Let \((a_n)\) be a bounded sequence of real numbers. Let us remind the notations

\[ R_n := \{a_k : k \geq n\}, \quad s_n := \sup R_n, \quad m_n := \inf R_n. \]

Let us also recall the following definition:

**Definition.** Limit superior of \((a_n)\) is defined as

\[ \limsup_{n \to \infty} a_n := \lim_{n \to \infty} s_n = \inf s_n. \]

Limit inferior of \((a_n)\) is defined as

\[ \liminf_{n \to \infty} a_n := \lim_{n \to \infty} m_n = \sup m_n. \]

**Theorem.** Let \((a_n)\) be a bounded sequence of real numbers. Then

1. For some subsequence \((a_{n_k})\),

\[ \lim_{k \to \infty} a_{n_k} = \limsup_{n \to \infty} a_n. \]

2. For some subsequence \((a_{n_k})\),

\[ \lim_{k \to \infty} a_{n_k} = \liminf_{n \to \infty} a_n. \]

3. If for some subsequence \((a_{n_k})\),

\[ a = \lim_{k \to \infty} a_{n_k} \]

then

\[ \liminf_{n \to \infty} a_n \leq a \leq \limsup_{n \to \infty} a_n. \]

**Proof.** To prove 1, let us construct recursively a sequence \(n_1 < n_2 < \ldots \) such that

\[ a_{n_k} > s_{n_k-1+1} - 2^{-k}. \]

Then, since \(a_{n_k} \in R_{n_k-1+1}\), we have

\[ a_{n_k} \leq s_{n_k-1+1}. \]

Observe that

\[ \lim_{k \to \infty} s_{n_k-1+1} = \lim_{k \to \infty} s_{n_k-1+1} - 2^{-k} = \lim_{n \to \infty} s_n = \limsup_{n \to \infty} a_n. \]

Thus, by the Squeezed sequence lemma (applied to \(s_{n_k-1+1}\) and \(s_{n_k-1+1} - 2^{-k}\))

To construct a subsequence with the property (*), let \(n_1\) be an index such that

\[ a_{n_1} > s_1 - 2^{-1}. \]

Such an index exists since \(s_1 = \sup R_1\).

If \(n_{k-1}\) is already constructed, let \(a_{n_k}\) be an element of \(R_{n_{k-1}+1}\) with \(a_{n_k} > s_{n_{k-1}+1} - 2^{-k}\).

Again, it exists since \(s_{n_{k-1}+1} = \sup R_{n_{k-1}+1}\). Also, \(n_k \geq n_{k-1} + 1 > n_{k-1}\).

The proof of 2 is the same as the proof of 1.

To prove 3, let us observe that \(m_{n_k} \leq a_{n_k} \leq s_{n_k}\), and both sequences \((m_{n_k})\) and \((s_{n_k})\) converge to \(\liminf_{n \to \infty} a_n\) and \(\limsup_{n \to \infty} a_n\) respectively, as subsequences of convergent sequences \((m_n)\) and \((s_n)\). Thus

\[ \liminf_{n \to \infty} a_n \leq \lim_{k \to \infty} a_{n_k} \leq \limsup_{n \to \infty} a_n. \]

\[ \square \]
Corollary (Bolzano-Weierstrass Theorem). Let \((a_n)\) be a bounded sequence of real numbers. Then it has a convergent subsequence.

Proof. Take a subsequence converging to \(\limsup_{n\to\infty} a_n\). □

Corollary. Let \((a_n)\) be a bounded sequence of real numbers and \(a \in \mathbb{R}\). Assume that every convergent subsequence of \((a_n)\) converges to \(a\). Then \((a_n)\) itself converges to \(a\).

Proof. There is a subsequence of \((a_n)\) convergent to \(\limsup_{n\to\infty} a_n\). Thus \(\limsup_{n\to\infty} a_n = a\). By the same reasoning, \(\liminf_{n\to\infty} a_n = a\). Therefore,

\[
\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a,
\]

and thus

\[
\lim_{n\to\infty} a_n = a.
\]

□