

PHASE TRANSITION FOR THE UNIVERSAL BOUNDS ON THE INTEGRAL MEANS SPECTRUM

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ABSTRACT. In this paper we generalize Beurling's estimate on the rate of decay of harmonic measure near a boundary point with given rotation. The generalization allows us to establish the existence of phase transition for the universal bounds on the integral means mixed spectrum of bounded simply connected domains.

1. INTRODUCTION

In what follows, $\Omega \subset \mathbb{C}$ is a simply connected domain, w_0 is an interior point of Ω , and $\phi : \mathbb{D} \rightarrow \Omega$ is the Riemann map with $\phi(0) = w_0$, $\phi'(0) > 0$. $\omega(E)$ will denote the harmonic measure of $E \subset \partial\Omega$ evaluated at w_0 .

It is known (see [GM08]), that for any $v \in \partial\Omega$, the harmonic measure of a small ball, $\omega(B(v, \delta))$ decays with the rate at least $\delta^{1/2}$ when $\delta \rightarrow 0$. It follows from Makarov's dimension theorem (see [Mak85]) that a.e. by harmonic measure, $\omega(B(v, \delta))$ decays like δ . More precisely, for ω - a.e. $v \in \partial\Omega$,

$$\lim_{\delta \rightarrow 0} \frac{\log \omega(B(v, \delta))}{\log \delta} = 1.$$

These results can be significantly refined using Multifractal Analysis (see, for example, [Fal90]). Roughly speaking, *dimension spectrum of ω* , $f(\alpha)$ is defined as the dimension of the set of points with the rate of decay (or *local dimension*) of harmonic measure equal to α . This "naive" version of the definition only makes sense for "nice" domains Ω , such as Carleson fractals and Julia sets (see [Mak98, Zin00]).

For general domains, there are a few ways to make the notion of the dimension spectrum useful. We will use the following version of the spectrum (see [Mak98]).

Definition 1. The *Minkowski dimension spectrum of a simply connected domain Ω* is defined as

$$f(\alpha) = f_\Omega(\alpha) = \lim_{\eta \rightarrow 0} \limsup_{\delta \rightarrow 0} \frac{N(\delta, \alpha, \eta)}{\log \frac{1}{\delta}},$$

where $N(\delta, \alpha, \eta)$ is a maximum number of disjoint balls B_n of radii δ with the centers $v_n \in \partial\Omega$ and $\delta^{\alpha+\eta} \leq \omega(B_n) \leq \delta^{\alpha-\eta}$.

Note that for each α either $f_\Omega(\alpha) \geq 0$ or $f_\Omega(\alpha) = -\infty$. The latter occurs if for all small δ and η there are no disks with the prescribed amount of harmonic measure.

The above-mentioned upper estimate of harmonic measure implies that $f(\alpha) = -\infty$ whenever $\alpha < 1/2$. Makarov's theorem implies that $f(1) = 1$ and $f(\alpha) < \alpha$ if $\alpha \neq 1$.

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Closely related to the dimension spectrum is one of the central objects of the Theory of Conformal maps, the *Integral Means Spectrum*, defined as

$$(1) \quad \beta_{\Omega}(t) = \beta_{\phi}(t) = \limsup_{r \rightarrow 1^-} \frac{\log \int_{r\mathbb{T}} |\phi^{it}(\zeta)| |d\zeta|}{\log \frac{1}{1-r}}.$$

The value of the spectrum is independent of the choice of the Riemann map $\phi : \mathbb{D} \rightarrow \Omega$. We refer to [Mak98] and [Pom92] for the discussion of the spectrum and its properties.

Integral means spectrum of a domain provide an upper bound on the dimension spectrum of harmonic measure in the form of the following Legendre-type transform (see [Mak98]):

$$(2) \quad f_{\Omega}(\alpha) \leq \inf_t (\alpha(\beta_{\Omega}(t) - t + 1) + t)$$

$$(3) \quad \beta_{\Omega}(t) \geq \sup_{\alpha} \left(\frac{f_{\Omega}(\alpha) - t}{\alpha} + t - 1 \right).$$

For Carleson fractals and Julia sets, these inequalities become equalities. Using this observation and Fractal Approximation, one can see that the questions about sharp upper bounds on the dimension spectrum and integral means spectrum can be studied simultaneously. More precisely, the *Universal Integral Means Spectrum*

$$B_b(t) = \sup_{\text{Bounded simply connected } \Omega} \beta_{\Omega}(t)$$

and the *Universal Dimension Spectrum* as

$$F(\alpha) = \sup_{\text{simply connected } \Omega} f_{\Omega}(\alpha)$$

are related by the Legendre-type transforms (2), (3) with the equalities instead of inequalities (see [Mak98]):

$$(4) \quad F(\alpha) = \inf_t (\alpha(B_b(t) - t + 1) + t)$$

$$(5) \quad B_b(t) = \sup_{\alpha} \left(\frac{F(\alpha) - t}{\alpha} + t - 1 \right).$$

A lot of classical questions about the boundary behavior of conformal maps and the fine properties of harmonic measure can be reformulated in terms of Universal Integral Means Spectrum or, equivalently, Universal Dimension Spectrum. For example, the celebrated Brennan conjecture can be restated as $B_b(-2) = 2$.

The strongest conjecture about the value of $B_b(t)$ was made by Ph. Kraetzer (see [Kra96]):

Conjecture 1.

$$B_b(t) = \begin{cases} \frac{t^2}{4}, & |t| \leq 2, \\ |t| - 1, & |t| > 2 \end{cases}.$$

Equivalently,

$$F(\alpha) = 2 - \frac{1}{\alpha}$$

whenever $\alpha \geq 1/2$

We refer to [BS05] and [HS08] for the survey of the recent progress related to Kraetzer conjecture.

One of the most interesting features of the conjectured behavior of $B_b(t)$ is the existence of the phase transition phenomenon: the function becomes linear when $|t| > 2$. Since $B_b(t) \geq |t| - 1$ and $B_b(t)$ is convex ([Pom92]), the existence of above-mentioned phase transition is equivalent to the equality $B_b(t) = t - 1$ for some $t < 0$ (the existence of the phase transition for $t > 0$ follows from the easy identity $B_b(2) = 1$). This property was established by Carleson and Makarov in [CM94]

using the estimates on the Universal Dimension Spectrum near $\alpha = 1/2$ and the Legendre-type relations (4), (5).

Various modern approaches to the verification of Conjecture 1 make use of a slight generalization of the integral means spectrum. Namely, one allows the exponent to be a complex number.

Definition 2. The *Integral Mixed Spectrum* of a simply connected domain Ω with the Riemann map $\phi : \mathbb{D} \rightarrow \Omega$ is

$$m_\Omega(z) = m_\phi(z) = \limsup_{r \rightarrow 1^-} \frac{\log \int_{r\mathbb{T}} |\phi'^z(\zeta)| d|\zeta|}{\log \frac{1}{1-r}}$$

where $z \in \mathbb{C}$.

Since $\phi'(\zeta) \neq 0$ in \mathbb{D} , the complex powers of $\phi'(\zeta)$ are well defined, and the rate of growth does not depend on the branch chosen. m_Ω depends not only on the boundary behavior of $|\phi'(\zeta)|$, but also on the growth of $\arg \phi'(\zeta)$, or the rotation of the image of the radius.

Universal Integral Mixed Spectrum is defined the same as the Integral Means counterpart:

$$M_b(z) = \sup_{\text{Bounded simply connected } \Omega} m_\Omega(z).$$

In [BP87], Becker and Pommerenke extended the Brennan conjecture to the complex case, asking whether $M_b(z) = 1$ whenever $|z| = 2$. As in the real-exponent case, since $M_b(z) \geq |z| - 1$ and $M_b(z)$ convex, this conjecture would imply that $M_b(z) = |z| - 1$ whenever $|z| \geq 2$. In this paper, we establish a weak form of the last conjecture.

Theorem 1. *There exists $T > 0$ such that $M_b(z) = |z| - 1$ for $|z| \geq T$.*

Roughly speaking, Theorem 1 states that for $|z| \geq T$, the extremal growth of the mean values of the complex powers of the derivatives occurs near a single spiral. Since it is known that $M_b(0) = 0 > 0 - 1$, this is not the case for small values of $|z|$. Furthermore, one can show that for $|z| < 2$ we have $M_b(z) > |z| - 1$. Consequently, we have a phase transition for the behavior of $M_b(z)$ along some curve surrounding zero. Becker-Pommerenke conjecture implies that this critical curve is the circle $|z| = 2$. We conjecture that the following generalization of Kraetzer conjecture actually holds:

Conjecture 2.

$$M_b(z) = \begin{cases} \frac{|z|^2}{4}, & |z| \leq 2, \\ |z| - 1, & |z| > 2 \end{cases}.$$

The smooth phase transition at $z = 2$, predicted by this conjecture, is shown to exist by Jones and Makarov in [JM95] for the real values of z , and by Baranov and Hedenmalm in [BH08] for the complex values of z .

The geometric counterpart of the Integral Mixed Spectrum is the Dimension Mixed Spectrum. To define it, we will need to introduce a notion of *rotation near a boundary point*, which essentially corresponds to the growth of $\arg \phi'$.

Definition 3. Let $v \in \partial\Omega$, $\delta > 0$, Ω_δ be the connected component of $\Omega \setminus B(v, 2\delta)$ containing w_0 . Define the *rotation of the domain Ω near v at the distance δ* as

$$\rho(v, \delta) = \exp\left(\inf_{w \in \partial\Omega_\delta \cap B(v, 2\delta)} \arg(w - v)\right),$$

where the branch of the function $g(w) = \arg(w - v)$ is selected so that $-\pi < \arg(w_0 - v) \leq \pi$.

Since Ω is simply-connected, the branch of $g(w)$ is well-defined in Ω . In other words, $\rho(v, \delta)$ measures how many times a curve should rotate around point v before it first gets 2δ -close to v within Ω . An easy topological observation ([Bin]) shows that

$$(6) \quad \text{For any two points } w_1, w_2 \in \partial\Omega_\delta \cap B(v, 2\delta), \text{ we have } |\arg(w_1 - v) - \arg(w_2 - v)| \leq 2\pi$$

Let us also note that we consider the exponent of the argument in Definition 3 to scale the rotation the same way as the harmonic measure of the corresponding disc.

At this point we can introduce a two parameter generalization of the dimension spectrum, the *Dimension Mixed Spectrum* (See [Bin] for further motivation and variants of the definition).

Definition 4. The *Dimension Mixed Spectrum* of a simply connected domain Ω is

$$f_{\Omega}(\alpha, \gamma) = \lim_{\eta \rightarrow 0} \limsup_{\delta \rightarrow 0} \frac{N(\delta, \alpha, \gamma, \eta)}{\log \frac{1}{\delta}},$$

where $N(\delta, \alpha, \gamma, \eta)$ is a maximum number of disjoint balls B_n of radii δ with the centers $v_n \in \partial\Omega$ and $\delta^{\alpha+\eta} \leq \omega(B_n) \leq \delta^{\alpha-\eta}$, $\delta^{\gamma+\eta} \leq \rho(v_n, \delta) \leq \delta^{\gamma-\eta}$.

Intuitively, $f(\alpha, \gamma)$ measures the dimension of the set of point with the local dimension of harmonic measure equal to α and the rate of boundary rotation is equal to γ . As in non-rotational case, $f_{\Omega}(\alpha, \gamma)$ is either non-negative or equals to $-\infty$.

The following geometric observation was essentially made by Beurling in [Beu89]. It generalizes the lower bound of $\delta^{1/2}$ on the rate of decay of harmonic measure.

Lemma 1. *Let $f_{\Omega}(\alpha, \gamma) \geq 0$ for some simply connected Ω . Then $\alpha \geq \frac{1}{2} + \frac{\gamma^2}{2}$. In other words, if $\alpha + \eta < \frac{1+(\gamma-\eta)^2}{2}$, then there is no $v \in \partial\Omega$ and $\delta > 0$ such that*

$$\omega(B(v, \delta)) \geq \delta^{\alpha+\eta} \text{ and } \delta^{\gamma+\eta} \leq \rho(v, \delta) \leq \delta^{\gamma-\eta}.$$

The characterization is sharp: for any $\alpha \geq \frac{1}{2} + \frac{\gamma^2}{2}$ there exists a bounded domain Ω with $f_{\Omega}(\alpha, \gamma) \geq 0$.

Remark. Lemma 1 is equivalent to the well-known Pommerenke-Ýoccoz-Levin inequality.

As shown in [Bin], the Legendre-type relations between the Integral Means Spectrum and Dimension Spectrum (2), (3) generalize to the rotational case. Namely, for any simply-connected domain Ω ,

$$(7) \quad f_{\Omega}(\alpha, \gamma) \leq \inf_z (\alpha m_{\Omega}(z) + (1 - \alpha) \operatorname{Re}(z) - \gamma \operatorname{Im}(z) + \alpha)$$

$$(8) \quad m_{\Omega}(z) \geq \sup_{\alpha, \gamma} \left(\frac{f_{\Omega}(\alpha, \gamma) - (1 - \alpha) \operatorname{Re}(z) + \gamma \operatorname{Im}(z) - \alpha}{\alpha} \right).$$

Using the Fractal Approximation, one can also generalize (4), (5) ([Bin]):

$$(9) \quad M_b(z) = \sup_{\alpha, \gamma} \left(\frac{F(\alpha, \gamma) - (1 - \alpha) \operatorname{Re}(z) + \gamma \operatorname{Im}(z) - \alpha}{\alpha} \right)$$

$$(10) \quad F(\alpha, \gamma) = \inf_z (\alpha M_b(z) + (1 - \alpha) \operatorname{Re}(z) - \gamma \operatorname{Im}(z) + \alpha).$$

where $F(\alpha, \gamma)$ denote the *Universal Dimension Mixed Spectrum*

$$F(\alpha, \gamma) = \sup_{\text{simply connected } \Omega} f_{\Omega}(\alpha, \gamma)$$

Using these relations, one can restate Theorem 1 as a generalization of the Beurling's estimate (Lemma 1):

Theorem 2. $F(\alpha, \gamma) \leq C(\alpha - \sqrt{(\alpha - 1)^2 + \gamma^2})$ whenever $\alpha \geq \frac{1+\gamma^2}{2}$.

Let us note that $C = 2$ in this inequality is equivalent to the Becker-Pommerenke conjecture. Conjecture 2 can be restated in terms of the Dimension Mixed Spectrum as

Conjecture 3.

$$F(\alpha, \gamma) = 2 - \frac{1 + \gamma^2}{\alpha}$$

whenever $\alpha \geq \frac{1 + \gamma^2}{2}$.

The rest of the paper is organized as follows. First, for the sake of completeness, as well as to motivate our further arguments, we give the proof of Lemma 1. Then we explain the equivalence of Theorem 1 and Theorem 2. Finally we provide a refinement of Lemma 1 and a combinatorial construction necessary for the proof of Theorem 2.

2. ESTIMATES ON THE HARMONIC MEASURE IN THE PRESENCE OF LARGE ROTATION.

The crucial technical tool for our proofs of both Lemma 1 and the main theorems is the *extremal distance*.

Definition 5. Let Ω be a domain, and $E, F \subset \bar{\Omega}$. For a non-negative continuous function ρ on Ω , let

$$A(\rho) = \iint_{\Omega} \rho^2 dx dy$$

and

$$L(\rho) = \inf_{\sigma} \int_{\sigma} \rho ds,$$

where the infimum is taken over all rectifiable crosscuts σ of Ω with one end on E and the other end on F . The *extremal distance from E to F within Ω* is defined as

$$\lambda_{\Omega}(E, F) = \sup_{\rho} \frac{L^2(\rho)}{A(\rho)},$$

where the supremum now is taken over all ρ with $A(\rho) > 0$.

We refer to, say, [GM08] for the discussion of the properties of the extremal distance and the connections with harmonic measure. We will make use of the conformal invariance of extremal distance, the sub additivity, and the fact that to estimate the extremal distance from below one just need to produce a “nice” metric ρ . Some finer properties of the extremal distance will be introduced as they are required.

Proof of the Lemma 1. Let us note that the use of symmetry and equation (6) imply that $F(\alpha, \gamma) = F(\alpha, -\gamma)$. So without loss of generality we can assume that γ is positive. Let us fix a simply connected domain Ω , a point $v \in \partial\Omega$, a point $w_0 \in \Omega$. Let L be the arc of the circle centered at v of the radius $R = |w_0 - v|$ which forms a crosscut of Ω containing w_0 .

Let now $B(v, \delta)$ be a ball with the harmonic measure $\omega(B(v, \delta)) > \delta^{\alpha + \eta}$ and the rotation $\rho(v, \delta) < \delta^{\gamma - \eta}$. Then (see [Mak87], Corollary 1.4) there exists an arc $l \subset \{w : |w - v| = 2\delta\} \cap \Omega$ such that

$$(11) \quad \lambda_{\Omega}(l, L) \leq \left(\frac{\alpha + \eta}{\pi} + \epsilon(\delta) \right) |\log \delta|,$$

where $\epsilon(\delta) \rightarrow 0$ when $\delta \rightarrow 0$.

Let us consider a branch $g_v(w)$ of the function $\log(w - v)$ in the domain Ω . To be consistent with the definition of the rotation, let us normalize g_v to take the principal value for $w = w_0$. By the conformal invariance of extremal distance, $\lambda_{\Omega}(l, L)$ is the same as the extremal distance between $g_v(L) \subset \{u : \text{Im}(u) = \log R\}$ and $g_v(l)$.

To obtain a lower estimate on $\lambda_{\Omega}(l, L)$, we consider the domain

$$\Omega' = g_v(\Omega_{\delta} \cap \{w : 2\delta < |w - v| < R\}).$$

In the standard Euclidean metric, the area of the domain is at most $2\pi \log \frac{R}{2\delta}$, since the projection of the Ω' to the real axis has the length $\log \frac{R}{2\delta}$, and the length of each vertical crosscut is at most 2π . On the other hand, since the rotation is at least $(\gamma - \eta) |\log 2\delta|$, the Euclidean distance between $g(l)$ and $g(L)$ is at least $\sqrt{1 + (\gamma - \eta)^2} \log \frac{R}{2\delta} - 4\pi$. Thus, comparing the extremal distance to the quantity obtained for the Euclidean metric on Ω' , we obtain the estimate

$$(12) \quad \lambda_{\Omega}(l, L) \geq \frac{1}{\pi} \left(\frac{1}{2} + \frac{(\gamma - \eta)^2}{2} - \epsilon'(\delta) \right) \left(\log \left(\frac{R}{2\delta} \right) \right),$$

where $\epsilon'(\delta) \rightarrow 0$ when $\delta \rightarrow 0$.

Combination of (11) and (12) gives the inequality

$$\frac{1}{\pi} \left(\frac{1}{2} + \frac{(\gamma - \eta)^2}{2} - \epsilon'(\delta) \right) \left(\log \left(\frac{R}{2\delta} \right) \right) \leq \left(\frac{\alpha + \eta}{\pi} + \epsilon(\delta) \right) |\log \delta|,$$

which implies the first assertion of the Lemma if δ is small enough.

To prove the exactness let us fix γ . Take $t = \arctan \gamma$ and consider the domain $h(\mathbb{D})$, where $h(z) = (1 - z)^{\lambda \cos t e^{it}}$. The map h is bounded and univalent in \mathbb{D} for $0 < \lambda \leq 2$. Near the point $v = h(1) = 0$ the local dimension of harmonic measure of $h(\mathbb{D})$ is equal to $\alpha = \frac{1}{\lambda \cos^2 t} = \frac{1 + \gamma^2}{\lambda}$ and the local rotation rate is equal to γ . \square

Proof that Theorem 2 is equivalent to Theorem 1. Let us fix θ and suppose that Theorem 2 holds. First let us observe that for any θ , $(\alpha - 1) \cos \theta + \gamma \sin \theta \leq \sqrt{(\alpha - 1)^2 + \gamma^2}$. Thus we have

$$F(\alpha, \gamma) \leq C(\alpha - (\alpha - 1) \cos \theta - \gamma \sin \theta)$$

for any θ . So, by (9),

$$M_b(t e^{i\theta}) \leq \sup_{\alpha, \gamma} (C - t) \left(1 - \left(1 - \frac{1}{\alpha} \right) \cos \theta - \frac{\gamma}{\alpha} \sin \theta \right) + t - 1 = t - 1$$

for $t = C =: T$. Thus, for $|z| = T$, $M_b(z) \geq |z| - 1$. Since M_b is a convex increasing in every direction function and $M_b(z) \geq |z| - 1$, this implies that $M_b(z) = |z| - 1$ as soon as $|z| \geq T$.

On the other hand, if Theorem 1 holds, then by (10)

$$F(\alpha, \gamma) = \inf_z (\alpha M_b(z) + (1 - \alpha) \operatorname{Re}(z) - \gamma \operatorname{Im}(z) + \alpha) \leq \inf_{\theta, |t| \geq T} (\alpha(t - 1) + (1 - \alpha)t \cos \theta - \gamma t \sin \theta + \alpha) = T(\alpha - \sqrt{(\alpha - 1)^2 + \gamma^2}).$$

\square

It will be more convenient for us to deal with the dimension spectrum. Let us note that because of the symmetry property of the universal spectrum, it is enough to consider only nonnegative γ . The case $\gamma = 0$ immediately follows from Theorem 1 of [CM94], which essentially states that $\sup_{\gamma} F(\alpha, \gamma) \leq C(\alpha - \frac{1}{2})$. Thus we can consider only the case $\gamma > 0$.

Let us fix $\gamma > 0$ and $\alpha = \frac{\gamma^2}{2} + \frac{1}{2} + \epsilon$, $\epsilon > 0$. It will be convenient to normalize Ω by the condition $\operatorname{diam} \Omega = 10\sqrt{\gamma^2 + 1}$.

As in the proof of Lemma 1, for a point $v \in \partial\Omega$ let $g_v(w)$ be a branch of $\log(w - v)$. Let us also fix a large constant A . Let L'_n be the line $x = \gamma y - \frac{A\gamma}{\sqrt{1 + \gamma^2}}$, P'_n be the half plane $x \geq \gamma y - \frac{A\gamma}{\sqrt{1 + \gamma^2}}$.

We denote by $P_n(v)$ the connected component of $g_v(\Omega) \cap P'_n$ containing $g_v(w_0)$, $R_n(v) = P_{n+1}(v) \setminus P_n(v)$ and $L_n(v) = \partial R_n(v) \cap L'_n$. Let

$$\nu_n(v) = \inf_{w_2 \in L_{n+1}} \operatorname{Im}(w_2) - \inf_{w_1 \in L_n} \operatorname{Im}(w_1).$$

$$\text{Let } \lambda(\nu) = \frac{A + \sqrt{1 + \gamma^2} \nu \gamma}{\sqrt{1 + \gamma^2} 2\pi}.$$

Lemma 2. *The extremal distance $\tilde{\lambda}(n, v)$ between $L_n(v)$ and $L_{n+1}(v)$ in $g_v(\Omega)$ is at least $\lambda(\nu_n(v))$. Equality is reached only when $\nu_n(av) = \frac{\gamma A}{1 + \gamma^2}$ and $R_n(v)$ is a rectangle with sides A and $\frac{2\pi}{\sqrt{1 + \gamma^2}}$.*

Proof. To obtain this estimate, we put a standard Euclidean metric on the $R_n(v)$. Then

$$\tilde{\lambda}(n, v) \geq \frac{A^2}{\text{Area of } R_n(v)} \geq \frac{A + \sqrt{1 + \gamma^2} \nu_n \gamma}{\sqrt{1 + \gamma^2} 2\pi}.$$

The equality in the first inequality is only achieved when $R_n(v)$ is a rectangle with sides A and $\frac{2\pi}{\sqrt{1 + \gamma^2}}$. \square

Let $Y_n(v) = \tilde{\lambda}(n, v) - \lambda(\nu_n(v))$ denote the “excess” of the extremal distance in $R_n(v)$. Note that by the lemma, $Y_n(v) = 0$ if and only if the domain Ω is a logarithmic spiral with parameter γ on the distance $e^{-\frac{kA}{\sqrt{1 + \gamma^2}}}$ from v . Lemma 4 gives the quantitative version of this characteristic.

The following two technical lemmas allow us to estimate the number of non intersecting balls with fixed rotation and large harmonic measure.

Lemma 3. *Let $\delta = \frac{1}{2} \exp(-\frac{An + A + 2\pi}{\sqrt{\gamma^2 + 1}})$ with some $n > N(\epsilon)$. Let $\rho(v, \delta) \leq \delta^\gamma$ and $\omega(B(v, \delta)) \geq \delta^\alpha$.*

Then

$$\sum_{i=1}^n Y_n(z) \leq C \frac{An}{\sqrt{1 + \gamma^2}} \epsilon$$

for some absolute constant $C > 0$.

Proof. Because $\rho(v, \delta) \leq \delta^\gamma$, we have $\sum_{j=1}^n \nu_j(v) \geq \frac{An\gamma}{\sqrt{1 + \gamma^2}}$. It implies that

$$(13) \quad \sum_{j=1}^n \lambda(\nu_n(v)) \geq \frac{A + \sqrt{1 + \gamma^2} \frac{An\gamma}{\sqrt{1 + \gamma^2} \gamma}}{\sqrt{1 + \gamma^2} 2\pi} = \frac{An\sqrt{1 + \gamma^2}}{2\pi}.$$

As in the proof of Lemma 1, we can use Corollary 1.4 in [Mak87] to see that since $\omega(B(v, 2\delta)) \geq \delta^\alpha$, there exist two arcs $l \subset \Omega \cap \partial B(v, \delta)$ and $L \subset \Omega \cap \partial B(v, 1)$ with the extremal distance between them no greater than $\frac{1}{\pi} \alpha \log \frac{1}{\delta}$.

So, by the sub additivity property of the extremal distance, we have

$$(14) \quad \sum_{j=1}^n \tilde{\lambda}(j, v) \leq \frac{1}{\pi} \alpha \log \frac{1}{r} = \frac{An + A + 2\pi}{\pi \sqrt{\gamma^2 + 1}} \alpha = \frac{An + A + \pi}{2\pi} \left(\sqrt{1 + \gamma^2} + \frac{2\epsilon}{\sqrt{\gamma^2 + 1}} \right).$$

Now to estimate $\sum_{j=1}^n Y_j(v) = \sum_{j=1}^n \lambda(\nu_n(v)) - \sum_{j=1}^n \tilde{\lambda}(\nu_n(v))$, we just subtract the estimate (13) from (14) and get the desired result for a large enough constant C . \square

Lemma 4. *There exist positive absolute constants $\sigma > 0$ and $\tau > 0$ such that if $v \in \partial\Omega$ and $Y_1(v) \leq e^{-\sigma kA}$, then for any $v_1 \in (v + \exp(\partial R_n(z))) \cap \partial\Omega$, we have $Y_i(v_1) \geq \tau A$ $i = 2, \dots, k$.*

Proof. Since $\text{diam } \partial\Omega = 10\sqrt{1 + \gamma^2}$, the boundary $g_v(\partial\Omega)$ intersects both $L_1(v)$ and $L_2(v)$.

Let $S(b, c) = \{(x, y) : \gamma x + b - c + 2\pi k \leq y \leq \gamma x + b + c + 2\pi k, k \in \mathbb{N}\}$ be the collection of strips of the length c mapped to a neighborhood of a logarithmic spiral by the exponential map.

Let c_0 be the minimal c for which $g_v(\partial\Omega) \cap R_1(z) \subset S(b, c)$ for some b . Then by an estimate from [Jen70],

$$(15) \quad e^{-\sigma kA} \geq Y_1(v) \geq \frac{\text{const}}{\lambda(\nu_1)} c_0^3.$$

So $\partial\Omega \cap (v + \exp(R_1(v)))$ lies in the neighborhood of a logarithmic spiral of the width $\text{const } e^{-\frac{\sigma}{3}kA}$, which is less than $\frac{1}{100}e^{-kA}$ if $\sigma > 3$ and A is large enough. It means that for $v_1 \in \partial\Omega \cap (v + \exp(R_1(v)))$, the spiral-like set $v_1 + \exp(R_i(v_1))$ is contained in an angle of the size $\frac{3\pi}{2}$ with the center in v_1 when $i = 2, \dots, k$. It implies that for some constant C we have $Y_i(v_1) \geq C\lambda(\nu_n) \geq \tau A$ if $\tau < \frac{C}{2\pi\sqrt{1+\gamma^2}}$. \square

Proof of Theorem 2. The combinatorial construction we use here is very much similar to the one from [CM94].

Let $N(n, x)$ denote the maximal number of non intersecting balls of the radius $\exp(-\sqrt{\gamma^2+1}(An+2\pi))$ with the centers $v_j \in \partial\Omega \cap \mathbb{D}$ satisfying $\sum_{i=1}^n Y_i(v_j) \leq x$ for all j . Let us prove that

$$(16) \quad N(n, x) \leq C_1(\gamma)e^{C_2x}$$

for some absolute constant C_2 and a constant $C_1(\gamma)$ depending only on γ , but not on the domain Ω .

Together with Lemma 3, (16) implies that

$$f(\alpha, \gamma) \leq \lim_{n \rightarrow \infty} \frac{\log \left(N \left(n, \frac{C}{\sqrt{\gamma^2+1}} An\epsilon \right) \right)}{\sqrt{\gamma^2+1}(An+2\pi)} \leq C_0\epsilon$$

for some absolute constant C_0 not depending on γ . Since

$$\alpha - \sqrt{(\alpha-1)^2 + \gamma^2} = \frac{2\alpha - 1 - \gamma^2}{\alpha + \sqrt{(\alpha-1)^2 + \gamma^2}} \leq 4 \left(\alpha - \frac{1}{2} - \frac{\gamma^2}{2} \right),$$

Theorem 2 follows from the estimate (16).

We prove (16) by induction on n . The area counting shows that $N(n, X) \leq Ce^{2nA\sqrt{\gamma^2+1}}$. So, for small n (say, for $n = 1, 2$), the desired inequality is true for large enough $C_1(\gamma)$.

Let us now fix n . Assume that (16) is proved for $n-1$.

Fix $N(n, x)$ non intersecting balls with the centers v_j and $\sum_{i=1}^n Y_i(v_j) \leq x$ for all j .

Let $\eta = \min_{v \in \partial\Omega \cap \mathbb{D}} Y_1(v)$ and $\psi = e^{-\sigma kA}$ from Lemma 4.

Three cases are possible:

- (1) $\eta \geq \psi$.
- (2) $\eta \leq \psi^n$.
- (3) $\psi^k \geq \eta \geq \psi^{k+1}$, for some $k = 1, \dots, n-1$.

We show that in each of them $N(n, x) \leq C_1e^{C_2x}$.

Case 1: $\eta \geq \psi$. For all v_j , we have

$$\sum_{i=2}^n Y_i(v_j) \leq x - \psi.$$

The domain $\Omega \cap \mathbb{D}$ can be covered by $\text{const } e^{2A}$ discs B of the radius e^{-A} . By rescaling we get that in each of the balls B , we have no more than $N(n-1, x-\psi)$ points v_j .

So, by the induction hypothesis,

$$N(n, x) \leq \text{const } N(n-1, x-\psi)e^{2A} \leq \text{const } C_1e^{2A}e^{C_2(x-\psi)} \leq C_1e^{C_2x}$$

if $C_2 > \text{const } Ae^{\sigma A}$.

Case 2: $\eta \leq \psi^n$. For some $v_0 \in \partial\Omega \cap \mathbb{D}$, we have $Y_1(v_0) \leq \psi^n$, so by Lemma 4, we have $Y_i z \geq A\tau, i = 2, \dots, n$ for all $v \in v_0 + \exp(R_1(v_0))$.

If $x < A\tau(n-1)$ then all the points v_j lie inside $\mathbb{D} \setminus (v_0 + \exp(R_1(v_0)))$. Thus, by rescaling, we have

$$N(n, x) \leq N(n-1, x) \leq C_1 e^{C_2 x}.$$

If $x \geq A\tau(n-1)$, then $N(n, x) \leq \text{const } e^{2nA} \leq C_1 e^{C_2 x}$, if $C_2 > \frac{10}{\tau}$.

Case 3: $\psi^k \geq \eta \geq \psi^{k+1}$, for some $k = 1, \dots, n-1$. Again we take a point v_0 with $\psi^{k+1} \leq Y_1(v_0) \leq \psi^k$. As above, by Lemma 4, we have $Y_i(v) \geq A\tau, i = 2, \dots, k$ for $v \in v_0 + \exp(R_1(v_0))$.

We cover $v_0 + \exp(R_1(v_0))$ by $\text{const}(\gamma)e^{kA}$ discs of the radius e^{-kA} . Then any point v_j either belongs to one of the discs or to $\mathbb{D} \setminus (v_0 + \exp(R_1(v_0)))$. This gives us, by the induction assumption,

$$\begin{aligned} N(n, x) &\leq N(n-1, x - \psi^{k+1}) + e^{kA} N(n-k, x - (k-1)A\tau) \\ &\leq C_1 e^{C_2 x} \left(e^{-C_2 \psi^{k+1}} + e^{kA - C_2(k-1)A\tau} \right) \leq C_1 e^{C_2 x} \end{aligned}$$

if $C_2 \psi > 2$ and $C_2 \tau > 4$. □

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