

THE COMPLEXITY OF SIMULATING BROWNIAN MOTION

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ABSTRACT. We analyze the complexity of the Walk on Spheres algorithm for simulating Brownian Motion in a domain $\Omega \subset \mathbb{R}^d$. The algorithm produces samples from the hitting probability distribution on $\partial\Omega$ within an error of ε . We introduce energy functions using Newton potentials to obtain an $O(\log^2 1/\varepsilon)$ upper bound on the convergence of the algorithm for a very rich class of domains Ω . In particular, we show this rate of convergence for all 3-dimensional domains with connected exterior.

We show that, in general, the convergence rate of the algorithm may be polynomial in $1/\varepsilon$, and give a tight worst-case bound of $O(\varepsilon^{4/d-2})$ on the convergence of the algorithm in $d \geq 3$ dimensions. For $d = 3$, this gives an optimal upper bound of $O(\varepsilon^{-2/3})$, improving on the previously known bound of $O(\varepsilon^{-1})$.

1. INTRODUCTION

Brownian Motion (BM) is the most important model of randomized motion in \mathbb{R}^d . It is the simplest (but, in a sense, generic) example of a continuous diffusion process. BM found an astonishing number of application to diverse areas of Mathematics and Science, including Biomathematics, Finance, Partial Differential Equations, and Statistical Physics [EKM97, KS98, Maz02, Nel67, Szn98].

Because of the ubiquity of BM, its effective simulation provides a way to efficiently solve a variety of problems, such as computation of the Conformal Maps, Tomography, and Stochastic PDEs. One of the main ways in which simulations of BM are used is to study its first hitting probabilities with respect to some stopping conditions. If the stopping condition is that of hitting the boundary of some domain Ω , then for any starting point x the *harmonic measure* h_x on $\partial\Omega$ gives the probability distribution of the first location where a BM originated at x hits $\partial\Omega$. In many of the BM's applications, its enough to obtain information about the harmonic measure, more specifically, to efficiently sample from it.

One of the immediate applications of the ability to sample from harmonic measures is solving the Dirichlet problem in \mathbb{R}^d . The Dirichlet problem on a domain $\Omega \subset \mathbb{R}^d$ with boundary condition $f : \partial\Omega \rightarrow \mathbb{R}$ is the problem of finding a function $u : \Omega \rightarrow \mathbb{R}$ satisfying

$$(1) \quad \begin{cases} \Delta u(x) = 0 & x \in \Omega \\ u(x) = f(x) & x \in \partial\Omega \end{cases}$$

In other words, finding a harmonic function u subject to the boundary conditions f . By the celebrated Kakutani's Theorem [GM04], the value of u at $x \in \Omega$ is exactly the expected value of f with respect to the harmonic measure h_x on $\partial\Omega$: $u(x) = \mathbf{E}_{h_x(z)}[f(z)]$.

In the present paper we study the amount of time it takes to sample from the harmonic measure with precision ε using the Walk on Spheres algorithm – the simplest and most commonly used algorithm for sampling from the harmonic measure.

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1.1. The Walk on Spheres algorithm. The Walk on Spheres (WoS) algorithm was first proposed in 1956 by M. Muller in [Mul56]. In the paper the method was applied to the solution of various boundary problems for the Laplace operator, including the Dirichlet problem. The logarithmic running time of the process X_t was established for convex domains by M. Motoo in [Mot59] and was later generalized for a wider, but still very restricted, class of planar and 3-dimensional domains by G.A. Mikhailov in [Mih79]. See also [EKMS80] and [Mil95] for additional historical background and the use of the algorithm for solving other types of boundary value problems.

In our earlier work [BB07], we established sharp bounds on the rate of convergence of WoS for arbitrary planar domains. Unfortunately, the techniques of [BB07] do not generalize well to higher dimensions. In the present paper, we introduce an energy function with logarithmic growth. It allows us to obtain a polylogarithmic rate of convergence of WoS for large class of domains in \mathbb{R}^d including, for example, all domains in \mathbb{R}^3 with connected complements. We also establish sharp bounds on the worst possible rate of convergence for general higher dimensional domains.

Let us now define the WoS. We would like to simulate a BM in a given bounded domain Ω until it gets ε -close to the boundary $\partial\Omega$. Of course one could simulate it using jumps of size δ in a random direction on each step, but this would require $O(1/\delta^2)$ steps. Since we must take $\delta = O(\varepsilon)$, this would also mean that the process may take $O(1/\varepsilon^2)$ steps to converge.

The idea of the WoS algorithm is very simple: we do not care about the path the BM takes, but only about the point at which it hits the boundary. Thus if we are currently at a point $X_n \in \Omega$ and we know that

$$d(X_n) := d(X_n, \partial\Omega) \geq r,$$

i.e. that X_n is at least r -away from the boundary, then we can just jump $r/2$ units in a random direction from X_n to a point X_{n+1} . To justify the jump we observe that a BM hitting the boundary would have to cross the sphere

$$S_n = \{x : |x - X_n| = r/2\}$$

at some point, and the first crossing location X_{n+1} is distributed uniformly on the sphere. There is nothing special about a jump of $d(X_n)/2$ and it can be replaced with any $\alpha d(X_n)$ where $0 < \alpha < 1$. Let $\{\gamma_n\}$ be a sequence of i.i.d. random variables each being a vector uniformly distributed on the unit sphere in \mathbb{R}^d . We could take, for example, $\gamma_n = \Gamma_n^d / |\Gamma_n^d|$, where Γ_n^d is a normally distributed d -dimensional Gaussian variable. Then, schematically, the Walk on Spheres algorithm can be presented as follows:

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WalkOnSpheres( $X_0, \varepsilon$ )
 $n := 0$ ;
while  $d(X_n) = d(X_n, \partial\Omega) > \varepsilon$  do
  compute  $r_n$ : a multiplicative estimate on  $d(X_n)$  such that  $\beta \cdot d(X_n) < r_n < d(X_n)$ ;
   $X_{n+1} := X_n + (r_n/2) \cdot \gamma_n$ ;
   $n := n + 1$ ;
endwhile
return  $X_n$ 

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Thus at each step of the algorithm we jump at least $\beta/2$ and at most $1/2$ -fraction of the distance to the boundary in a random direction. An example of running the WoS algorithm in 2-d is illustrated on Figure 1.

As mentioned earlier, it is clear that the algorithm is correct. Moreover, it is not hard to see that it converges in $O(1/\varepsilon^2)$ steps. However, in many situations, this rate of convergence is unsatisfactory. In particular, if we wanted to get 2^{-n} -close to the boundary, it would take us a number of steps *exponential* in n . As it turns out, in many natural situations, the rate of convergence is *polynomial* in n (i.e. polylogarithmic in $1/\varepsilon$). The object of the paper is to prove that this is the case.

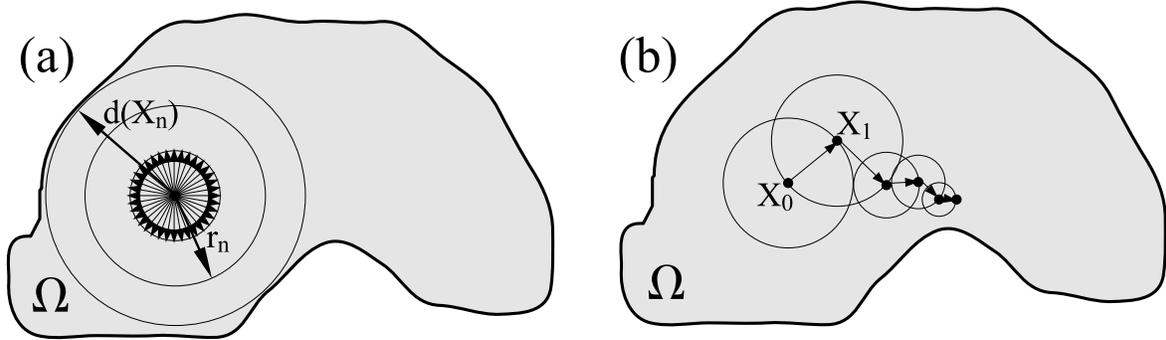


FIGURE 1. An illustration of the WoS algorithm for $d = 2$: one step jump (a), and a possible run of the algorithm for several steps (b)

While an actual implementation of the WoS would involve round-off errors introduced through an imperfect simulation, we will ignore those to simplify the presentation as they do not affect any of the main results. Thus the problem becomes purely that of analyzing the stochastic process X_t and its convergence speed to $\partial\Omega$.

Providing the domain Ω to the algorithm. It is worth noting that the algorithm needs access to the input domain Ω in a very weak sense. We need an oracle $\text{dist}_\Omega(x)$ that satisfy the following:

$$(2) \quad \text{dist}_\Omega(x) \in \begin{cases} (\beta d(x), d(x)) & \text{if } x \in \Omega, d(x) > \beta\varepsilon \\ [0, \beta\varepsilon) & \text{if } x \in \Omega, d(x) \leq \beta\varepsilon \\ 0 & \text{if } x \notin \Omega \end{cases}$$

for some $0 < \beta < 1$. Note that dist_Ω would also allow us to decide both the size of the jump on step n and whether X_n is sufficiently close to $\partial\Omega$ for the algorithm to terminate.

If Ω is given to the algorithm as a union of squares on a ε -fine grid, then dist_Ω can be computed in time $\text{poly}(1/\varepsilon)$. In many applications, however, this function can be computed in time $\text{poly}(\log 1/\varepsilon)$, because we only need to estimate the distance within a *multiplicative* error of β . The precise condition for this is that the complement set Ω^c is poly-time computable as a subset of \mathbb{R}^d in the sense of Computable Analysis. See for example [BW99, Wei00, BC06] for more details on poly-time computability of real sets. The vast majority of domains in applications satisfy this condition.

Thus, in cases when the domain Ω is sufficiently nice for Ω^c to be poly-time computable, the rate of convergence of the WoS becomes the crucial component in the running time of its execution. In particular, depending on whether the rate of convergence is $\text{poly}(1/\varepsilon)$ or $\text{poly}(\log 1/\varepsilon)$ it could take time that is either exponential or polynomial in n to sample points that are 2^{-n} -away from $\partial\Omega$.

1.2. The results.

Definition 1. A domain $\Omega \subset \mathbb{R}^d$ is said to be c -thick for $0 < c < 1$ if for every $x \in \partial\Omega$ there is a Borel measure μ_x which satisfies the following conditions:

- (1) $\text{supp}(\mu_x) \cap \Omega = \emptyset$, or equivalently $\mu_x(\Omega) = 0$;
- (2) for any $y \in \mathbb{R}^d$ and $r > 0$, $\mu_x(B(y, r)) \leq r^{d-2}$;
- (3) for any $r > 0$, $\mu_x(B(x, r)) \geq c \cdot r^{d-2}$.

The class of c -thick domains is very rich, even for specific constant values of c . In particular we have the following interesting special cases.

Claim 2. (1) All 2-dimensional domains are 1-thick;

(2) all bounded d -dimensional domains Ω satisfying the following condition:

for each $x \in \partial\Omega$ there is a $(d-2)$ -dimensional hypercube L_x of some size α with $L_x \cap \overline{\Omega} = \{x\}$

are c -thick for some $c = c(\alpha)$;

(3) all bounded 3-dimensional domains Ω such that the complement Ω^c is connected are $\frac{1}{2}$ -thick.

Proof. The first statement simply follows by placing a δ -measure $\mu_x(\{x\}) = 1$ at x . The second statement follows similarly by using the normalized Lebesgue measure on the hypercube L_x as the measure μ_x .

To prove the third statement, consider a measurable function $f_x : [0, \infty) \rightarrow \Omega^c$ such that $|x - f_x(r)| = r$. Existence of such a function follows from the connectedness of $\Omega^c \ni x$ by a standard topological argument. Define the measure μ_x as

$$\mu_x(B) = \frac{1}{2}m_1(f_x^{-1}(B)),$$

where m_1 is the standard Lebesgue measure on $[0, \infty)$. Let $y \in \mathbb{R}^3$ with $|y - x| = a$ and $r > 0$, then

$$f_x^{-1}(B(y, r)) \subset [a - r, a + r],$$

and hence $\mu_x(B(y, r)) \leq r$. On the other hand, for each $r > 0$,

$$\mu_x(B(x, r)) = \frac{1}{2}m_1(f_x^{-1}(B(x, r))) = \frac{1}{2}m_1([0, r)) = \frac{1}{2}r.$$

□

Our main result states that for c -thick domains the Walk on Spheres will reach the $\frac{1}{n}$ -neighborhood of $\partial\Omega$ in time polylogarithmic in n :

Theorem 3. *There is a constant $M = M(c, d)$ such that for any c -thick domain $\Omega \subset B(0, 1) \subset \mathbb{R}^d$ and any $x_0 \in \Omega$ the following holds. Let X_t be the Walk on Spheres process with $X_0 = x_0$. Let T_n be the first time such that $d(X_t, \partial\Omega) < 1/n$ then*

$$\mathbf{P}[T_n > 2M \log^2 n] < 1/2.$$

By using the Markov property of the Walk on Spheres and repeating it C times we obtain that

$$\mathbf{P}[T_n > 2CM \log^2 n] < 2^{-C},$$

which also implies $\mathbf{E}[T_n] \leq M \cdot \log^2 n$.

In particular, by Claim 2, we obtain polylogarithmic convergence of the WoS on all planar domains and all domains in 3-d space with connected exterior. As the following lower bound demonstrates, the lack of c -thickness in dimensions 3 and higher is an obstacle to fast convergence.

Theorem 4. *For every $d > 2$ there is a domain $\Omega \subset B(0, 1) \subset \mathbb{R}^d$ and an n such that if T_n is the first time the WoS is $1/n$ -close to $\partial\Omega$ as above, then there is a constant $A = A(d) > 0$ such that*

$$(3) \quad \mathbf{P}[T_n > A \cdot n^{2-4/d}] > 1/2.$$

Note that, as expected, (3) says nothing about $d = 2$. Also, the domains Ω in Theorem 4 are extremely “thin”, having a thickness of $c = O(n^{4/d-2}) \ll 1$. It turns out that Theorem 4 is tight, and $O(n^{2-4/d})$ is the worst number of steps one can expect, as demonstrated by the following matching upper bound.

Theorem 5. *For every $d > 2$ there is a constant $B = B(d) > 0$ such that for any domain $\Omega \subset B(0, 1) \subset \mathbb{R}^d$,*

$$(4) \quad \mathbf{P}[T_n > B \cdot n^{2-4/d}] < 1/2.$$

Theorem 5 is somewhat surprising, not only because it beats the $O(n^2)$ bound guaranteed by general diffusion properties of Brownian Motions, but also because for dimension $d = 3$ it gives an upper bound of $O(n^{2/3})$ steps beating the $O(n)$ bound that would follow from analysis using subharmonic functions as in prior works.

1.3. Techniques. The main technical contribution of the paper is constructing an energy function $U : \Omega \rightarrow [0, \infty)$ for the proof of Theorem 3. The energy function U is subharmonic, i.e. $\Delta U \geq 0$. This means that *on average* when the WoS makes a jump of magnitude $r_n/2$ in a random direction, the value of the function does not decrease: $\mathbf{E}[U(X_{k+1})|X_k] \geq U(X_k)$. Moreover, while $U(y)$ will tend to ∞ as y tends to $\partial\Omega$, we will make sure that it does not grow too fast: $U(y) \leq \log(2/d(y, \partial\Omega))$. In other words, once the value of $U(X_t)$ reaches $\log(2n)$, we can be sure that X_t is at least $1/n$ -close to $\partial\Omega$, and thus the WoS must have terminated before time t .

Hence the process $U(X_k)$ terminated when X_k reaches the $1/n$ -neighborhood of $\partial\Omega$ is a submartingale on the interval $[0, \log(2n)]$. Note that by definition the process $U(X_k)$ can never fall below 0. The main technical challenge is to show that the function U can be constructed in a way that $\mathbf{E}[(U(X_{k+1}) - U(X_k))^2] > \alpha > 0$ for some fixed α . This bounds the average step size of the process $U(X_k)$ from below, and implies that with probability $> 1/2$ the process will take $O(\frac{1}{\alpha} \log^2 n)$ to reach a value bigger than $\log(2n)$.

We construct the function U as a supremum of a family of *Newton potentials* of a certain set of Borel measures. As Newton potentials are harmonic, the subharmonicity of U is guaranteed. The bulk of the work then goes into bounding the average deviation α of the process $U(X_k)$ in terms of the thickness of the domain. Note that this analysis is absolutely necessary, because by Theorem 4 we know that in general the rate of convergence could be $\Omega(n^{2-4/d})$, far worse than the $O(\log^2 n)$ bound we are trying to prove.

2. UPPER BOUNDS ON THE RATE OF CONVERGENCE

2.1. Constructing an energy function of logarithmic growth. For a finite Borel measure μ on \mathbb{R}^d , the *Newton potential of the measure μ* is defined by

$$U^\mu(x) = \frac{1}{d-2} \int \frac{d\mu(z)}{\|z-x\|^{d-2}},$$

if $d \geq 3$, and by

$$U^\mu(x) = \int \log \frac{1}{\|z-x\|} d\mu(z).$$

if $d = 2$. The value $U^\mu(x) = \infty$ is allowed when the integral diverges.

It is well known (e.g. see [Lan72]) that the function U^μ is superharmonic on \mathbb{R}^d , and harmonic outside of $\text{supp } \mu$.

Let us now fix a c -thick domain $\Omega \subset B(0, 1) \subset \mathbb{R}^d$. Let us consider the set \mathcal{M} of all Borel measures μ supported inside the $B(0, 2)$ and outside of Ω (i.e. $\mu(\Omega) = 0$), satisfying the following condition:

$$(5) \quad \text{for any } y \in \mathbb{R}^d \text{ and } r > 0, \mu(B(y, r)) \leq r^{d-2}$$

Let us recall that $d(y) = \text{dist}(y, \partial\Omega)$.

Lemma 6. For $y \in \Omega$ and $\mu \in \mathcal{M}$,

$$U^\mu(y) \leq \log \frac{2}{d(y)}.$$

Proof. The following identity follows from Fubini Theorem and substitution,

$$(6) \quad U^\mu(y) = \frac{1}{d-2} \int_0^\infty \mu(B(y, t^{-1/(d-2)})) dt = \int_0^\infty \frac{\mu(B(y, r))}{r^{d-1}} dr$$

for $d \geq 3$. Similarly,

$$(7) \quad U^\mu(y) = \int_{-\infty}^{\infty} \mu(B(y, e^{-t})) dt = \int_0^{\infty} \frac{\mu(B(y, r))}{r} dr$$

for $d = 2$. By (5) and $\text{supp } \mu \cap \Omega = \emptyset$, these equations imply that for any $d \geq 2$,

$$(8) \quad U^\mu(y) \leq \int_{d(y)}^2 \frac{1}{t} dt = \log \frac{2}{d(y)}.$$

□

Let us now introduce the energy function $U(y) = \sup_{\mu \in \mathcal{M}} U^\mu(y)$. It is a subharmonic function on Ω , as a supremum of locally bounded family of harmonic functions. Furthermore, by Lemma 6 it satisfies $U(y) \leq \log \frac{2}{d(y)}$ for all $y \in \Omega$.

2.2. Proof of the upper bound. Let X_t be the WoS process initiated at some point $X_0 = y \in \Omega$. Let us define a new process $U_t = U(X_t)$, the value of the energy function at the n -th step of the process. Note that because U is subharmonic and positive, U_t is a positive submartingale, that is $\mathbf{E}[U_{t+1}|U_t] \geq U_t$. From Lemma 6 it follows that if $U_t > \log 2n$ then $d(X_t) < 1/n$.

The main technical step in the proof of Theorem 3 is contained in the following lemma.

Lemma 7. *There are constants k and L , depending only on the thickness c , the precision of the distance estimate β , and the dimension d , such that*

$$\mathbf{E}[(U_{t+k} - U_t)^2 | U_t] > L.$$

Proof. Let us fix t . Since \mathcal{M} is a compact set, $U_t = U^\mu(X_t)$ for some $\mu \in \mathcal{M}$. Let x be the point of $\partial\Omega$ closest to X_t .

First, let us consider the case $d = 2$. Then $U_{t+k} \geq U^{\delta x}(X_{t+k}) = \log \frac{1}{|x - X_{t+k}|}$. Given X_t , with probability at least $1/4^k$, $|X_{t+k} - x| \leq d(X_t)/3$, for large enough k . Hence $U_{t+k} \geq \log \frac{3}{d(X_t)}$ w.p. $> 1/4^k$. By Lemma 6, $U_t \leq \log \frac{2}{d(X_t)}$, and thus with probability at least $1/4^k$,

$$|U_{t+k} - U_t|^2 \geq (\log 3/2)^2,$$

implying $\mathbf{E}[(U_{t+k} - U_t)^2 | U_t] > \frac{1}{4^k} (\log 3/2)^2$, for some constant $k = k(\beta)$.

Let us now consider case when $d > 2$. Fix X_t and the corresponding measure μ . We will construct a δ , dependent only on d , c , and β , and a measure $\nu \in \mathcal{M}$, such that

$$(9) \quad U^\nu(y) > U^\mu(X_t) + 1 \text{ whenever } |y - x| < \delta \cdot d(X_t).$$

This would imply $\|U_{t+k} - U_t\|^2 \geq U^\nu(X_{t+k}) - U^\mu(X_t) > 1$ whenever $|X_{t+k} - x| < \delta$. Note that for some $p > 0$ dependent only on d and β ,

$$\mathbf{P}[|X_{t+k} - x| < (1 - \beta/2)^{-k} d(X_t)] > p^k.$$

Hence, for sufficiently large k , $\mathbf{P}[|X_{t+k} - x| < \delta] > p^k$, which, in turn, implies the statement of the lemma.

Our goal now is to construct ν satisfying (9). Let measure μ_1 be the measure μ_x from Definition 1 restricted to $B(X_t, 2d(X_t))$, $\mu_2 = \mu_3 = 0$, and for $k \geq 4$, let μ_k be the measure μ restricted to the d -dimensional annulus $A_k = \{z : 2^{k-1}d(X_t) \leq \|z - X_t\| \leq 2^k d(X_t)\}$ scaled by the factor $1 - \alpha_k := 1 - 2^{(3-k)(d-2)}$. Let us also put $\alpha_1 = \alpha_2 = \alpha_3 = 1$. We define $\nu = \sum_k \mu_k$. The ingredients of the construction are illustrated on Figure 2(a).

Let us first prove that $\nu \in \mathcal{M}$. Consider any disk $B(z, r)$. Let K be the largest number such that $B(z, r)$ intersects A_K . If $B(z, r)$ does not intersect $B(X_t, 2d(X_t))$, the measure ν is no greater than

μ on $B(z, r)$, and thus $\nu(B(z, r)) \leq r^{d-2}$. If $K = 1$, $\nu(B(z, r)) = \mu_x(B(z, r)) \leq r^{d-2}$. For all other cases, $r \geq 2^{K-2}d(X_t)$, which, by the choice of α_K , implies that $\alpha_K r^{d-2} \geq (2d(X_t))^{d-2}$. Thus

$$\begin{aligned} \nu(B(z, r)) &\leq \mu_x(B(X_t, 2d(X_t))) + \mu(B(x, r)) - \sum_{k=1}^K \alpha_k \mu(B(z, r) \cap A_k) \leq \\ &(2d(X_t))^{d-2} + (1 - \alpha_K)\mu(B(z, r)) \leq \alpha_K r^{d-2} + (1 - \alpha_K)r^{d-2} = r^{d-2}. \end{aligned}$$

Next, we will show that

$$(10) \quad U^\nu(y) \geq U^\mu(y) - C_1 + c \cdot 2^{2-d} \log \frac{1}{\delta}$$

and

$$(11) \quad U^\mu(y) \geq U^\mu(X_t) - C_2$$

for some constants C_1 and C_2 depending only on d and c . These inequalities provide the estimate (9) whenever $\log 1/\delta > 2^{d-2}(C_1 + C_2 + 1)/c$.

To establish (10), let us note that for any k we have

$$\mu(A_1) + \mu(A_2) + \cdots + \mu(A_k) = \mu(B(X_t, 2^k d(X_t))) \leq (2^k d(X_t))^{d-2}.$$

By the Abel summation formula,

$$\sum_k \alpha_k 2^{k(2-d)} \mu(A_k) \leq \sum_k d(X_t)^{d-2} (2^{d-2} \alpha_{k-1} - \alpha_k) \cdot 2^{2-d} \leq 2^{(d-2)} (d(X_t))^{d-2}.$$

This implies

$$(12) \quad \frac{1}{d-2} \sum_k \alpha_k \int_{A_k} \frac{1}{\|z - X_t\|^{d-2}} d\mu(z) \leq 2^{d-2} \sum_k \alpha_k \mu(A_k) (2^k d(X_t))^{2-d} \leq 4^{d-2}.$$

Thus we obtain

$$\begin{aligned} U^\nu(y) &\geq \int_{2\delta d(X_t)}^{d(X_t)} \frac{\mu_x(B(y, r))}{r^{d-1}} dr + \sum_{k \geq 2} \int_0^\infty \frac{\mu_k(B(y, r))}{r^{d-1}} dr \geq \\ &\int_{2\delta d(X_t)}^{d(X_t)} \frac{\mu_x(B(x, r - \delta d(X_t)))}{r^{d-1}} dr + \int_0^\infty \frac{\mu(B(y, r))}{r^{d-1}} dr - \frac{1}{d-2} \sum_{k \geq 1} \alpha_k \int_{A_k} \frac{d\mu(z)}{\|z - X_t\|^{d-2}} \geq \\ &\int_{2\delta d(X_t)}^{d(X_t)} c \cdot \left(\frac{r - \delta r}{r} \right)^{d-2} \frac{dr}{r} + \int_0^\infty \frac{\mu(B(y, r))}{r^{d-1}} dr - 4^{d-2} \geq c \cdot 2^{2-d} \log \frac{1}{2\delta} + U^\mu(y) - 4^{d-2}. \end{aligned}$$

which implies (10).

To obtain (11), we just need to note that

$$\begin{aligned} U^\mu(y) - U^\mu(X_t) &\geq \frac{1}{d-2} \int_{|z - X_t| > 4d(X_t)} \left(\frac{1}{\|z - y\|^{d-2}} - \frac{1}{\|z - X_t\|^{d-2}} \right) d\mu(z) - \\ &\int_{d(X_t)}^{4d(X_t)} \frac{\mu(B(X_t, r))}{r^{d-1}} dr \geq -d - \log 4. \end{aligned}$$

We defer the details of the last derivation to the full version of the paper. \square

We can now use Lemma 7 to prove the main theorem. Let us replace the submartingale U_t by a stopped submartingale

$$V_t = \begin{cases} U_t, & t < T_n \\ U_{T_n}, & t \geq T_n \end{cases}$$

By the optional stopping time theorem (see [KS91]), V_t is also a positive submartingale; $V_t \leq \log \frac{4}{n}$. This implies, in particular, that

$$(13) \quad \mathbf{E}[V_t(V_{t+k} - V_t)] = \mathbf{E}[\mathbf{E}[V_t(V_{t+k} - V_t)|V_t]] \geq \mathbf{E}[\mathbf{E}[V_t(V_t - V_t)|V_t]] = 0$$

Lemma 7 implies that

$$(14) \quad \mathbf{E}[(V_{t+k} - V_t)^2] > L \cdot \mathbf{P}[T_n > t + k].$$

We are now in a position to prove Theorem 3.

Proof of Theorem 3. Assume first that for some M ,

$$\mathbf{P}[T_n > M \log^2 n] \geq 1/2.$$

It means that for all $t \leq M \log^2 n - k$, $\mathbf{P}[T_n \geq t + k] \geq 1/2$. This implies

$$\mathbf{E}[V_{t+k}^2] = \mathbf{E}[(V_{t+k} - V_t + V_t)^2] = \mathbf{E}[V_t^2] + \mathbf{E}[(V_{t+k} - V_t)^2] + 2\mathbf{E}[V_t(V_{t+k} - V_t)] \geq \mathbf{E}[V_t^2] + L/2.$$

The last inequality follows from (13) and (14). Hence $\mathbf{E}[V_{M \log^2 n}^2] \geq \frac{LM \log^2 n}{2k}$. Since $V_t \leq \log \frac{4}{n}$, this leads to a contradiction for large enough M . \square

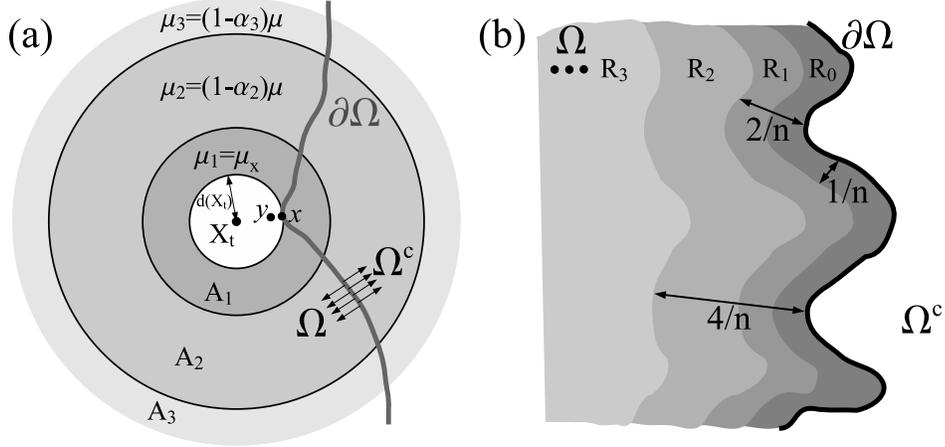


FIGURE 2. (a) Construction of the measure ν in Theorem 3; (b) The regions R_k from the proof of Theorem 5

2.3. The upper bound in the general case: proof of Theorem 5. The goal of Theorem 5 is to give a tight unconditional upper bound on the convergence of the WoS. The idea of the proof is as follows. When the WoS is far from the boundary $\partial\Omega$ it makes fairly big steps and when it is close it makes small steps. There are not too many big steps because the number of big steps of length $> \varepsilon$ confined to $B(0, 1)$ is bounded by $O(1/\varepsilon^2)$. On the other hand, there are not too many small steps, because a small step means that the WoS is very close to $\partial\Omega$, and should converge before having an opportunity to make many more steps. The proof follows from two claims. The first bounds the number of big jumps.

Claim 8. *There is a constant $C_1 = C_1(d)$ such that for $N \geq C_1 \cdot n^{2\ell}$ if $\gamma_1, \gamma_2, \dots, \gamma_N$ is a sequence of i.i.d. random drawings uniformly distributed on the unit sphere and a_1, a_2, \dots, a_N is a positive random process such that a_{k+1} is adapted to $\{\gamma_i, a_i\}_{i=1}^k$ and $a_i \in [0, 1]$. Then*

$$(15) \quad \mathbf{P} \left[a_i > n^{-\ell} \text{ for at least } N/2 \text{ of the } i\text{'s and } \max_{k=1}^N \left| \sum_{i=1}^k a_i \gamma_i \right| < 2 \right] < 1/4.$$

Proof. We observe that

$$M_k = \left(\sum_{i=1}^k a_i \gamma_i \right)^2 - \sum_{i=1}^k |a_i|^2$$

is a martingale. The claim is obtained by stopping it when $k = N$ or when the condition $\left| \sum_{i=1}^k a_i \gamma_i \right| < 2$ is violated, and using an argument similar to one used in the proof of Theorem 3. We defer the details to the full version of the paper. \square

Claim 8 bounds the number of big jumps made by the WoS. To bound the number of small jumps, we denote by $R_0 \subset \Omega$ the $1/n$ -neighborhood of $\partial\Omega$, and more generally, by

$$R_k := \{x \in \Omega : 2^{k-1}/n < d(x, \partial\Omega) \leq 2^k/n\}$$

(see Figure 2(b)). Recall that X_t is the WoS process and T is its stopping time (i.e. the first time when $X_t \in R_0$), we claim:

Claim 9. *Denote*

$$v_k = \#\{t < T : X_t \in R_k\},$$

then

$$\mathbf{P}[v_k > C_2 \cdot 2^{k(d-2)} M] < 1/4^M,$$

for some constant $C_2 = C_2(d, \beta)$ and for any $M > 1$.

Proof. Let us assume, for simplicity, that the WoS at each step makes a jump in a random direction of at least $7/8$ the distance to the boundary. Removing the assumption will only affect the value of the constant C_2 . Suppose that at some point t , $X_t \in R_k$. We estimate the probability that this is the *last* time the WoS visits R_k from below.

First of all, with some constant probability $p > 0$, $X_{t+1} \in R_{k+2}$, i.e. the first jump brings us much closer to $\partial\Omega$. Let $x \in \partial\Omega$ be the nearest point to X_{t+1} in $\partial\Omega$. We have $|x - X_{t+1}| < 2^{k-2}/n$. Consider the harmonic function

$$\Phi_x(y) = \frac{(2n)^{2-d}}{|y-x|^{d-2}} - \frac{2^{2-d}}{2^{(k-1)(d-2)}}.$$

in \mathbb{R}^d . Then the process $\Phi_x(X_{t+j})$ is a martingale. We stop it at time $t + \tau$ when either the WoS terminates or when $X_{t+\tau} \in R_k$ (i.e. the process gets back to R_k), whichever comes first. If $X_{t+\tau}$ is $1/n$ -close to $\partial\Omega$ (but not closer than $1/2n$), then $\Phi_x(X_{t+\tau}) \leq 1$. If $X_{t+\tau} \in R_k$, then $\Phi_x(X_{t+\tau}) \leq 0$. Thus the probability that the WoS terminates at $X_{t+\tau}$ (i.e. we never visit R_k again) is at least

$$\mathbf{P}[X_{t+\tau} \notin R_k] \geq \mathbf{E}[\Phi_x(X_{t+\tau})] = \Phi_x(X_{t+1}) \geq \frac{2^{2-d}}{2^{(k-2)(d-2)+1}} = \alpha \cdot 2^{-k(d-2)},$$

for a constant α . Thus the probability that the visit X_t to R_k is the last one is at least $p \cdot \alpha \cdot 2^{-k(d-2)}$. The claim now follows from an estimate of the probability of having at least v_k returns to R_k , each of them *not* being the last one. \square

Claims 8 and 9 together imply Theorem 5.

Proof of Theorem 5. By Claim 9, for any k , we have that

$$\mathbf{P}[v_{k-s} > C_2 \cdot 2^{(k-s)(d-2)} \cdot (3/2 + s/2)] < 1/4^{3/2+s/2} = (1/8) \cdot 2^{-s}.$$

Hence, by union bound $v_{k-s} \leq C_2 \cdot 2^{(k-s)(d-2)} \cdot (3/2 + s/2)$ for all $s \geq 0$ with probability at least $3/4$. Let k be such that $2^k \approx n^{2/d}$. Then, with the probability at least $3/4$, we have the total number of jumps smaller than $2^k/n$ bounded by

$$(16) \quad \sum_{s=0}^k v_{k-s} \leq \sum_{s=0}^k C_2 \cdot 2^{(k-s)(d-2)} \cdot (3/2 + s/2) < 4C_2 \cdot 2^{k(d-2)} \approx 4C_2 \cdot n^{2-4/d}.$$

If we take $N = (C_1 + 8C_2)n^{2-4/d}$ steps of the WoS, (16) implies that at least half the steps would be of magnitude at least $2^k/n \approx n^{2/d-1}$, except with probability $< 1/4$. Applying Claim 8 with $\ell = 1 - 2/d$, we see that with the probability at least $1/2$, a WoS with more than N steps would escape the unit ball, contradicting the fact that $\partial\Omega \subset B(0, 1)$. Hence with probability $\geq 1/2$ the WoS terminates after $O(n^{2-4/d})$ steps. \square

3. PROOF OF THE LOWER BOUND

In this section we will prove Theorem 4, giving an example of a “thin” d -dimensional domain Ω_d for which the WoS will likely take $\Omega(n^{2-4/d})$ steps to converge within $1/n$ from the boundary $\partial\Omega_d$. The domain Ω_d is comprised of a d -dimensional unit ball $B(0, 1)$ with a set S of $\alpha \cdot n^{d-2}$ points for some small α removed from it. The points are removed from the region $B(0, 2/3) - B(0, 1/3)$ in a way such that for each $x \in \mathbb{R}^d$ with $1/3 < |x| < 2/3$, there is an $s \in S$ with $|x - s| < \ell := An^{2/d-1}$. It is not hard to see that just taking the intersection of the grid $(An^{2/d-1} \cdot \mathbb{Z})^d$ with $B(0, 2/3) - B(0, 1/3)$ attains this goal. By making A a bigger constant we can make α arbitrarily small.

We first claim that for an appropriately chosen α , the probability that the WoS originating at 0 would first reach a $1/n$ -neighborhood of $\partial\Omega_d$ near the unit sphere $U = \partial B(0, 1)$ is at least $3/4$. Let s be any point in S . We first estimate the probability that a WoS would reach a $1/n$ -neighborhood of s before reaching the unit sphere U . Let X_T be the location of the first visit of the WoS either in a $1/n$ -neighborhood of s or near U . Consider the harmonic function

$$\Psi_s(x) = \frac{1}{(n \cdot |x - s|)^{d-2}} \geq 0$$

on \mathbb{R}^d . Then $\Psi_s(x) \geq 1$ in the $1/n$ -neighborhood of s . By the harmonicity of Ψ_s , $\Psi_s(X_t)$ is a martingale, and thus

$$\mathbf{E}[\Psi_s(X_T)] = \Psi_s(X_0) = \Psi_s(0) < \frac{B}{n^{d-2}}$$

for some constant B . Thus the probability that X_T is in the $1/n$ -neighborhood of s is bounded by B/n^{d-2} . By applying the union bound, we see that the probability that the WoS will visit the $1/n$ -neighborhood of *any* point in S before reaching U is at most $B \cdot \alpha$. By taking $\alpha = 1/(4B)$ we obtain that with probability $> 3/4$ the WoS will reach U .

Each step in the portion of the WoS originating at 0 and terminating at U that crosses the region $B(0, 2/3) - B(0, 1/3)$ has a magnitude limited by $\ell = An^{2/d-1}$ by the definition of the process. Thus, by standard properties of the diffusion process, the crossing (and thus the entire WoS) will take at least

$$\gamma \cdot (1/\ell)^2 = (\gamma/A^2) \cdot n^{2-4/d}$$

steps for some $\gamma > 0$ w.h.p., concluding the proof. We defer the full details of the last step to the full version of the paper.

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