# Projections

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Let $K \subset \mathbb{R}^2$ - compact, $\Pi_\theta$ - orthogonal projection to $L_\theta = (\cos\theta, \sin\theta)t$.

Since $\Pi_\theta$ - Lipshitz, $\mathrm{Hdim}\, \Pi_\theta(K) \le \min(\mathrm{Hdim}\, K, 1)$.

**Thm.** (Marstrand Projection Thm) A.e. $\theta$,
$\mathrm{Hdim}\, \Pi_\theta K = \min(\mathrm{Hdim}\, K, 1)$.
Enough to prove for $\mathrm{Hdim}\, K \le 1$, since every $K$ with $\mathrm{Hdim}\, K > 1$ contain $K'$ with $\mathrm{Hdim}\, K = 1$.
Will follow from

**Thm** (Kaufman). If $\mathrm{Cap}_\alpha K > 0$ for some $0 < \alpha < 1$, then for a.e. $\theta$, $\mathrm{Cap}_\alpha(\Pi_\theta K) > 0$.

**Pf** (Kaufman $\Rightarrow$ Marstrand).
Take $\alpha < \mathrm{Hdim}\, K$. Then $\mathrm{Cap}_\alpha(K) > 0$ so $\mathrm{Cap}_\alpha(\Pi_\theta K) > 0$ a.e. $\theta$ (Kaufman!), so $\mathrm{Hdim}(\Pi_\theta K) \ge \alpha$.
Now take $\alpha_n \uparrow \mathrm{Hdim}\, K$ then
$\{\theta : \forall \alpha_n\, \mathrm{Hdim}(\Pi_\theta K) \ge \alpha_n\}$ has full measure, as countable intersections of the sets of full measure.

**Pf.** (Kaufman).
Let $\mu$ be a measure, $I_\alpha(\mu) < \infty$, $\mu_\theta := \Pi_\theta \mu$
$\left(\mu_\theta(A) := \mu(\Pi_\theta^{-1}(A)), \text{ or } \int_{\mathbb{R}} f(x)\, d\mu_\theta(x) = \int_{\mathbb{R}^2} f(\Pi_\theta(y))\, d\mu(y)\right)$.

$$\int_0^\pi I_\alpha(\mu_\theta)\, d\theta = \int_0^\pi \iint \frac{d\mu_\theta(t)\, d\mu_\theta(s)}{|t-s|^\alpha}\, d\theta =$$

$$\int_0^\pi \int_K \int_K \frac{d\mu(x)\, d\mu(y)\, d\theta}{|\Pi_\theta(x-y)|^\alpha} \underset{\text{Tonelli; } K\, K\, 0}{=} \iint \int_0^\pi \frac{d\theta}{|\Pi_\theta(x-y)|^\alpha}\, d\mu(x)\, d\mu(y) =$$

$$\underbrace{\int_K \int_K \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha}}_{I_\alpha(\mu)} \cdot \underbrace{\int_0^\pi \frac{d\theta}{|\Pi_\theta(u)|^\alpha}}_{\text{does not depend on } x, y, \text{ as long as } x \ne y} \quad \text{where } u = \frac{(x-y)}{|x-y|}$$

$< \infty$ since $\alpha < 1$! $\quad \int^\pi \frac{d\theta}{\sin\theta^\alpha}$

Thus, $I_\alpha(\mu_\theta) < \infty$ a.s., $\mathrm{supp}\, \mu_\theta \subset \Pi_\theta K \Rightarrow$ A.s. $\mathrm{Cap}_\alpha \Pi_\theta K < \infty$