Problem 1 of 5. Assume $f$ is continuous on $[-1, 1]$ and differentiable on $(-1, 0) \cup (0, 1)$. If $\lim_{x \to 0} f'(x) = L$, show that $f'(0)$ exists and equals $L$.

Problem 2 of 5.

1. Let $(f_n)$ be a sequence of continuous functions that converges uniformly to a function $f$ on a compact set $A$. If $f(x) \neq 0$ on $A$, show that $(1/f_n)$ converges uniformly on $A$ to $1/f$.

2. Give an example of a sequence $(f_n)$ of continuous functions that converges uniformly to a function $f$ on a $(0, 1]$, $f(x) \neq 0$ on $(0, 1]$, but $(1/f_n)$ does not converge uniformly on $(0, 1]$ to $1/f$.

Problem 3 of 5. Assume the sequence of functions $f_n(x)$ converges to a function $f(x)$ pointwise on a compact set $A$ and assume that for each $x \in A$ the sequence $f_n(x)$ is increasing. Assume that $f_n$ and $f$ are continuous on $A$.

1. Set $g_n := f - f_n$ and translate the preceding hypothesis into statements about the sequence $(g_n)$.

2. Let $\varepsilon > 0$ be arbitrary, and define

$$K_n := \{x \in A : g_n(x) \geq \varepsilon\}.$$ 

show that $K_{n+1} \subset K_n$.

3. Show that each $K_n$ is compact.

4. Use the pointwise convergence of the sequence $(g_n)$ to show that $\bigcap_{n=1}^{\infty} K_n = \emptyset$.

5. Conclude that for some $N$, $K_N = \emptyset$.

6. Derive that for $n \geq N$ and for all $x \in A$

$$|f_n(x) - f(x)| < \varepsilon.$$ 

This proves that $(f_n)$ converges to $f$ uniformly.

Problem 4 of 5. Let $f$ be a continuous function on $\mathbb{R}$ and let $(a_n)$ be a real sequence converging to zero. Let the sequence of functions $(f_n(x))$ be defined by $f_n(x) := f(x + a_n)$.

1. Show that the sequence of functions $(f_n(x))$ converges to $f$ uniformly on every bounded $A \subset \mathbb{R}$.

2. Show that if the function $f$ is uniformly continuous on $\mathbb{R}$, then $(f_n(x))$ converges to $f$ uniformly on $\mathbb{R}$.

3. Show that a function $f$ is not uniformly continuous on $\mathbb{R}$ if and only if for some sequence $(a_n)$, $f_n(x)$ does not converge to $f(x)$ uniformly.

Hint: You can use Theorem 4.4.5.
Problem 5 of 5. Let \((f_n)\) be a sequence of bounded (not necessarily continuous functions) on \([0, 1]\).

(1) Construct such a sequence \((f_n)\) converging pointwise to an unbounded function \(f\) on \([0, 1]\).

(2) Assume that \((f_n)\) converges uniformly to a function \(f\). Show that \(f\) is also bounded.