Syllabus

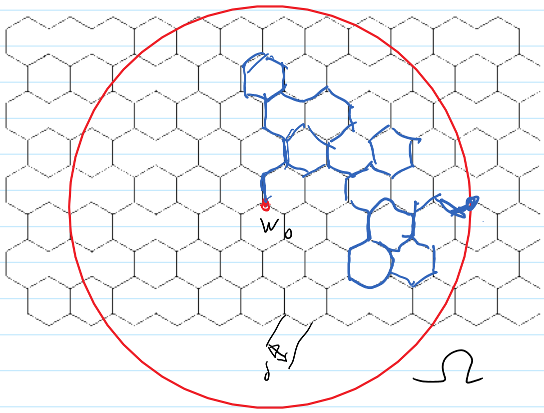
Schramm Loewner Evolution and Lattice Models

In recent years, significant progress has been obtained in the rigorous understanding of the scaling limits of the various lattice models of statistical physics. One of the instrumental tools in this development is the Schramm Loewner Evolution (SLE), invented by Oded Schramm in 1998.
The course will introduce the students to these developments. The topics will include the definition and geometric properties of SLE, including the necessary background in Geometric Function Theory; basic properties of the lattice models, such as Percolation, Ising, Potts, and Self Avoiding Random Walk; proofs of the existence of scaling limits and their relations to Schramm Loewner Evolution; the rate of convergence of critical interfaces to SLE curves and obtaining Schramm Loewner Evolution by welding.

References:

1. "Conformal Maps and Geometry", by D. Beliaev
2. "Conformally Invariant Processes in the Plane", by Gregory F. Lawler
3. "Schramm-Loewner Evolution," by Antti Kemppainen

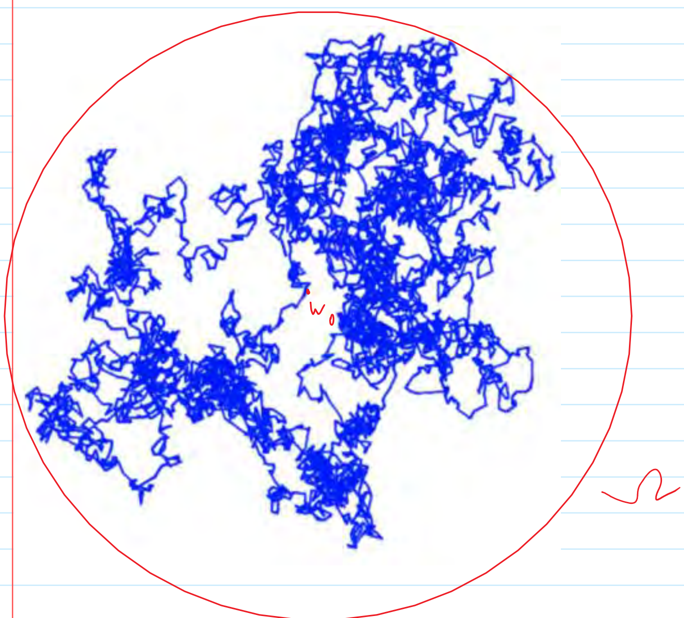
Example 0: Random Walk and Brownian Motion



Creates a random curve in (Ω, w_0)

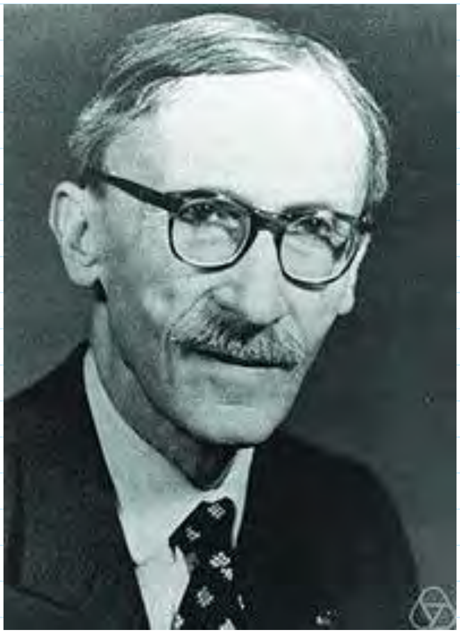
A probability (≥ 0, total mass $=1$) measure on the space of curves from w_0 to $\partial\Omega$.

Not rotationally invariant.



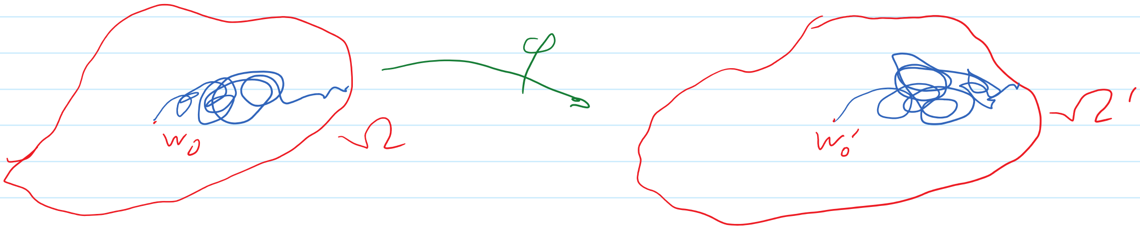
As $\delta \to 0$, converges (weekly) to the law of trajectory of 2D Brownian motion, $\mu_{(\Omega, w_0)}$

It is rotationally invariant



Paul Pierre Lévy (1886-1971)

Theorem (Levy). 2 D Brownian motion is conformally invariant.



$\varphi : (\Omega, w_0) \to (\Omega', w_0')$ - conformal

$\varphi_* \mu_{(\Omega, w_0)} = \mu_{(\Omega', w_0')}$

Shizuo Kakutani (1911-2004)

One of the proofs is based on the Kakutani's observation:

Theorem (Kakutani). If $B_t^{w_0}$ is 2D Brownian motion started at w_0, $T = \inf\{t : B_t^{w_0} \notin \Omega\}$, $f \in C(\partial\Omega)$, u - solution of corresponding Dirichlet problem, then $u(w_0) = E(f(B_T^{w_0}))$

Restatement. $B_T^{w_0}$ is distributed according to harmonic measure

It is conformally invariant.

Recurring theme: the conformal invariance of one observable leads to conformal invariance of random curve!

Further examples: Loop Erased Random Walk and Percolation.

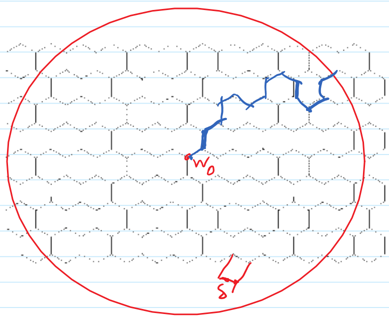
1. Loop Erased Random Walk.


Gregory Lawler


Oded Schramm (1961-2008)


Wendelin Werner

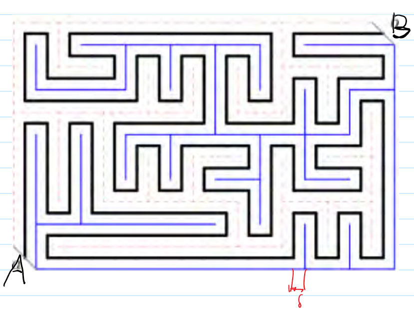
Model proposed by Greg Lawler.



$LERW_\delta$.

As $\delta \to 0$, $LERW_\delta$ converges weakly to a conformally invariant law on simple curves from w_0 to $\partial\Omega$. (Lawler-Schramm-Werner).

2. Uniform Spanning Tree.

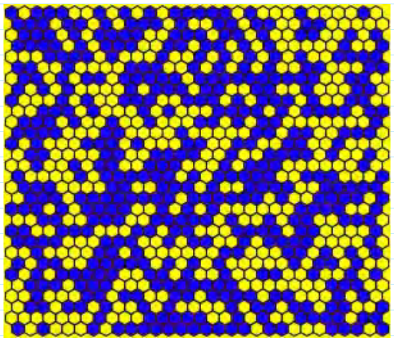


UST_δ

As $\delta \to 0$, UST_δ converges to a conformally

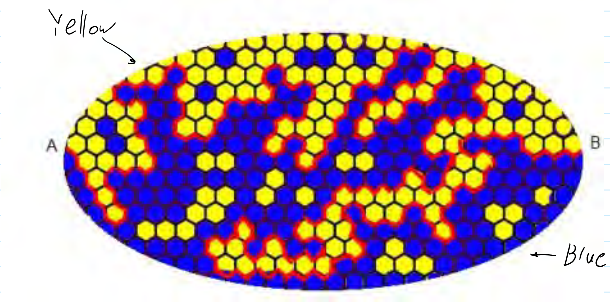
invariant law on space-filling curves in Ω from A to B.
(Lawler- Schramm- Werner)

3. Critical Percolation on Hexagonal Lattice.



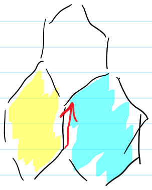
Color each hexagonal face blue or yellow independently with probability $\frac{1}{2}$.

Now, let us do it in a domain Ω:



Perc_δ

Red path: Exploration Process



Turn left on Blue, right on Yellow; toss coin on new hexagon

Stanislav Smirnov

Theorem (Smirnov)

As $\delta \to 0$, Perc_δ converges to a conformally invariant law on self-touching curves from A to B.

What is this limiting process?
How to describe it?

It is supposed to be conformally invariant, so enough to describe it in some canonical domain

For the case like Example 1 (curve from interior point to $\partial\Omega$), the natural domain is $(\mathbb{D}, 0)$.

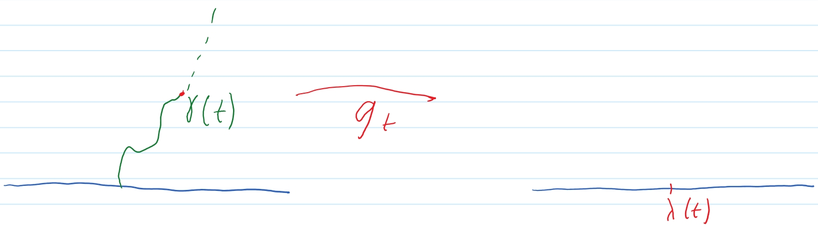
$(\mathbb{D} = \{z : |z|<1\})$.

For the case like Examples 2 and 3 (curve between two boundary points), the natural domain is $(\mathbb{H}, 0, \infty)$

$(\mathbb{H} = \{z : \operatorname{Im} z > 0\})$.

Another important feature, parameterization does not matter!

Use conformal maps to describe the curve!



g_t: conformal map from $\mathbb{H} \setminus \gamma([0,t]) \to \mathbb{H}$, which has hydrodynamic normalization at ∞:

$g_t(z) = z + \frac{2a(t)}{z} + \dots$ Re-parametrize to make $a(t)=t$

$\lambda(t) := g_t(\gamma(t)) \in \mathbb{R}$.

Then $$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \lambda(t)}$$ (Löwner Equation, Chordal form).

Charles Löwner(1893-1968)

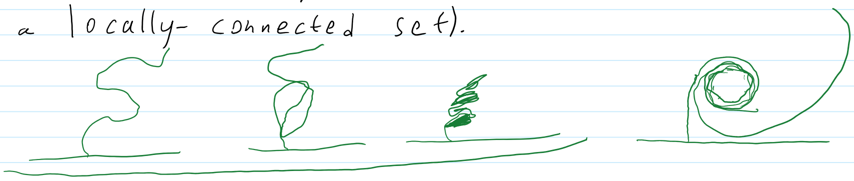
Invented in 1923 to solve Bieberbach Conjecture:

If $f(z) = z + a_2 z^2 + \dots + a_n z^n + \dots$ - univalent in $\mathbb{D}$, then $|a_n| \le n$

Was the main tool in 1984 solution of this conjecture by de Branges!

One can restore $\gamma(t)$ from $\lambda(t)$! But sometimes γ is not quite a curve (doesn't have to be a locally-connected set).

U is not quite a curve (doesn't have to be a locally-connected set).

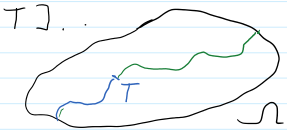


Oded Schramm observation.

Assume that γ is a random curve which satisfies

1) Conformal invariance.

2) Domain Markov Property: the law of $\gamma(t+T)_{t\geq 0}$ in Ω is the same as the law of $\gamma(t)$ in $\Omega \setminus \gamma[0,T]$..



Then γ is generated by

$\lambda(t) = B(\varkappa t)$, where $B(t)$ - standard 1D Brownian Motion,

$\varkappa > 0$.

$SLE_\varkappa$ - Schramm-Löwner Evolution.

$LERW_\delta \to SLE_2$

$Perc_\delta \to SLE_6$

$UST_\delta \to SLE_8$

Steffen Rohde

Theorem (Rohde-Schramm).

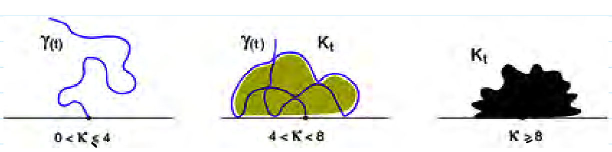
1) $SLE_\varkappa$ is a.s. a curve.

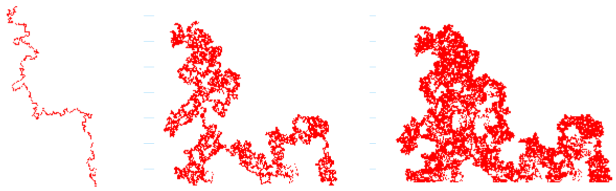
2) $0 < \varkappa \leq 4$ - simple curve

$4 < \varkappa < 8$ - self-touching curve

$\varkappa \geq 8$ - space-filling curve.

($\varkappa = 8$ - special! It is known to be a curve only because $UST_\delta \to SLE_8$).

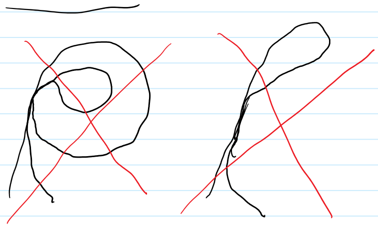




How to prove convergence?

Step 1. Find an observable which converges to a conformally invariant limit

Step 2. Establish pre-compactness in your family of curves



(There is an axiomatic theory for this now!)

Step 3. Use observable to prove that the driving function is $B(\kappa t)$ for some κ.
(there is axiomatic theory for this also).

Course plan:

1) Self-Avoiding Walk (SAW).
We know the observable, but convergence is unknown.
But it helps establish a new, purely combinatorial, property.

2) Other models and their observables.

3) Background in conformal maps and Löwner evolution

4) Background in Itô calculus and SLE. Some

geometric properties of SLE.

5) Convergence of critical interfaces: the framework

6) Gaussian Free Field and its relation to SLE.