Problem 1 of 10. Let $A$ be a nonempty bounded set. Let
\[ B := \{ x \in \mathbb{R} : x \text{ is a lower bound for } A \}; \quad C := \{ x \in \mathbb{R} : x \text{ is an upper bound for } A \}. \]
Prove that $B$ is bounded above, $C$ is bounded below, and
\[ \inf A = \sup B; \quad \sup A = \inf C. \]
Can either $B$ or $C$ be bounded?

Problem 2 of 10. Let $A_1, A_2, A_3, \ldots$ be a collection of bounded nonempty sets.
1. Show that $\bigcup_{k=1}^n A_k$ is also bounded. Compute its supremum and infimum in terms of suprema and infima of $\{A_k\}_{k=1}^n$.
2. Give an example of an infinite collection of bounded nonempty sets $\{A_k\}_{k=1}^\infty$ such that $\bigcup_{k=1}^\infty A_k$ is not bounded above or below.
3. Assuming now that $\bigcup_{k=1}^\infty A_k$ is also bounded, compute its supremum and infimum in terms of suprema and infima of $\{A_k\}_{k=1}^\infty$.

Problem 3 of 10. Let $A$ be a nonempty bounded set, $c \in \mathbb{R}$. Let $cA := \{ cx : x \in A \}$.
Prove that $cA$ is also bounded and compute its supremum and infimum.

Problem 4 of 10. Assume that $\inf A > \inf B$. Show that there is $\varepsilon > 0$ and $b \in B$, such that $b + \varepsilon$ is a lower bound for $A$.

Problem 5 of 10. Let $\sup A < \inf B$. Show that there is $\varepsilon > 0$ and $c \in \mathbb{R}$, such that $c + \varepsilon$ is a lower bound for $B$ and $c - \varepsilon$ is an upper bound for $A$.

Problem 6 of 10. Give an example of a sequence of nested open intervals $((a_n, b_n))_{n=1}^\infty$, $(a_{n+1}, b_{n+1}) \subset (a_n, b_n)$, such that $\bigcap_{n=1}^\infty (a_n, b_n) = \emptyset$.

Problem 7 of 10. Assume that the sequence $a_n$ is strictly increasing, i.e. $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. Assume also that the sequence $b_n$ is strictly decreasing, i.e. $b_{n+1} < b_n$ and $a_n < b_n$ for all $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^\infty (a_n, b_n) \neq \emptyset$.

Hint: Let $I_n = [a_n, b_n]$ be a closed interval. Observe that $I_{n+1} \subset (a_n, b_n)$ and thus $\bigcap_{n=1}^\infty I_{n+1} \subset \bigcap_{n=1}^\infty (a_n, b_n)$.

Problem 8 of 10. A set $A \subset \mathbb{R}$ is called dense in $\mathbb{R}$ if for every real $x < y \in \mathbb{R}$ one can find $a \in A$ with $x < a < y$.

1. Let $B$ be an infinite subset of $\mathbb{N}$. Prove that the set of all rational numbers of the form $\frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in B$, is dense in $\mathbb{R}$.

2. Prove that the set of all rational numbers of the form $\frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and $239|p| > q$ is not dense in $\mathbb{R}$.
Problem 9 of 10. A sequence \((a_n)\) is called wrongverging to \(a\) if
\[
\forall \varepsilon \in \mathbb{R} \exists N : n > N \implies a - a_n < \varepsilon.
\]

(1) Give an example of a wrongverging sequence.
(2) Prove that if a sequence wrongverges to some \(a \in \mathbb{R}\) then it also wrongverges to any \(x \in \mathbb{R}\).

Problem 10 of 10. Let \((a_n)\) be a sequence of strictly positive numbers \(a_n > 0\) converging to 0. Let \((b_n)\) be a sequence of real numbers and \(b \in \mathbb{R}\).

(1) Assume that \(\lim_{n \to \infty} b_n = b\). Prove that
\[
\forall k \exists N : n > N \implies |b - b_n| < a_k.
\]
**Hint:** For a fixed \(k\), \(a_k\) is just a positive number.

(2) Assume now that
\[
\forall k \exists N : n > N \implies |b - b_n| < a_k.
\]
Prove that \(\lim_{n \to \infty} b_n = b\).
**Hint:** Fix \(\varepsilon > 0\). Then you can always find \(a_k < \varepsilon\). (Why?)

You just established that \(\lim_{n \to \infty} b_n = b\) if and only if
\[
\forall k \exists N : n > N \implies |b - b_n| < a_k.
\]