calculus.}}{\Longrightarrow}$

$u \equiv$ const, $v \equiv$ const ∎

**Remark** Can assume less: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \equiv 0$ (without assuming differentiability apriori). Differentiability follows from continuity.

**Proof.** $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ – continuous $\overset{\text{Calculus}}{\Longrightarrow}$ $f$ is real-differentiable

$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0 \Rightarrow f$ is analytic ∎

---

We will prove later: every analytic function in $B(z_0, r)$ is infinitely differentiable.

We will prove later: every analytic function in $B(z_0, r)$ is infinitely differentiable.

A corollary of this:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

By Cauchy-Riemann

$\Delta u := \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ – Laplace operator.

Same way:

$\Delta v \equiv 0$

$\Delta f = \Delta u + i\Delta v = 0.$

Examples.

$u = x^2 - y^2$

$u = xy$

$u = e^x \cos y$

Def $u \in C^2$ is called harmonic on a set $k$ if $\Delta u \equiv 0$.

Theorem. Let $u$ be real and harmonic in some $B(z_0, r)$. Then $\exists f$ – analytic in $B(z_0, r)$, $u = \operatorname{Re} f$.

We will prove a more general version later.