

DIMENSION OF PLURIHARMONIC MEASURE AND POLYNOMIAL ENDOMORPHISMS OF \mathbf{C}^n .

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ABSTRACT. Let F be a polynomial endomorphism of \mathbf{C}^n which extends holomorphically to \mathbf{CP}^n . We prove that the dimension of μ_F , the pluriharmonic measure on the boundary of the filled Julia set of F , is bounded above by $2n - 1$.

1. INTRODUCTION

The *dimension* of a probability measure on a metric space is defined as the minimal Hausdorff dimension of a set of full measure. In this paper, we show that the dimension of pluriharmonic measure in \mathbf{C}^n is bounded above by $2n - 1$ when it arises as the measure of maximal entropy for a regular polynomial endomorphism.

For a compact set K in \mathbf{C}^n , *pluriharmonic measure* is defined as

$$\mu_K := dd^c G_K \wedge \cdots \wedge dd^c G_K,$$

where G_K is the pluricomplex Green's function of K with pole at infinity, $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$. The support of μ_K is contained in the Shilov boundary of K . See §2.

Let $F : \mathbf{C}^n \rightarrow \mathbf{C}^n$ be a regular polynomial endomorphism; i.e., one which extends holomorphically to \mathbf{CP}^n . The filled Julia set of F is the compact set

$$K_F = \{z \in \mathbf{C}^n : F^m(z) \not\rightarrow \infty \text{ as } m \rightarrow \infty\}.$$

Pluriharmonic measure μ_F on K_F is F -invariant and of maximal entropy [FS95]. It is not difficult to construct examples where the Hausdorff dimension of the support of μ_F is any value up to and including $2n$. In answer to a question posed in [FS01], we prove:

Theorem 1. *The dimension of pluriharmonic measure on the filled Julia set of a regular polynomial endomorphism of \mathbf{C}^n is at most $2n - 1$.*

In [Oks81], Oksendal conjectured that the dimension of harmonic measure in \mathbf{C} would never exceed 1, though the Hausdorff dimension of its support can take values up to and including 2. Makarov addressed this question for simply connected domains, showing that the dimension of harmonic measure is always equal to one [Mak85]. The theorem was extended by Jones and Wolff, establishing that the dimension is no greater than 1 for general planar domains [JW88]. Moreover, Wolff

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proved that there is always a set of full harmonic measure with σ -finite Hausdorff 1-measure [Wol93].

The complex structure on the plane plays a crucial role in the proof of these theorems. Namely, one relies heavily on the subharmonicity of the function $\log |\nabla u|$ for harmonic u .

It is also possible to take a dynamical approach to the dimension estimates. It follows from the results of Carleson, Jones, and Makarov ([CJ92] and [Mak89]) that any planar domain can be approximated in some sense by domains invariant under hyperbolic dynamical systems (the *fractal approximation*). In the special case of *conformal Cantor sets*, Carleson obtained dimension estimates using the dynamics [Car85]. Recently, it was shown that it suffices to consider polynomial Julia sets in the fractal approximation [BJ02].

Harmonic measure (evaluated at infinity) on the Julia set of a polynomial is the unique measure of maximal entropy [Bro65], [Lju83]. The estimate on dimension in this case follows from a relation to the Lyapunov exponent and the entropy. Indeed, for any polynomial map F on \mathbf{C} , we have

$$(1) \quad \dim \mu_F = \frac{\log(\deg F)}{L(F)},$$

where μ_F denotes harmonic measure on the Julia set of F and $L(F) = \int \log |F'| d\mu_F$ is the Lyapunov exponent [Mn88], [Man84]. The Lyapunov exponent of a polynomial is bounded below by $\log(\deg F)$ with equality if and only if the Julia set is connected [Prz85].

On the other hand, for harmonic measure in \mathbf{R}^n , the methods applied to the study of dimension in \mathbf{C} fail dramatically. The logarithm of the gradient of a harmonic function in \mathbf{R}^n , $n > 2$, is not subharmonic, and there is no dynamical interpretation of harmonic measure. Furthermore, in [Wol95], Wolff showed that for each $n > 2$ there exists a domain in \mathbf{R}^n with the dimension of harmonic measure strictly greater than $n - 1$. A result of Bourgain, however, gives an upper bound on the dimension of harmonic measure in the form $n - \varepsilon(n)$ [Bou87].

Because of the harmonicity of $|\nabla u|^{\frac{n-2}{n-1}}$ for a harmonic function u in \mathbf{R}^n (see [Ste70]), it is conjectured that the dimension of harmonic measure in \mathbf{R}^n does not exceed $n - 1 + \frac{n-2}{n-1}$.

For pluriharmonic measure in \mathbf{C}^n , both of the observations which led to proofs of the Oksendal conjecture in \mathbf{C} are valid: the measure depends on the complex structure of \mathbf{C}^n and is the measure of maximal entropy for polynomial dynamics. Theorem 1 should be the first step in the proof of the following conjecture:

Conjecture 1. *The dimension of pluriharmonic measure of domains in \mathbf{C}^n is at most $2n - 1$.*

The example of area measure on the surface of the unit sphere in \mathbf{C}^n shows that this estimate is sharp.

We believe also that one can give a precise formula for the dimension of pluriharmonic measure in the dynamical case, just as in dimension one (see formula

(1) above). For a diffeomorphism of a real compact manifold, such a formula was obtained by Ledrappier and Young in [LY85]:

$$(2) \quad \dim \mu = \sum \frac{h_i(\mu)}{\lambda_i(\mu)},$$

where μ is an ergodic invariant measure without zero Lyapunov exponents, $\lambda_i(\mu)$ are its positive Lyapunov exponents, and $h_i(\mu)$ are the corresponding *directional entropies* (defined in [LY85]). A similar formula was established by Bedford, Lyubich, and Smillie for polynomial diffeomorphisms of \mathbf{C}^2 [BLS93]. We make the following conjecture, which would imply Theorem 1.

Conjecture 2. *For any holomorphic $F : \mathbf{CP}^n \rightarrow \mathbf{CP}^n$ of (algebraic) degree $d > 1$,*

$$\dim \mu_F = \log d \sum_{i=1}^n \frac{1}{\lambda_i},$$

where λ_i , $i = 1 \dots n$ are the Lyapunov exponents of F with respect to μ_F repeated with multiplicities.

Sketch proof of Theorem 1. We rely on estimates on the Lyapunov exponents of F with respect to μ_F . In particular, Briend and Duval showed that the minimal Lyapunov exponent λ_{\min} is bounded below by $\frac{1}{2} \log d$ (where d is the degree of F) [BD99]. Bedford and Jonsson proved that the sum Λ of the Lyapunov exponents satisfies $\Lambda \geq \frac{n+1}{2} \log d$ [BJ00]. Combining these, we have $\Lambda \geq \frac{n-1}{2} + \lambda_0$, where $\lambda_0 = \max\{\lambda_{\max}, \log d\}$.

We define an invariant set Y of full measure so that preimages of small balls centered at points in Y scale in a way governed by the Lyapunov exponents: the m -th preimage of a ball of radius r should contain a ball of radius $\approx r e^{-m\lambda_{\max}}$ and have volume $\approx r^{2n} e^{-2m\Lambda}$. If we cover Y by N balls of radius r , then the m -th preimages define a cover by Nd^{mn} regions of controlled shape. Call their union U_m .

The set Y will be shown to have Hausdorff dimension $\leq 2n - 1$. We consider cubes which intersect Y of diameter about $\frac{1}{4} r e^{-m\lambda_0}$ for all m sufficiently large. Each cube of size m lies in U_m and the number of disjoint m -th cubes which fit into U_m is about $\text{Vol } U_m / \text{Vol}(\text{cube})$. By the estimates above, this is $\leq Nd^m e^{2m(n-1)\lambda_0}$. It is straightforward to compute now that the Hausdorff measure of Y in dimension $2n - 1 + \varepsilon$, which is no greater than $\sum_m \#(\text{mesh cubes of size } m) (\text{diam})^{2n-1+\varepsilon}$, is finite for all $\varepsilon > 0$. □

2. PLURIHARMONIC MEASURE IN \mathbf{C}^n AND DYNAMICS

In this section, we give some of the necessary background on pluripotential theory in \mathbf{C}^n and its relation to polynomial dynamics. More details can be found in [Kli91], [BT87] and [Bed93].

Let $\text{PSH}(\mathbf{C}^n)$ denote the class of plurisubharmonic functions in \mathbf{C}^n . For a compact set K in \mathbf{C}^n , the pluricomplex Green's function with pole at infinity is defined as

$$G_K(z) = \sup \{v(z) : v \in \text{PSH}(\mathbf{C}^n), v \leq 0 \text{ on } K, v(z) \leq \log \|z\| + O(1) \text{ near } \infty\}.$$

See, for example, [Kli91] or [Bed93]. If G_K is continuous, the set K is said to be *regular*.

In contrast to the one dimensional setting, G_K is not necessarily pluriharmonic (or even harmonic) outside of K . It is, however, *maximal plurisubharmonic*, meaning that if v is any plurisubharmonic function on a domain Ω compactly contained in $\mathbf{C}^n - K$ with $v \leq G_K$ on $\partial\Omega$, then $v \leq G_K$ on Ω . Equivalently, the Monge-Ampere mass of G_K ,

$$\mu_K = (dd^c G_K)^n,$$

vanishes in $\mathbf{C}^n - K$. We call the measure μ_K the *pluriharmonic measure* on K , and note that it is supported in the Shilov boundary of K . In fact, if K is regular, then its support is equal to the Shilov boundary [BT87].

Pluriharmonic measure arises in the study of dynamics just as in the one dimensional setting. A polynomial endomorphism $F : \mathbf{C}^n \rightarrow \mathbf{C}^n$ is called *regular* if it can be extended holomorphically to \mathbf{CP}^n . The *degree* of F is the degree of its polynomial coordinate functions. We consider only those F of degree > 1 . The *escape rate function* of F is defined by

$$G_F(z) = \lim_{m \rightarrow \infty} \frac{1}{d^m} \log^+ \|F^m(z)\|,$$

where d is the degree of F and $\log^+ = \max\{\log, 0\}$. The function G_F agrees with the pluricomplex Green's function for the filled Julia set $K_F = \{z \in \mathbf{C}^n : F^m(z) \not\rightarrow \infty\}$. Fornæss and Sibony showed that the pluriharmonic measure μ_F on K_F is ergodic for F and a measure of maximal entropy [FS95].

By the Oseledec Ergodic Theorem, F has n Lyapunov exponents $\lambda_{\min} \leq \dots \leq \lambda_{\max}$ almost everywhere with respect to μ_F [Ose68]. We will only need the existence of the minimal, maximal, and the sum Λ of the Lyapunov exponents, which we can define as follows:

$$\begin{aligned} \lambda_{\min} &= - \lim_{m \rightarrow \infty} \frac{1}{m} \int \log \|(DF^m)^{-1}\| d\mu_F, \\ \lambda_{\max} &= \lim_{m \rightarrow \infty} \frac{1}{m} \int \log \|DF^m\| d\mu_F, \text{ and} \\ \Lambda &= \int \log |\det DF| d\mu_F. \end{aligned}$$

Briend and Duval proved that the Lyapunov exponents are all positive [BD99]; they show

$$(3) \quad \lambda_{\min} \geq \frac{1}{2} \log d,$$

where d is the degree of F . Bedford and Jonsson studied the sum of the Lyapunov exponents, and demonstrate that ([BJ00])

$$(4) \quad \Lambda \geq \frac{n+1}{2} \log d.$$

For the proof of Theorem 1, it is convenient to work in the *natural extension* (\hat{X}, F) where F is invertible (see [CFS82] and [BD99]). Let $P(F) = \bigcup_{m \geq 0} F^m(C(F))$

be the postcritical set of F and set $X = \mathbf{C}^n - \bigcup_{m \geq 0} F^{-m}(P(F))$. The space (\hat{X}, F) is the set of all bi-infinite sequences

$$\{\hat{x} = (\cdots x_{-1}x_0x_1\cdots) \in \prod_{-\infty}^{\infty} X : F(x_i) = x_{i+1}\}.$$

The map F acts on (\hat{X}, F) by the left shift. We define projections $\pi_i : (\hat{X}, F) \rightarrow X$ for all i by $\pi_i(\hat{x}) = x_i$. Since μ_F does not charge the critical locus of F , we have $\mu_F(X) = 1$. The measure μ_F lifts to a unique F -invariant probability measure $\hat{\mu}$ on (\hat{X}, F) so that $\pi_{0*}\hat{\mu} = \mu_F$.

3. PROOF OF THE MAIN THEOREM.

In this section we give a proof of the following theorem which clearly implies Theorem 1.

Theorem 2. *Pluriharmonic measure μ_F on the filled Julia set of a degree d regular polynomial endomorphism $F : \mathbf{C}^n \rightarrow \mathbf{C}^n$ satisfies*

$$\dim \mu_F \leq 2n - 2 + \frac{\log d}{\max\{\log d, \lambda_{\max}\}},$$

where λ_{\max} is the largest Lyapunov exponent of F with respect to μ_F .

We begin with a classical lemma. Statements (a) and (c) are exactly as in [BD99, Lemma 2]. We first observe that there exists a constant $C(n)$ so that for any $n \times n$ matrix A with $\|A - I\| < 1$, we have

$$(5) \quad |\det A - 1| \leq \frac{C(n)}{2} \|A - I\|.$$

Lemma 1. *Let $g : \Omega \rightarrow \mathbf{C}^n$ be a function with bounded C^2 -norm on a domain $\Omega \subset \mathbf{C}^n$ and set $M = C(n)(\|g\|_{C^2} + 1)$. Let $x \in \Omega$ be a noncritical point of g . Given $\varepsilon > 0$, let $r(x) = \frac{1 - e^{-\varepsilon/3}}{2M\|(D_xg)^{-1}\|^2}$, set $B_0 = B(g(x), r(x))$ and let B_1 be the preimage of B_0 under g containing x . Then*

- (a) g^{-1} is well defined in B_0 ,
- (b) $\text{Lip}(g|_{B_1}) \leq \|(D_xg)\|e^{\varepsilon/3}$,
- (c) $\text{Lip}(g^{-1}|_{B_0}) \leq \|(D_xg)^{-1}\|e^{\varepsilon/3}$, and
- (d) $\inf_{y \in B_1} |\det(D_yg)| \geq |\det(D_xg)|e^{-\varepsilon/3}$.

Proof. Let us consider a ball $B_2 = B(x, \rho)$, where

$$\rho = \frac{e^{\varepsilon/3} - 1}{M\|(D_xg)^{-1}\|}.$$

For each $y \in B_2$, we have

$$(6) \quad \|\mathbf{I} - (D_xg)^{-1}(D_yg)\| \leq (\|g\|_{C^2} + 1)\|(D_xg)^{-1}\|\rho \leq \frac{e^{\varepsilon/3} - 1}{C(n)},$$

and in particular, $\text{Lip}(\mathbf{I} - (D_x g)^{-1} \circ g) < 1$. If $g(y_1) = g(y_2)$ for some $y_1 \neq y_2 \in B_2$, then

$$\|y_1 - y_2\| = \|(y_1 - (D_x g)^{-1}g(y_1)) - (y_2 - (D_x g)^{-1}g(y_2))\| < \|y_1 - y_2\|,$$

which is a contradiction, and therefore g is injective on B_2 .

To establish (a), we need to know that $B_0 \subset g(B_2)$. Map g is open on B_2 , so it is enough to check that if $|y_1 - x| = \rho$, then $|g(y_1) - g(x)| > r(x)$. But this is again a direct consequence of (6).

Now, since $B_1 \subset B_2$, we have for all $y \in B_1$,

$$\|D_x g - D_y g\| \leq \|D_x g\| \|\mathbf{I} - (D_x g)^{-1}(D_y g)\| \leq \|D_x g\| \frac{e^{\varepsilon/3} - 1}{C(n)},$$

and we conclude that

$$\|D_y g\| \leq \|D_x g\| + \frac{e^{\varepsilon/3} - 1}{C(n)} \|D_x g\| \leq e^{\varepsilon/3} \|D_x g\|,$$

establishing (b).

To prove (c), observe that by (6) for $y \in B_2$,

$$\begin{aligned} \|(D_y g)^{-1}\| &\leq \|(D_x g)^{-1}\| \|\mathbf{I} - (\mathbf{I} - (D_x g)^{-1} D_y g)^{-1}\| \\ &\leq \frac{\|(D_x g)^{-1}\|}{1 - \|\mathbf{I} - (D_x g)^{-1} D_y g\|} \\ &\leq \|(D_x g)^{-1}\| e^{\varepsilon/3}. \end{aligned}$$

For (d), we compute for all $y \in B_1$ (using (5) above),

$$\begin{aligned} |\det D_y g - \det D_x g| &= |\det D_x g| |1 - \det(D_x g)^{-1} D_y g| \\ &\leq |\det D_x g| \frac{C(n)}{2} \|\mathbf{I} - (D_x g)^{-1} D_y g\| \\ &\leq |\det D_x g| \frac{1}{2} (e^{\varepsilon/3} - 1) \\ &\leq |\det D_x g| (1 - e^{-\varepsilon/3}), \end{aligned}$$

and therefore,

$$\inf_{y \in B_1} |\det D_y g| \geq |\det D_x g| e^{-\varepsilon/3}.$$

□

Let F be a regular polynomial endomorphism of \mathbf{C}^n and μ the pluriharmonic measure on the boundary of the filled Julia set of F . Denote by λ_{\min} , λ_{\max} , and Λ , the minimal, maximal, and sum of the n Lyapunov exponents of F with respect to μ_F . The space (\hat{X}, F) denotes the natural extension of F . See §2.

Lemma 2. *Given $\varepsilon > 0$, there exist an integer $N > 0$ and measurable functions r and κ on (\hat{X}, F^N) so that for almost every point \hat{x} , we have $r(\hat{x}) > 0$ and $\kappa(\hat{x}) < \infty$, and for each $m \geq 0$, a well-defined branch of F^{-mN} sending x_0 to x_{-m} with*

- (a) $F^{-mN}(B(x_0, s)) \supset B(x_{-m}, \frac{s}{\kappa(\hat{x})})e^{-mN(\lambda_{\max} + \varepsilon)}$ for all $s \leq r(\hat{x})$, and
 (b) $\text{Vol } F^{-mN}B(x_0, r(\hat{x})) \leq \kappa(\hat{x})e^{-mN(2\Lambda - \varepsilon)}$.

Proof. Choose N so that

$$0 < \lambda_{\min} - \varepsilon \leq -\frac{1}{N} \int \log \|(DF^N)^{-1}\| d\mu_F \leq \lambda_{\min},$$

and

$$\lambda_{\max} \leq \frac{1}{N} \int \log \|DF^N\| d\mu_F \leq \lambda_{\max} + \varepsilon.$$

Observe that

$$\Lambda = \frac{1}{N} \int \log |\det DF^N| d\mu_F$$

for any $N \geq 0$.

For notational simplicity, set $g = F^N$. Fix $\hat{x} \in (\hat{X}, g)$. Let

$$r(x_{-m}) = \frac{1 - e^{-\varepsilon/3}}{2M\|(D_{x_{-m}}g)^{-1}\|^2},$$

as in Lemma 1 where Ω is a large ball containing the filled Julia set of F . By the ergodic theorem applied to the function

$$\hat{x} \mapsto \log \|(D_{x_0}g)^{-1}\|,$$

we have

$$\lim_{m \rightarrow \infty} \frac{1}{m} \log \|(D_{x_{-m}}g)^{-1}\| = 0,$$

and therefore there exists a measurable function $\eta > 0$ on (\hat{X}, g) with

$$(7) \quad r(x_{-m}) \geq \eta(\hat{x})e^{-m\varepsilon/2}$$

for all $m \geq 0$ and almost every \hat{x} .

Let $B_m = B(x_0, r(x_{-1})) \cap \dots \cap g^m B(x_{-m}, r(x_{-m-1}))$. Let g^{-m} denote the inverse branch of g^m taking x_0 to x_{-m} , well-defined on B_m by Lemma 1(a). Iterating results (b), (c), and (d) of Lemma 1, we have

$$\text{Lip}(g^{-m}|B_m) \leq \|(D_{x_{-m}}g)^{-1}\| \cdots \|(D_{x_{-1}}g)^{-1}\| e^{m\varepsilon/3},$$

$$\text{Lip}(g^m|g^{-1}B_m) \leq \|D_{x_{-m}}g\| \cdots \|D_{x_{-1}}g\| e^{m\varepsilon/3},$$

and

$$\inf_{y \in g^{-m}B_m} |\det D_y g^m| \geq |\det D_{x_{-m}} g^m| e^{-m\varepsilon/3}.$$

Applying the ergodic theorem to the functions $\hat{x} \mapsto \log \|(D_{x_0}g)^{-1}\|$, $\hat{x} \mapsto \log \|D_{x_0}g\|$, and $\hat{x} \mapsto \log |\det D_{x_0}g|$, we see that there exists a measurable function $1 \leq C(\hat{x}) < \infty$ so that

$$(8) \quad \text{Lip}(g^{-m}|B_m) \leq C(\hat{x})e^{-m(N\lambda_{\min} - \varepsilon/2)},$$

$$(9) \quad \text{Lip}(g^m|g^{-m}B_m) \leq C(\hat{x})e^{m(N\lambda_{\max} + \varepsilon/2)},$$

and

$$(10) \quad \inf_{y \in g^{-m}B_m} |\det D_y g^m| \geq \frac{1}{C(\hat{x})} e^{m(N\Lambda - \varepsilon/2)}$$

for almost every \hat{x} .

Let $r(\hat{x}) = \min\{\eta(\hat{x})/C(\hat{x}), 1\}$. By induction and the estimates (7) and (8), we establish that $B(x_0, r(\hat{x}))$ is contained in B_m for all $m \geq 0$. By (9), we have

$$B(x_0, r(\hat{x})) \supset g^m B(x_{-m}, \frac{r(\hat{x})}{2C(\hat{x})} e^{-m(N\lambda_{\max} + \varepsilon/2)}).$$

By (10), the volume of $g^{-m}B(x_0, r(\hat{x}))$ is bounded by

$$\text{Vol}(g^{-m}B(x_0, r(\hat{x}))) \leq \text{Vol}(B(x_0, r(\hat{x}))) C(\hat{x})^2 e^{-m(2N\Lambda - \varepsilon)}.$$

The Lemma is proved upon setting $\kappa(\hat{x}) = 2C(\hat{x})^2 \text{Vol } B_1$. \square

Proof of Theorem 1. For fixed $\varepsilon > 0$, let N , r , and κ be as in Lemma 2. Set $g = F^N$ and $d = \deg g$. Let λ_{\min} , λ_{\max} , and Λ denote the minimal, maximal, and sum of the Lyapunov exponents of the iterate g .

Choose a set $\hat{A} \subset (\hat{X}, g)$ and $r_0, \kappa_0 > 0$ so that

$$\hat{A} \subset \{\hat{x} \in (\hat{X}, g) : r(\hat{x}) \geq r_0 \text{ and } \kappa(\hat{x}) \leq \kappa_0\},$$

$\pi_0(\hat{A})$ has compact closure in \mathbf{C}^n and $\hat{\mu}(\hat{A}) > 0$. Let $\hat{Y} \subset (\hat{X}, g)$ be the set of all points whose forward orbit under g lands in \hat{A} infinitely often. By ergodicity, $\hat{\mu}(\hat{Y}) = 1$. Let $Y = \pi_0(\hat{Y}) = \{x_0 : \hat{x} \in \hat{Y}\}$, so $\mu(Y) = 1$, and let $A_i = \pi_{-i}(\hat{A})$. Observe that

$$Y = \bigcap_{l \geq 0} \bigcup_{m \geq l} A_m.$$

We will show that the Hausdorff dimension of Y is bounded above by $2n - 2 + \frac{\log d}{\lambda_0} + \frac{4\varepsilon}{\lambda_0}$, where $\lambda_0 = \max\{\lambda_{\max}, \log d\}$. As Y has full measure and ε is arbitrary, this will prove the theorem.

For a ball B in \mathbf{C}^n , let $\frac{1}{2}B$ denote a concentric ball with half the radius. Let Σ denote a finite collection of balls B of radius r_0 so that the balls $\frac{1}{2}B$ cover A_0 . For each point $y \in A_m$, select $\hat{y} \in \hat{A}$ so that $y = \pi_{-m}(\hat{y})$. Choose element B of Σ so that $\pi_0(\hat{y})$ lies in $\frac{1}{2}B$. Let B_y be the preimage of $g^{-m}B$ containing y . The collection of these B_y for all $y \in A_m$ defines the finite cover Σ_m of A_m .

If σ is the number of elements in Σ , then the number of elements of Σ_m is no greater than σd^{mn} . Let $\lambda_0 = \max\{\log d, \lambda_{\max}\}$. We will establish the following two properties of the cover Σ_m :

- (I) The union $\bigcup_{B \in \Sigma_m} B$ contains an $\frac{r_0}{4\kappa_0} e^{-m(\lambda_0 + \varepsilon)}$ -neighborhood of A_m , and
- (II) $\text{Vol}(\bigcup_{B \in \Sigma_m} B) \leq \sigma d^m \kappa_0 e^{-m(2\lambda_0 - \varepsilon)}$.

Observe first that for each $y \in A_m$, the set $B_y \in \Sigma_m$ contains a ball of radius $\frac{r_0}{4\kappa_0} e^{-m(\lambda_{\max} + \varepsilon)}$ around y by Lemma 2. Of course, $\lambda_{\max} \leq \lambda_0$, giving (I).

To establish (II), we observe that as $g^m B_y \subset B(\pi_0(\hat{y}), r_0)$ for each $B_y \in \Sigma_m$, Lemma 2(b) implies that

$$\text{Vol } B_y \leq \kappa_0 e^{-m(2\Lambda - \varepsilon)}.$$

Summing over the volumes of all elements in Σ_m , we write

$$\text{Vol} \left(\bigcup_{B \in \Sigma_m} B \right) \leq \sigma d^{mn} \kappa_0 e^{-m(2\Lambda - \varepsilon)}.$$

By (4), Λ is bounded below by $\frac{n+1}{2} \log d$, and by (3), each Lyapunov exponent is bounded below by $\frac{1}{2} \log d$. Combining these gives $\Lambda \geq \frac{n-1}{2} \log d + \lambda_0$ and we obtain statement (II).

We define a covering \mathcal{M}_m of A_m to be the collection of all mesh cubes of edge length $\frac{1}{\sqrt{2n}} \frac{r_0}{4\kappa_0} e^{-m(\lambda_0 + \varepsilon)}$ which intersect A_m . Let $c = \frac{1}{\sqrt{2n}} \frac{r_0}{4\kappa_0}$. By property (I), each cube is contained in an element of Σ_m . The number of cubes in \mathcal{M}_m is bounded above by the volume $\text{Vol}(\cup_{B \in \Sigma_m} B)$ divided by the volume of each cube. That is,

$$\begin{aligned} |\mathcal{M}_m| &\leq \frac{\sigma d^{mn} \kappa_0 e^{-m(2\lambda_0 - \varepsilon)}}{c^{2n} e^{-2mn(\lambda_0 + \varepsilon)}} \\ &= \frac{\sigma \kappa_0}{c^{2n}} d^m e^{2(n-1)m\lambda_0} e^{(2n+1)m\varepsilon}. \end{aligned}$$

We now show that Hausdorff measure of Y in dimension $2n - 2 + \frac{\log d}{\lambda_0} + \frac{4\varepsilon}{\lambda_0}$ is finite, completing the proof. Fix $\delta > 0$. Choose $l \geq 0$ so that the mesh cubes in \mathcal{M}_m are of diameter $\delta_m \leq \delta$ for each $m \geq l$. The union of the elements of \mathcal{M}_m for $m \geq l$ covers Y . Therefore,

$$\begin{aligned} m_{2n-2+\frac{\log d}{\lambda_0}+\frac{4\varepsilon}{\lambda_0}}(Y) &\leq \sum_{m \geq l} |\mathcal{M}_m| (\delta_m)^{2n-2+\frac{\log d}{\lambda_0}+\frac{4\varepsilon}{\lambda_0}} \\ &\leq C \sum_{m \geq l} d^m e^{2(n-1)m\lambda_0} e^{(2n+1)m\varepsilon} e^{-m(\lambda_0 + \varepsilon)} \left(2n-2+\frac{\log d}{\lambda_0}+\frac{4\varepsilon}{\lambda_0} \right) \\ &= C \sum_{m \geq l} e^{-m\varepsilon} \left(1+\frac{\log d}{\lambda_0}+\frac{4\varepsilon}{\lambda_0} \right) \\ &\leq C \sum_0^\infty e^{-m\varepsilon} < \infty \end{aligned}$$

□

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