Problem 1 of 5. Let \((a_n)\) be a sequence of positive numbers. Let \(b_n := \frac{a_n}{1 + a_n}\)

1. Prove that the sequence \(b_n\) is bounded.
2. Prove that if the sequence \(a_n\) is bounded then \(\lim \sup b_n < 1\).
3. Prove that if the sequence \(a_n\) is not bounded then \(\lim \sup b_n = 1\).
4. Prove that \(\lim \inf b_n = 0\) if and only if \(\lim \inf a_n = 0\).

Problem 2 of 5. Let \((a_n)\) be a bounded sequence. Define

\[ S := \{ x : x < a_n \text{ for infinitely many } n \} \]

Prove that \(S\) is bounded above and \(\sup S = \lim \sup a_n\).

Problem 3 of 5. Let \((a_n)\) be a real sequence and \(a \in \mathbb{R}\). Assume that every subsequence \((a_{n_k})\) of \((a_n)\) contains a (sub-)subsequence \((a_{n_{k_l}})\) converging to \(a\).

1. Prove that the sequence \((a_n)\) is bounded.
2. Prove that \((a_n)\) converges to \(a\).

Problem 4 of 5.

1. Let \((a_n)\) be a sequence. Assume that the series \(\sum_{n=1}^\infty |a_{n+1} - a_n|\) converges. Prove that the sequence \((a_n)\) converges.
2. The map \(f : \mathbb{R} \mapsto \mathbb{R}\) is called a contraction if for some \(q < 1\), and for all \(x, y \in \mathbb{R}\),

\[ |f(x) - f(y)| \leq q|x - y|. \]

Let the sequence \((a_n)\) be defined recursively: \(a_1 \in \mathbb{R}, a_{n+1} = f(a_n)\). Use the first part to show that the sequence \((a_n)\) converges.
3. Prove that \(a := \lim_{n \to \infty} a_n\) is the unique fixed point of \(f\), i.e. the unique \(a \in \mathbb{R}\) with \(f(a) = a\).

Problem 5 of 5 (Raabe’s Summability Test). Assume that \((a_n)\) is a positive sequence and

\[ \lim \inf n \left( \frac{a_n}{a_{n+1}} - 1 \right) > 1. \]

1. Show that for some \(\varepsilon > 0\) one can find \(N\) so that for \(n \geq N\),

\[ na_n - (n + 1)a_{n+1} > \varepsilon a_{n+1}. \]
2. Let \(s_n = \sum_{k=1}^n a_k\) be the sequence of partial sums. Sum up the inequalities in the previous part for \(n = N + 1, N + 2, \ldots, N + p - 1\) to obtain that for any \(p \in \mathbb{N}\)

\[ \varepsilon (s_{N+p} - s_N) < Na_N - (N + p)a_{N+p} < Na_N. \]
(3) Prove that for any \( p \in \mathbb{N} \):
\[
s_{N+p} < s_N + \frac{N a_N}{\varepsilon}.
\]
Conclude that the sequence \((s_n)\) is bounded, and thus the series \(\sum_{n=1}^{\infty} a_n\) converges.

(4) Let now
\[
a_n := \left[ \frac{1 \cdot 4 \cdot 7 \cdots (3n - 2)}{3 \cdot 6 \cdot 9 \cdots 3n} \right]^2.
\]
Use the previous part to show that the series \(\sum_{n=1}^{\infty} a_n\) converges.