

Conformal Invariance for Certain Models of the Bond–Triangular Type

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Abstract: Convergence to SLE_6 of the percolation exploration process for a correlated bond–triangular type model studied in [5] is established, which puts the said model in the same universality class as the standard site percolation model on the triangular lattice [13]. The result is proven for all domains with boundary (upper) Minkowski dimension less than 2, following the general streamlined approach outlined in [12].

Keywords: Universality, percolation, conformal invariance.

1 Introduction

In this note, we investigate the percolation type model invented in [5] and prove the existence of the scaling limit for the associated exploration process (the model will be briefly reviewed in subsection 3.1). As anticipated, the limiting process is described by chordal SLE_6 – which coincides with the well-known limiting behavior of ordinary site percolation on the triangular lattice ([13], [4]). The result, while not unexpected, establishes for the first time a non-trivial example of universality of the sort anticipated since the 1960s: in particular, via the common continuum limit, the behavior of the present model on the lattice at long distances is asymptotically identical to that of the critical triangular site percolation model.

In the previous work [5], Cardy’s formula was established for the model under consideration – at least for a certain class of domains. According to the methodology outlined by S. Smirnov in [12], to be recapitulated in Section 2.3, the existence of this conformal invariant in a model of the percolation–type should, in principle, lead directly to the above–mentioned scaling limit. While by and large this is the substance of the present work, many of the technical results necessary to implement this program were not *a priori* available. Specifically, tightness estimates which ensure the existence of limiting paths require an elaborate

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proof and, further, Cardy’s formula for the system at hand has to be established in greater generality than had been done in [5]. Finally, needless to say, it is necessary to actually define an exploration process at the microscopic level.

We remark that the present rendition of the program in [12] actually provides a general framework from which chordal SLE_6 can be shown to describe the limit of critical 2D percolation explorer processes. While no specific (axiomatized) claims will be made in this direction, as will become clear, no microscopic details of the model in [5] are essential. Indeed, what is actually required boils down to the above–mentioned plus various well-known properties typical of critical 2D percolation models. Specifically, for a general percolation model one needs

- Russo–Seymour–Welsh (RSW) theory: uniform estimates for probabilities of crosscuts (of either type) on all scales plus the ability to stitch smaller crossings together without substantial degradation of the estimates – FKG–type inequalities.
- BK–type inequalities whereby probabilities of separated path type events can be estimated in terms of the individual probabilities.
- A self–replicating definition of an explorer process and a class of *admissible* domains with the property that this class is preserved under the operation of deleting the beginning of a typical explorer path in an admissible domain.
- The validity of Cardy’s Formula for the above–mentioned admissible domains.

Since it is conceivable that various general results established here will be useful in other contexts, the model specifics and the general methodologies will be segregated into separate sections – Sections 3 and 4, respectively.

2 Conformal Invariance of the Scaling Limit

2.1 SLE: Definitions and Notations

As the title of this subsection indicates, we will briefly review the relevant notions of Löwner evolution – mostly for the purpose of fixing notation. Let Ω be a domain with two boundary prime ends a and c .

Definition 2.1. Let $\{\Omega_t\}_{t=0}^\infty$ be a strictly decreasing family of subdomains of Ω ($t \in [0, \infty)$) which is Carathéodory continuous with respect to c , such that $\Omega_0 = \Omega$ and $\bigcap_{t=0}^\infty \Omega_t = c$. Then we call $\{\Omega_t\}_{t=0}^\infty$ a Löwner chain.

Let \mathbb{H} denote the upper–half plane of \mathbb{C} . We can select some conformal map $g_0 : \Omega \rightarrow \mathbb{H}$ such that $g_0(a) = 0$ and $g_0(c) = \infty$. The family of conformal maps $g_t : \Omega_t \rightarrow \mathbb{H}$ normalized

such that $g_t(b) = \infty$ and $g_t \circ g_0^{-1}(z) = z + \frac{A(t)}{z} + o(1/z)$ are continuous in t . We now reparameterize time so that $A(t) = 2t$.

We call γ a *crosscut* in Ω from a to c if it is the preimage of a non-self-crossing curve from 0 to ∞ in \mathbb{H} under g_0 . Note that γ is allowed to touch itself but not to cross itself. We define Ω_t to be the connected component of $\Omega \setminus \gamma_{[0,t]}$ containing c . It's easy to see that Ω_t is a Löwner chain if and only if the following two conditions are satisfied for every $t > 0$:

(L1) $\gamma_t \in \overline{\Omega_{t-\varepsilon}}$, $\forall \varepsilon > 0$

and

(L2) $\exists \delta_n \rightarrow 0$, $\forall \varepsilon > 0$, $\gamma_{t-\delta_n} \in \Omega_{t-\delta_n-\varepsilon}$.

Under these conditions, we can reparametrize γ so that the maps g_t 's satisfy the following celebrated Löwner equation:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t},$$

where $\lambda_t = g_t(\gamma(t))$ is a continuous function.

On the other hand, the solution of the Löwner equation for any initial conformal map $g_0 : \Omega \rightarrow \mathbb{H}$ and any continuous function $\lambda(t)$ defines a Löwner chain, but not necessarily a curve (see [8] for a complete discussion). λ_t is called the *driving function* of Ω_t .

If we take the very special function $\lambda_t = B(\kappa t)$, where $B(t)$ is one-dimension Brownian motion started at zero, then the corresponding random Löwner chain is called the Stochastic (or Schramm) Löwner Evolution with parameter κ , SLE_κ . It is known that at least for Jordan domains it is a.s. generated by a random curve (see [11]). We would be particularly interested in the case $\kappa = 6$.

2.2 Statement of the Main Theorem and Lemmas

The principal result of this note may be stated as follows:

Main Theorem. *Let Ω be a domain with $M(\partial\Omega) < 2$. Let a and c be two prime ends at the boundary of the domain. Let us endow the space of curves from a to c in Ω with the sup-norm metric defined in Definition 4.3. Let μ_ε be the probability measure on random curves inherited from the exploration process of the percolation problem described in §3.1 and §3.2 with lattice spacing ε . Then*

$$\mu_\varepsilon \xrightarrow{\mathcal{L}} \mu_0,$$

where μ_0 has the law of chordal SLE_6 from a to c .

The main non-trivial property of the percolation model which will be used in the proof of the theorem is Cardy's Formula:

Lemma 2.2 (Cardy's Formula). *Let (Ω, a, b, c, d) be a conformal rectangle – that is to say, a domain with boundary prime ends a, b, c, d , listed in counter-clockwise order. Assume*

that $M(\partial\Omega) < 2$. Let $C_\varepsilon(\Omega, a, b, c, d)$ denote the probability that there exists a blue crossing from $[a, b]$ to $[c, d]$ on the ε -lattice approximation of Ω . Consider the (unique) conformal map which takes (Ω, a, b, c, d) to $(\mathbb{H}, 1-x, 1, \infty, 0)$, where, clearly, $0 < x < 1$ and $x = x(\Omega, a, b, c, d)$. Then

$$\lim_{\varepsilon \rightarrow 0} C_\varepsilon(\Omega, a, b, c, d) = F(x) := \frac{\int_0^x (s(1-s))^{-2/3} ds}{\int_0^1 (s(1-s))^{-2/3} ds} \quad (1)$$

We will prove this lemma in the necessary generality in §3.4.

We start by listing the properties of a typical exploration process. We prove the properties in §4.1. In what follows, let μ' be any limit point, in the weak*-Hausdorff topology on compact sets, of μ_ε .

Definition 2.3. Let Ω be a domain. Let $\delta \gg \eta > 0$ and let $\gamma : [0, 1] \rightarrow \Omega$ be a parametrized curve. We say that γ has a δ - η doubleback if there exists disjoint subsegments I_1 and I_2 of $[0, 1]$, with $\text{diam}(\gamma(I_1)) \geq \delta$, $\text{diam}(\gamma(I_2)) \geq \delta$, and such that the segments $\gamma(I_1)$ and $\gamma(I_2)$ are η -close in the sup-norm.

Lemma 2.4 (No Doubleback). *Let Ω denote a domain of the type described, and let $\gamma \in \text{supp}(\mu')$. Let δ, η satisfy $\eta < c_1\delta$, with a particular c_1 of order unity. Then for all δ sufficiently small, there are additional constants c_2 and c_3 of order unity such that for all ε sufficiently small, the μ_ε -probability (and hence the μ' -probability) of a δ - η doubleback is bounded above by*

$$\frac{c_2}{\delta^2} \cdot e^{-c_3\delta/\eta}.$$

Definition 2.5. Let Ω be a domain. Let $\Delta_2 \gg \Delta_1 > 0$ and let $\gamma : [0, 1] \rightarrow \Omega$ be a parameterized curve. We say γ has no Δ_2 - Δ_1 double visits to the boundary if it is the case that whenever there exist $t_1 < t_2$ such that $\text{dist}(\gamma(t_1), \partial\Omega) < \Delta_1$ and $\text{dist}(\gamma(t_1), \gamma(t_2)) > \Delta_2$, then $\forall t > t_2$, $\text{dist}(\gamma(t), \gamma(t_1)) > \Delta_2$.

Lemma 2.6 (No Double Visits Near the Boundary). *Let Ω denote a domain of the type described. The μ' -probability of a Δ_2 - Δ_1 double visit to the boundary tends to zero as $\frac{\Delta_1}{\Delta_2} \rightarrow 0$.*

Lemma 2.7 (Multi-Arm Estimates). *Let $D(\eta, l)$ denote the circular annulus with inner radius η and outer radius l . Consider the event of a (i) 5-arm crossing of $D(\eta, l)$ and (ii) 6-arm crossing of $D(\eta, l)$. Then the 5-arm event has probability bounded above by $(\eta/l)^2$ while the 6-arm event has probability bounded above by $(\eta/l)^{2+\kappa}$ for some $\kappa > 0$.*

Using the previous general estimates, we establish the following important properties of any weak*-limiting point μ' . The proofs can be found in §4.2, §4.3, and §4.4, respectively.

Lemma 2.8 (Tightness). *Let μ' be any limit point, in the weak* Hausdorff topology on compact sets, of μ_ε . Then μ' gives full measure to Löwner curves in Ω from a to c .*

Furthermore,

Lemma 2.9 (Admissibility). *The limit point μ' gives full measure to curves with upper Minkowski dimension less than $2 - \psi'$ for some $\psi' > 0$.*

Finally, we need a lemma providing some notion of uniform continuity of crossing probabilities. Before we can state the lemma, we note that in Lemma 2.8 (and 2.9), a stronger notion of convergence is available. Indeed, for domains which are regular enough, the results of [1] provide weak convergence in the distance provided by the sup-norm:

$$\text{dist}(\gamma_1, \gamma_2) = \inf_{\varphi_1, \varphi_2} \sup_t |\gamma_1(\varphi_1(t)) - \gamma_2(\varphi_2(t))|,$$

where the infimum is over all possible parametrizations. For our purposes – where prime ends are a concern – we will consider a weighted sum of the distances within various regions between the curves. The construction is standard and precise details will be spelled out in Subsection 4.2. We will denote the appropriate distance by **Dist** (see Definition 4.3).

Lemma 2.10 (Restricted Uniform Continuity). *Let Ω be a domain with $M(\partial\Omega) < 2$ and a, b, c, d be four points or prime ends on $\partial\Omega$. Let $\vartheta, \Delta > 0$ and consider curves which start at a and end at c . Then there exists a set $\Xi_{\vartheta, \Delta}$ of such curves and $\eta > 0$, such that if $\gamma_1 \in \Xi_{\vartheta, \Delta}$, $M(\gamma_2) < 2$, $M(\gamma_1) < 2$, and **Dist** $(\gamma_1, \gamma_2) < \eta$, then $\forall T \geq 0$ such that $\gamma_1([0, T])$ and $\gamma_2([0, T])$ do not visit the Δ neighborhood of c , provided that b, c, d are at the boundaries of both domains $\Omega \setminus \gamma_1((0, T])$ and $\Omega \setminus \gamma_2((0, T])$,*

$$|C_\varepsilon(\Omega \setminus \gamma_1([0, T]), \gamma_1(T), b, c, d) - C_\varepsilon(\Omega \setminus \gamma_2([0, T]), \gamma_2(T), b, c, d)| < \frac{1}{2}\vartheta$$

and for all ε sufficiently small,

$$\mu_\varepsilon(\Xi_{\vartheta, \Delta}) > 1 - \frac{1}{2}\vartheta,$$

with the same for μ' .

2.3 Proof of the Main Theorem

Let us show how to derive our Main Theorem from the preceding lemmas. We closely follow the strategic initiative outlined in the expositions of [12]; moreover, the “expansion at infinity” technique we will use here first appeared in [9] in the proof of the convergence of the loop-erased random walk to SLE₂.

Fix Ω with $M(\partial\Omega) < 2$ and two boundary prime ends a and c . Note that here we denote by c (instead of b) the finishing point of the Exploration Process. Let us note that

the collection of measures (μ_ε) defined by the Exploration Process on ε -lattice is weakly precompact as a set of regular measures defined on the space of compact subsets of $\text{Clos}(\Omega)$ with the Hausdorff metric. Thus to prove the Main Theorem it is enough to show that any weak limit point μ' , of μ_ε , has the law of SLE_6 from a to c in Ω .

By Lemma 2.8, μ' gives full measure to Löwner curves. Let w_t be the random driving function of the curve. To finish the proof, we need to show that w_t has the law of B_{6t} , where B_t is the standard one dimensional Brownian Motion started at 0.

We will use the following a priori estimate on w_t .

Lemma 2.11 (*A priori estimate*).

$$\mathbb{P}[w_t > n] \leq C_1 \exp\left(-C_2 \frac{n}{\sqrt{t}}\right),$$

for some absolute constants C_1 and C_2 .

The lemma implies, in particular, that all moments of w_t are finite. We postpone the proof of this lemma until §4.5.

Let us add two boundary prime ends b and d so that (a, b, c, d) are listed counter-clockwise and give the discrete Exploration Process from a to c , \mathbb{X}_t^ε , the Löwner parametrization. Now fix $t > 0$. By definition, the faces on the right side of the Exploration Process are blue, and the faces on the left side are yellow. A blue crossing from $[a, b]$ to $[c, d]$ can either touch the exploration path $\mathbb{X}_{[0,t]}^\varepsilon$, or avoid it. In any case, it produces a blue crossing between $[\mathbb{X}_t^\varepsilon, b]$ to $[c, d]$ in $\Omega \setminus \mathbb{X}_{[0,t]}^\varepsilon$ (note that since we are only deleting the line segments of the explorer process and not the associated hexagons, this set is always connected). And vice versa, any blue crossing between $[\mathbb{X}_t^\varepsilon, b]$ to $[c, d]$ in $\Omega \setminus \mathbb{X}_{[0,t]}^\varepsilon$ produces a blue crossing from $[a, b]$ to $[c, d]$ in Ω .

This means that we can write the following Markov identity for the crossing probabilities

$$C_\varepsilon\left(\Omega \setminus \mathbb{X}_{[0,t]}^\varepsilon, \mathbb{X}_t^\varepsilon, b, c, d\right) = C_\varepsilon\left(\Omega, a, b, c, d \mid \mathbb{X}_{[0,t]}^\varepsilon\right), \quad (2)$$

so

$$\mathbb{E}_{\mu_\varepsilon}\left[C_\varepsilon\left(\Omega \setminus \mathbb{X}_{[0,t]}^\varepsilon, \mathbb{X}_t^\varepsilon, b, c, d\right)\right] = C_\varepsilon(\Omega, a, b, c, d). \quad (3)$$

Now let us fix $t > s > 0$. Then by the same reasoning we have

$$\mathbb{E}_{\mu_\varepsilon}\left[C_\varepsilon\left(\Omega \setminus \mathbb{X}_{[0,t]}^\varepsilon, \mathbb{X}_t^\varepsilon, b, c, d\right) \mid \mathbb{X}_{[0,s]}^\varepsilon\right] = C_\varepsilon\left(\Omega \setminus \mathbb{X}_{[0,s]}^\varepsilon, \mathbb{X}_s^\varepsilon, b, c, d\right). \quad (4)$$

We now pass to the limit as $\varepsilon \rightarrow 0$. Let μ' be any limiting point of the measures μ_ε . Consider some $\mathbb{X}_{[0,t]} \in \Xi_{\vartheta, \Delta}$ (see statement of Lemma 2.10) and its Löwner parametrization η -neighborhood $\mathcal{N}_{\eta,t}$. Note that $\mathcal{N}_{\eta,t}$ is certainly contained in the sup-norm η neighborhood, and in fact has positive measure: by examining the inverse image of the η neighborhood

after a conformal map, we see that it contains a small sup-norm neighborhood. Let Ω_t be the connected component of $\Omega \setminus \mathbb{X}_{[0,t]}$ such that $c \in \partial\Omega_t$. By Lemma 2.9, we can assume that $M(\mathbb{X}) < 2$.

The capacity at c of the curve $\mathbb{X}_{[0,t]}$ is $2t$ so either the curve stays at some distance Δ away from c , or it has to pass through a fixed rectangle of large conformal modulus, and hence for small enough Δ , $\mathbb{X}_{[0,t]} \in \Xi_{\vartheta,\Delta}$, with large probability. This means that Lemma 2.10 applies and hence if η and ε are small enough and $\mathbb{X}_{[0,s]}^\varepsilon \in \mathcal{N}_{\eta,s}$, then $C_\varepsilon(\Omega \setminus \mathbb{X}_{[0,s]}^\varepsilon, \mathbb{X}_s^\varepsilon, b, c, d)$ is close to $C_\varepsilon(\Omega_s, \mathbb{X}_s, b, c, d)$. A second application of Lemma 2.10 implies that for any $\mathbb{X}_{[0,t]}^\varepsilon \in \mathcal{N}_{\eta,t}$, the value of $C_\varepsilon(\Omega \setminus \mathbb{X}_{[0,t]}^\varepsilon, \mathbb{X}_t^\varepsilon, b, c, d)$ is close to $C_\varepsilon(\Omega_t, \mathbb{X}_t, b, c, d)$, provided that ε and η are small enough *and provided that b and d are still in $\partial\Omega_t$* . Averaging over all possible continuations of $\mathbb{X}_{[0,s]}$ and taking small enough η and ε , we get from equation (4)

$$\begin{aligned} & |C_\varepsilon(\Omega_s, \mathbb{X}_s, b, c, d \mid b, d \in \partial\Omega_s) - \mathbb{E}_{\mu_\varepsilon} C_\varepsilon(\Omega_t, \mathbb{X}_t, b, c, d \mid \mathbb{X}_{[0,s]}, b, d \in \partial\Omega_t)| \\ & \leq \mathbb{P}_{\mu_\varepsilon}(b \notin \partial\Omega_t \text{ or } d \notin \partial\Omega_t) + \delta \end{aligned} \quad (5)$$

for an arbitrarily small δ .

By Lemma 2.2, $\lim_{\varepsilon \rightarrow 0} C_\varepsilon(\Omega_s, \mathbb{X}_s, b, c, d) = C_0(\Omega_s, \mathbb{X}_s, b, c, d)$. If we now let $\varepsilon \rightarrow 0$ and $\eta \rightarrow 0$ in (5) and use the bounded convergence theorem, we get the approximate Markov property for the measure μ'

$$\begin{aligned} & |C_0(\Omega_s, \mathbb{X}_s, b, c, d \mid b, d \in \partial\Omega_s) - \mathbb{E}_{\mu'} C_0(\Omega_t, \mathbb{X}_t, b, c, d \mid \mathbb{X}_{[0,s]}, b, d \in \partial\Omega_t)| \\ & \leq \mathbb{P}_{\mu'}(b \notin \partial\Omega_t \text{ or } d \notin \partial\Omega_t). \end{aligned} \quad (6)$$

Notice that the map

$$h_t(z) = \frac{g_t(z) - g_t(d)}{g_t(b) - g_t(d)},$$

where $g_t(z)$ is the Löwner map, maps the rectangle $(\Omega_t, \mathbb{X}_t, b, c, d)$ conformally onto

$$\left(\mathbb{H}, \frac{w_t - g_t(d)}{g_t(b) - g_t(d)}, 1, \infty, 0 \right).$$

By Cardy's identity (Lemma 2.2),

$$C_0(\Omega_t, \mathbb{X}_t, b, c, d) = F\left(\frac{g_t(b) - w_t}{g_t(b) - g_t(d)}\right). \quad (7)$$

Using equation (7), we can rewrite the equation (6) as

$$\begin{aligned} & \left| F\left(\frac{g_s(b) - w_s}{g_s(b) - g_s(d)}\right) - \mathbb{E}_{\mathbb{X}_{[s,t]}} \left(F\left(\frac{g_t(b) - w_t}{g_t(b) - g_t(d)} \mid \mathbb{X}_{[0,s]}\right) \mid b, d \in \partial\Omega_t \right) \right| \\ & \leq \mathbb{P}(b \notin \partial\Omega_t \text{ or } d \notin \partial\Omega_t). \end{aligned} \quad (8)$$

Let us now define the flows $b_t := g_t(b) - w_t$, $d_t := g_t(d) - w_t$. We also put $b_t = 0$ if $b \notin \partial\Omega_t$ and $d_t = 0$ if $d \notin \partial\Omega_t$. Thus $b_0 = g_0(b) > 0$, $d_0 = g_0(d) < 0$. Now let us start moving b and d closer to c . We will move them in such a manner that $d_0 = -2b_0$. By the Löwner parametrization in the half plane, $g_t(g_0^{-1}(z)) = z + 2t/z + \mathcal{O}(1/z^2)$ for $z \rightarrow \infty$.

Let us first provide the expansion of the first term of the left hand side of (8), when b_0 is close to infinity.

$$\begin{aligned} F\left(\frac{g_s(b) - w_s}{g_s(b) - g_s(d)}\right) &= F\left(\frac{b_0 - w_s + 2s/b_0 + \mathbf{O}(1/b_0^2)}{b_0 + 2s/b_0 + \mathbf{O}(1/b_0^2) + 2b_0 + s/b_0 + \mathbf{O}(1/b_0^2)}\right) \\ &= F(1/3) - w_s/3F'(1/3)1/b_0 + (s/3F'(1/3) + w_s^2/12F''(1/3))1/b_0^2 + \mathbf{O}((w_s/b_0)^3) \\ &= A + \frac{Bw_s}{b_0} + \frac{C(w_s^2 - 6s)}{b_0^2} + \mathbf{O}((w_s/b_0)^3) \end{aligned} \quad (9)$$

for some A , B , and C .

By the same reasoning, the second expression on the left hand side of (8) is equal to

$$\begin{aligned} \mathbb{E}_{\mathbb{X}_{[s,t]}}\left(F\left(\frac{g_t(b) - w_t}{g_t(b) - g_t(d)}\right) \Big| \mathbb{X}_{[0,s]}\right) & \Big| b, d \in \partial\Omega_t \\ &= A + \frac{B\mathbb{E}(w_t|w_s)}{b_0} + \frac{C\mathbb{E}((w_t^2 - 6t)|w_s)}{b_0^2} + \mathbf{O}(|\mathbb{E}[w_t/b_0^3]|) \end{aligned} \quad (10)$$

The last expression of (8) decays exponentially with b_0 . Indeed, assume that $b \notin \partial\Omega_t$ (the case $d \notin \partial\Omega_t$ is analogous). The boundedness of the derivative at infinity of the map $g_t(g_0^{-1}(z))$ forces the image of the exploration process up to time t under the map g_0 to stay inside the box $[0, b_0] \times [0, 2\sqrt{t}]$. By Cardy's formula, such a probability decays exponentially with b_0 .

We can compare the coefficients of $1/b_0$ and $1/b_0^2$ of the two terms on the left hand side equation (8) with the corresponding coefficients in equations (9) and (10) to get

$$\mathbb{E}(w_t|w_s) = w_s, \quad \mathbb{E}(w_t^2 - 6t|w_s) = w_s^2 - 6s. \quad (11)$$

Therefore both w_t and $w_t^2 - 6t$ are continuous martingales, which, by the usual characterization of Brownian Motion, implies that w_t has the law of B_{6t} . \square

3 The Model

3.1 Review of Model

Here we give a quick description of the model under study. For more details see Section 2.2 of [5]. The model takes place on the hexagon tiling of the 2D triangular site lattice:

hexagons are yellow, blue and sometimes split; half and half. Connectivity for us is defined by adjacent shapes (of the same color) sharing an edge segment in common. Our description of the model starts with a particular local arrangement of hexagons:

Definition 3.1. A *flower* is the union of a particular hexagon with its six neighbors. The central hexagon we call an *iris* and the outer hexagons we call *petals*. We number the petals from 1 to 6, starting from the one directly to the right of the iris. All hexagons which are not flowers will be referred to as *filler*.

Let $\Omega \subset \mathbb{C}$ be a domain, which for simplicity we may regard as being a finite connected subset of the hexagon lattice, with certain hexagons being designated as irises (this determines the flowers) which is known as a *floral arrangement*. There are three restrictions on placement of irises: (i) no iris is a boundary hexagon, (ii) there are at least two non-iris hexagons between each pair of irises, and (iii) ultimately in infinite volume the irises have a periodic structure with 60° symmetries.

We are now ready to define the statistical properties of our model.

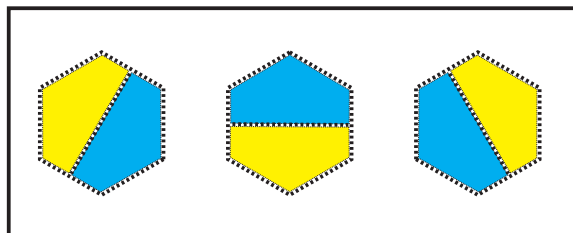


Figure 1: The three allowed “split” states of the hexagon. Note that these correspond to single bond occupancy events in the corresponding up-pointing triangle in the bond-triangular lattice percolation problem.

Definition 3.2. Let Ω be a domain with floral arrangement $\Omega_{\mathfrak{F}}$.

- Petals and hexagons in the complement of flowers are only allowed to be blue or yellow with probability $1/2$.
- Irises can be blue, yellow, or mixed (one of three ways c.f. Figure 1) with probabilities a , a , or s , so that $2a + 3s = 1$ and in addition,

$$a^2 \geq 2s^2.$$

- The exceptional configurations on the flowers, which we call *triggers*, are configurations where there are three yellow and three blue petals, with one pair of blue (and hence also yellow) petals contiguous. In these configurations, the irises can now only be blue or yellow, each with probability $1/2$.

Note that triggering is the only source of (very short range) correlation in this model. Everything else is configured independently.

Finally, it is remarked that the total of five possible configurations on a hexagon correspond to the eight possible configurations on (up-pointing) triangles – of which there are five distinct connectivity classes. It is not hard to see, by checking local connectivity properties, that the model described is a representation of a correlated percolation model on the triangular bond lattice.

It was shown in [5] Theorem 3.10 that our model exhibits all the typical properties of a 2D percolation model at criticality. Cardy’s formula for this model was the main result of [5] (Theorem 2.4). More specifically, let $\Omega \subset \mathbb{C}$ be a domain with piecewise smooth boundary which is conformally equivalent to a triangle. Let us denote the three boundaries and prime ends of interest by $\mathcal{A}, e_{AB}, \mathcal{B}, e_{BC}, \mathcal{C}, e_{AC}$, in counterclockwise order. We endow Ω with an approximate discretization (with hexagons) on a lattice of scale $\varepsilon = 1/N$ and a floral arrangement $\Omega_{\mathfrak{F}_\varepsilon}$. Let z be the vertex of a hexagon in $\Omega_{\mathfrak{F}_\varepsilon}$. We define the discrete crossing probability function $U_\varepsilon^Y(z)$ to be the indicator function of the event that there is a blue path connecting \mathcal{A} and \mathcal{B} , separating z from \mathcal{C} , with similar definitions for $V_\varepsilon^Y(z)$ and $W_\varepsilon^Y(z)$ and the blue versions of these functions. Then taking the scaling limit in an appropriate fashion (for more details see Section 2.3 of [5]), we have, e.g.

$$\lim_{\varepsilon \rightarrow 0} U_\varepsilon^Y = h_{\mathcal{C}},$$

where $h_{\mathcal{C}}$ is a so-called Carleson–Cardy function: It is harmonic, and on the up-pointing equilateral triangle with base \mathcal{C} being the unit interval, it is equal to $\frac{2}{\sqrt{3}} \cdot y$ which satisfies Cardy’s formula. The functions $h_{\mathcal{A}}$ and $h_{\mathcal{B}}$ are defined similarly.

3.2 The Exploration Process

We now give a (microscopic) definition of the percolation exploration process tailored to our system at hand. We must start with a precise prescription of how to construct our domains.

We start with a bounded domain $\Omega \subset \mathbb{C}$ which has $M(\partial\Omega) < 2$. In the preceding, $M(S)$ denotes the (upper) Minkowski dimension of the set S which, as usual, is defined as

$$M(S) = \limsup_{\vartheta \rightarrow 0} \frac{\log \mathcal{N}(\vartheta)}{\log(1/\vartheta)},$$

where $\mathcal{N}(\vartheta)$ is the number of boxes of side length ϑ needed to cover the set. Let a and b be two prime ends and consider hexagons of the ε -tiling of \mathbb{C} . It is assumed that within this tiling (with fixed origin of coordinates) the locations of all irises/flowers/fillers are predetermined. We define Ω_ε to be the union of all fillers and flowers whose closure lies in

the interior of Ω . It is assumed that ε is small enough that both a and b are in the same lattice connected component of the tiling. Other components, if any, will not be discarded but will only play a peripheral rôle. With the exception of flowers, the boundary of the domain will be taken as the usual internal lattice boundary, which consists of the points of the set which have neighbors not belonging to the set. If the lattice boundary cuts through a flower, then the whole flower is included as part of the boundary. The notation for this lattice boundary will be $\partial_\varepsilon\Omega_\varepsilon$.

Consider points $a_\varepsilon, b_\varepsilon$ which are on $\partial_\varepsilon\Omega_\varepsilon$ and are vertices of hexagons. We call $(\Omega_\varepsilon, \partial_\varepsilon\Omega_\varepsilon, a_\varepsilon, b_\varepsilon)$ *admissible* if

- Ω_ε contains no partial flowers.
- $\partial_\varepsilon\Omega_\varepsilon$ can be decomposed into two lattice connected sets consisting of hexagons and/or halves of boundary irises, one of which is colored blue and one of which is colored yellow, such that a_ε and b_ε lie at the points where the two sets join and such that the blue and yellow paths are valid paths following the connectivity and statistical rules of our model; in particular, the coloring of these paths do not lead to flower configurations that have probability zero.
- a_ε and b_ε lie at the vertices of hexagons, such that of the three hexagons sharing the vertex, one of them is blue, one of them is yellow, and the third is in the interior of the domain. (See Figure 2).

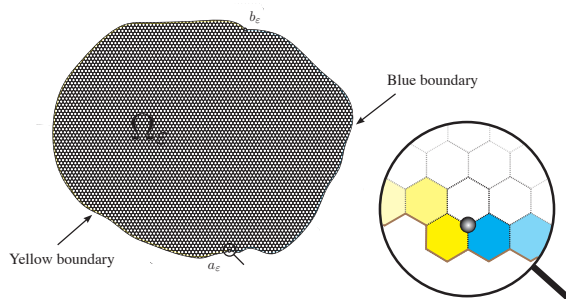


Figure 2: The setup for the definition of the exploration process.

We remark that in the case of boundary flowers (and other sorts of clusters on the boundary) it is not necessary to color *all* the hexagons/irises. Indeed the coloring scheme need not be unique – it is only required that a boundary coloring of the requisite type can be selected.

It is not hard to see that the domains $(\Omega_\varepsilon, \partial_\varepsilon\Omega_\varepsilon, a_\varepsilon, b_\varepsilon)$ converges to $(\Omega, \partial\Omega, a, b)$ in the sense that $\partial_\varepsilon\Omega_\varepsilon$ and Ω_ε converge respectively to $\partial\Omega$ and Ω in the Hausdorff metric and in

the Caratheodory metric with respect to any point inside Ω . Also, there exists a_ε and b_ε which converge respectively to a and b as $\varepsilon \rightarrow 0$. Notice that the latter convergence is really in terms of the preimages under the Riemann map of the relevant domain.

Geometrically, the *exploration process* produces, in any percolation configuration on Ω_ε , the unique interface connecting a_ε to b_ε , i.e. the curve separating the blue lattice connected cluster of the boundary from that of the yellow. We denote this interface by γ^ε . Dynamically, the exploration process is defined as follows: Let $\mathbb{X}_0^\varepsilon = a_\varepsilon$. Given $\mathbb{X}_{t-1}^\varepsilon$, it may be necessary to color new hexagons in order to determine the next step of the process. (In particular, $\mathbb{X}_{t-1}^\varepsilon$ is “usually” at the vertex of a hexagon which has not yet been colored.) We color any necessary undetermined hexagons according to the following rules:

- If the undetermined hexagon is a filler hexagon, we color it blue or yellow with probability $1/2$.
- If the undetermined hexagon is a petal or an iris, we color it blue or yellow or mixed with the conditional distribution given by the hexagons of the flower which are already determined.
- If a further (petal) hexagon is needed, it is colored according to the conditional distribution given by the iris and the other hexagons of the flower which have already been determined.

We are now ready to describe how to determine \mathbb{X}_t^ε :

- If $\mathbb{X}_{t-1}^\varepsilon$ is not adjacent to an iris, \mathbb{X}_t^ε will be equal to the next hexagon vertex we can get to in such a way that blue is always on the right of the segment $[\mathbb{X}_{t-1}^\varepsilon, \mathbb{X}_t^\varepsilon]$.
- If $\mathbb{X}_{t-1}^\varepsilon$ is adjacent to an iris, then the state of the iris is determined as described above, after which the exploration path can be continued (keeping blue on the right) until a petal is hit. The color of the petal will now be determined (according to the proper conditional distribution) and \mathbb{X}_t^ε will equal one of the two possible vertices common to the iris and the new petal which keeps the blue region to the right of the final portion of the segments joining $\mathbb{X}_{t-1}^\varepsilon$ to \mathbb{X}_t^ε .

In particular, it is noted that at the end of each step, we always wind up on the vertex of a hexagon. We denote by γ_t^ε the actual value taken by the random variable \mathbb{X}_t^ε .

We state without proof some properties of our exploration process.

Proposition 3.3. *Let $\gamma_{[0,t]}^\varepsilon$ be the line segments formed by the process up till time t , and $\Gamma_{[0,t]}^\varepsilon$ the hexagons revealed by the exploration process. Let $\partial_\varepsilon \Omega_\varepsilon^t = \partial_\varepsilon \Omega_\varepsilon \cup \Gamma_{[0,t]}^\varepsilon$ and let $\Omega_\varepsilon^t = \Omega_\varepsilon \setminus \Gamma_{[0,t]}^\varepsilon$. Then, the quadruple $(\Omega_\varepsilon^t, \partial_\varepsilon \Omega_\varepsilon^t, \mathbb{X}_t^\varepsilon, b_\varepsilon)$ is admissible. Furthermore, the exploration process in Ω_ε^t from \mathbb{X}_t^ε to b_ε has the same law as the original exploration process from a_ε to b_ε in Ω_ε conditioned on $\Gamma_{[0,t]}^\varepsilon$.*

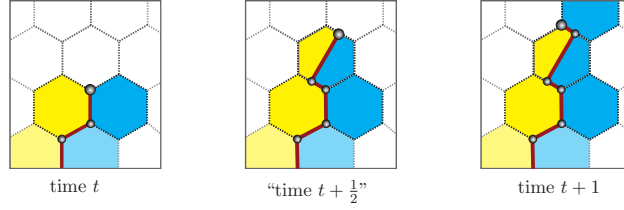


Figure 3: “Multistep” procedure by which the exploration process gets through a mixed hexagon.

3.3 A Restricted BK–Inequality

Here we will prove an inequality that will be needed for proofs in several other places.

Suppose A and B are two events. Then the BK inequality [2] states that (for suitable probability spaces) the probability of the *disjoint* occurrence of A and B is bounded above by the product of their probabilities. The most general version of this is Reimer’s inequality [10] (see also [3] for more background and a self–contained proof), which holds for arbitrary product probability spaces. For the model at hand, we do not have a product probability space; Reimer’s inequality would, in the present context, yield the desired result only for *flower* disjoint events. Unfortunately, we have need of a stronger statement; specifically, for disjoint path–type events where the individual paths may use the same flower. In fact, as the following example demonstrates, a general BK inequality does not hold in our system. However, as we later show, an abridged version holds for path–type events.

Example 3.4. Let A be the event of a blue connection between petals 1, 4, and 5 (without any requirement on the color of the petals 1, 4, and 5), and let $B = \{\text{petals 1, 4, 5 are blue}\}$. Observe that B and B^c are defined entirely on the petals 1, 4, 5, whereas A is defined on the complementary set. Therefore we have $A \cap B^c = A \circ B^c$. By Example 6.1 of [5], we know that $\mathbb{P}(A \cap B) < \mathbb{P}(A)\mathbb{P}(B)$. But this immediately implies that $\mathbb{P}(A \circ B^c) > \mathbb{P}(A)\mathbb{P}(B^c)$.

Before tending to the detailed analysis of flowers, let us first introduce the notion of disjoint occurrence for non–negative random variables.

Definition 3.5. Let $a_i, b_j \geq 0$ and let

$$X = \sum_1^n a_i \mathbf{1}_{A_i}, \quad Y = \sum_1^m b_j \mathbf{1}_{B_j},$$

where $A_i \cap A_k = \emptyset$ for $i \neq k$ and $B_j \cap B_l = \emptyset$ for $j \neq l$. We define

$$X \circ Y = \sum_{i,j} a_i b_j \mathbf{1}_{A_i \circ B_j}.$$

If the usual BK inequality holds then linearity immediately gives

$$\mathbb{E}(X \circ Y) \leq \mathbb{E}(X)\mathbb{E}(Y).$$

We will be working with this slight generalization; what we have in mind is the hexagon disjoint occurrence of paths, and in the case of paths of different colors, sharing of the iris may occur. To be precise, we have the following definition:

Definition 3.6. Let $\Lambda_{\mathfrak{F}}$ denote a flower arrangement and let S and T denote sets in $\Lambda_{\mathfrak{F}}$ which contain no irises. Let $X_{S,T}^b$ denote the indicator of the event that all hexagons in S and T are blue and that there is a blue path – possibly including irises – connecting S and T . Similarly we define $X_{S,T}^y$ to be the yellow version of this event. Now if S' and T' are two other sets of $\Lambda_{\mathfrak{F}}$ which are disjoint from S and T and also do not contain irises, then we may define $X_{S,T}^b \circ X_{S',T'}^b$ in accord with the usual fashion. However, for present purposes, in the event corresponding to $X_{S,T}^b \circ X_{S',T'}^y$, the two paths may share a mixed iris.

Lemma 3.7. *Let $X_{S_1,T_1}^{\ell_1}, X_{S_2,T_2}^{\ell_2}, \dots, X_{S_n,T_n}^{\ell_n}$ be the indicator functions of path-type events as described in Definition 3.6, where $\ell_i \in b, y$, then*

$$\mathbb{E}(X_{S_1,T_1}^{\ell_1} \circ X_{S_2,T_2}^{\ell_2} \circ \dots \circ X_{S_n,T_n}^{\ell_n}) \leq \mathbb{E}(X_{S_1,T_1}^{\ell_1})\mathbb{E}(X_{S_2,T_2}^{\ell_2}) \dots \mathbb{E}(X_{S_n,T_n}^{\ell_n}).$$

Proof. Our proof is slightly reminiscent of the proof of Lemma 6.2 in [5]. Let σ denote a configuration of petals and filler and let I denote a configuration of irises. We will use induction; first we prove the statement for the case of exactly one flower (i.e. supposing there is only one flower in all of $\Lambda_{\mathfrak{F}}$) and two paths, whose indicator functions we denote by X and Y . We write

$$\mathbb{E}(X \circ Y) = \mathbb{E}_{\sigma}[\mathbb{E}_I(X \circ Y | \sigma)].$$

If we can show that $\mathbb{E}_I(X \circ Y | \sigma) \leq \mathbb{E}_I(X | \sigma) \circ \mathbb{E}_I(Y | \sigma)$, then we may apply the BK-inequality to the outer expectation to yield the desired result since, on the outside, the measure is independent. It is clear that the function $\mathbb{E}(X \circ Y | \sigma)$ can only take on five different values; we write

$$\begin{aligned} \mathbb{E}(X \circ Y | \sigma) &= 1 \cdot \mathbf{1}_{\mathcal{O}(X \circ Y)}(\sigma) \\ &\quad + (a + s) \cdot \mathbf{1}_{A_1(X \circ Y)}(\sigma) \\ &\quad + (1/2) \cdot \mathbf{1}_{A_2(X \circ Y)}(\sigma) \\ &\quad + (a + 2s) \cdot \mathbf{1}_{A_3(X \circ Y)}(\sigma) \\ &\quad + s \cdot \mathbf{1}_{\mathcal{F}(X \circ Y)}(\sigma), \end{aligned} \tag{12}$$

where e.g.

$$\mathcal{O}(X \circ Y) = \{\sigma \mid \mathbb{E}(X \circ Y | \sigma) = 1\}.$$

It is not difficult to see that $\mathcal{O}(X \circ Y)$ is the set of σ configurations where $X \circ Y$ has occurred on the complement of the iris. The remaining terms warrant some discussion. We first point out that these terms correspond to configurations where the flower is pivotal

for the achievement of at least one of X and Y , and, due to the nature of the events in question, petal arrangements in these configurations satisfy certain constraints. For instance, configurations in A_3 must exhibit a petal arrangement such that one of the paths is in a position where it must transmit through the iris, which can be accomplished by the preferred color or two of the split configurations; the flower must *not* be in a triggering configuration and, needless to say, the other path has already occurred (independent of the iris).

Finally we observe that $\sigma \in \mathcal{F}(X \circ Y)$ implies that both paths must use the iris and therefore can only occur when the paths in question have different colors. It is not hard to see, via petal counting, that $\mathcal{F}(X \circ Y)$ forces the alternating configuration of petals and that indeed, we have a situation of a “parallel transmission” through the iris, with exactly one iris configuration which achieves both desired transmissions. We also note that in similar expressions for $\mathbb{E}(X|\sigma)$ and $\mathbb{E}(Y|\sigma)$, the corresponding terms $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ will be empty, since e.g. if the path is blue and some iris is capable of achieving the transmission, then certainly the pure blue iris will achieve the transmission.

Let us expand $\mathbb{E}(X|\sigma) \circ \mathbb{E}(Y|\sigma)$ in the sense defined above:

$$\begin{aligned}
\mathbb{E}(X|\sigma) \circ \mathbb{E}(Y|\sigma) &= 1 \cdot \mathbf{1}_{\mathcal{O}(X) \circ \mathcal{O}(Y)}(\sigma) \\
&\quad + (a + s) \cdot [\mathbf{1}_{\mathcal{O}(X) \circ A_1(Y)}(\sigma) + \mathbf{1}_{A_1(X) \circ \mathcal{O}(Y)}(\sigma)] \\
&\quad + (1/2) \cdot [\mathbf{1}_{\mathcal{O}(X) \circ A_2(Y)}(\sigma) + \mathbf{1}_{A_2(X) \circ \mathcal{O}(Y)}(\sigma)] \\
&\quad + (a + 2s) \cdot [\mathbf{1}_{\mathcal{O}(X) \circ A_3(Y)}(\sigma) + \mathbf{1}_{A_3(X) \circ \mathcal{O}(Y)}(\sigma)] \\
&\quad + (a + s)^2 \cdot [\mathbf{1}_{A_1(X) \circ A_1(Y)}(\sigma)] \\
&\quad + \mathcal{R}(a, s, \sigma),
\end{aligned} \tag{13}$$

where $\mathcal{R}(a, s, \sigma)$ contains all the remaining terms in the expansion, e.g. the terms

$$(1/2)(a + s) \cdot [\mathbf{1}_{A_1(X) \circ A_2(Y)}(\sigma) + \mathbf{1}_{A_2(X) \circ A_1(Y)}(\sigma)] \tag{14}$$

and

$$(a + s)(a + 2s) \cdot [\mathbf{1}_{A_1(X) \circ A_3(Y)}(\sigma) + \mathbf{1}_{A_3(X) \circ A_1(Y)}(\sigma)]. \tag{15}$$

We claim that (14) will evaluate to zero for each σ : In the first term, $A_1(X)$ requires that the petals exhibit a configuration which precludes a trigger and $A_2(Y)$ requires the petals to exhibit a configuration which leads to a trigger, and similarly for the second term. The terms in (15) may or may not evaluate to zero for all σ *a priori*, but in any case will not be needed.

Now we match up the terms in (12) and (13) and demonstrate that indeed $\mathbb{E}(X \circ Y|\sigma) \leq \mathbb{E}(X|\sigma) \circ \mathbb{E}(Y|\sigma)$. First note that $\mathcal{O}(X \circ Y) = \mathcal{O}(X) \circ \mathcal{O}(Y)$. Next, as discussed previously, we see that $A_i(X \circ Y) \subset (A_i(X) \circ \mathcal{O}(Y)) \cup (\mathcal{O}(X) \circ A_i(Y))$, $1 \leq i \leq 3$. Finally, and this is the key case, we claim that $\mathcal{F}(X \circ Y) \subset A_1(X) \circ A_1(Y)$. This follows from the observation we

made before, which is that if $\sigma \in \mathcal{F}(X \circ Y)$, then we must see the alternating configuration on the flower, requiring next to nearest neighbor transmissions through the iris for both paths; such a σ certainly lies in $A_1(X) \circ A_1(Y)$. Thus we are done, assuming that $(a+s)^2 \geq s$ – but this is equivalent to the statement that $a^2 \geq 2s^2$.

We have established the claim for the case of a single flower and two paths. Next we may induct on the number of flowers, as follows. Suppose now the claim is established for $K - 1$ flowers. We can now let σ denote the configuration of all petals, filler, and irises of the first $K - 1$ flowers. We condition on σ as above and adapt the notation so that the sets \mathcal{O} , A_i 's, and \mathcal{F} correspond to the K^{th} flower. The argument can then be carried out exactly as above to yield the result for K flowers and two paths. Finally we induct on the number of paths. Suppose the claim is true for $n - 1$ paths. Since the \circ operation is associative, we consider $(X_1 \circ \cdots \circ X_{n-1}) \circ X_n$, where the X_i 's are indicator functions of the n paths. We simply view $(X_1 \circ \cdots \circ X_{n-1})$ as a single path-type event and repeat the proof (note that the analogue of equation (13) may now contain non-trivial \mathcal{F} -type terms; these are immaterial since what is listed is already enough for an upper bound). This argument is sufficient since no more than two paths may share an iris under any circumstance. \square

3.4 Generalization of Cardy's Formula for $M(\partial\Omega) < 2$

Here we provide a proof of Lemma 2.2. As described in §3.1, [5] contains a proof of Cardy's formula for piecewise smooth domains, so what is needed here is a generalization to domains Ω with $M(\partial\Omega) < 2$. What we will prove is the following:

Lemma 3.8. *Let Ω denote any conformal triangular domain with $M(\partial\Omega) < 2$. Let U_ε^Y , V_ε^Y and W_ε^Y denote the crossing probability functions as defined in Ω for the lattice at scale ε . Then for the model as defined in §3.1, we have*

$$\lim_{\varepsilon \rightarrow 0} U_\varepsilon^Y = h_C,$$

with similar results for V_ε^Y and W_ε^Y and the corresponding blue versions of these functions, where h_A , h_B and h_C are the Cardy–Carleson functions.

To prove the current statement, we start by repeating the proof in [5] up to Lemma 7.2 and Corollary 7.4 – the one place where the assumption on a piecewise smooth boundary is used. We now give a quick exposition of the (relevant portions of the) strategy of proof in [5]. The idea (directly inherited from [13]) is to represent the derivative of the crossing probability functions as a “three-arm” event, e.g. two blue paths and one yellow path from some point to the boundaries, with all paths disjoint, and then derive Cauchy–Riemann type identities by switching the color of one of the arms.

In order to accomplish this color switching in our model, it was necessary to introduce a *stochastic* notion of disjointness. This amounted to the introduction of a large class of

random variables which indicate whether or not a percolation configuration contributes to the event of interest (e.g. a blue path from \mathcal{A} to \mathcal{B} , separating z from \mathcal{C}). We call the restrictions and permissions given by these random variables **-rules*. The *-rules may at times call a self-avoiding path illegitimate if it contains *close encounters*, i.e. comes within one unit of itself; on the other hand, the *-rules may at other times permit a path which is not self-avoiding but in fact shares a hexagon. Thus the *-rules are invoked only at shared hexagons and close encounter points of a path. When a close encounter or sharing at a hexagon is required to achieve the desired path event it is called an *essential lasso point*.

The fact that these *-rules may be implemented by random variables in a fashion which allows color switching is the content of Lemma 3.17 in [5]. The strategy was then to first prove that the *-version of e.g. the function U_ε , denoted U_ε^* , converges to h_C , then show that in the limit the starred and unstarred versions of the function coincide. For the current work, the precise statement is as follows:

Lemma 3.9. *Let Ω be a domain such that*

$$M \equiv M(\partial\Omega) < 2.$$

Let z denote a point in Ω . Consider the (blue version of the) function $U_\varepsilon(z)$ as defined in §3.1. Let $U_\varepsilon^(z)$ denote the version of U_ε with the *-rules enforced. Then,*

$$\lim_{\varepsilon \rightarrow \infty} |U_\varepsilon^*(z) - U_\varepsilon(z)| = 0.$$

In particular, on closed subsets of Ω , the above is uniformly bounded by a constant times a power of ε .

Before we begin the proof we need some standard percolation notation.

Definition 3.10. Back on the unit hexagon lattice, if L is a positive integer, let B_L denote a box of side L centered at the origin. Further, let $\Pi_5(L)$ denotes the event of five disjoint paths, not all of the same color, starting from the origin and ending on ∂B_L . Now let $m < n$ be positive integers, and let $\Pi(n, m)$ denote the event of five long arms, not all of the same color, connecting ∂B_m and ∂B_n . We use the notation $\pi_5(n)$ and $\pi_5(n, m)$ for the probabilities of $\Pi_5(n)$ and $\Pi_5(n, m)$, respectively.

Proof of Lemma 3.9. We set $N = \varepsilon^{-1}$ and, without apology, we will denote the relevant functions by U_N . For convenience we recap the proof of Lemma 7.2 in [5] (with one minor modification). Let us first consider the event which is contained in both the starred and unstarred versions of the u-function, namely the event of a self-avoiding, non-self-touching path separating z from \mathcal{C} , etc. We will denote the indicator function of this event by u_N^- . Similarly, let us define an event, whose indicator is u_N^{*+} , that contains both the starred and unstarred versions: this is the event that a separating path of the required type exists, with

no restrictions on self-touching, and is allowed to share hexagons provided that permissions are granted. It is obvious that

$$\mathbb{E}[\mathfrak{u}_N^{*+} - \mathfrak{u}_N^-] \geq |U_N^* - U_N|. \quad (16)$$

We turn to a description of the configurations, technically on (ω, X) (the enlarged probability space which include the permissions), for which $\mathfrak{u}_N^{*+} = 1$ while $\mathfrak{u}_N^- = 0$. In such a configuration, the only separating paths contain an *essential* lasso point which, we remind the reader, could be either a shared hexagon or a closed encounter pair. Let us specify the lasso point under study to be the last such point on the journey from \mathcal{A} to \mathcal{B} (i.e. immediately after leaving this point, the path must capture z without any further sharing or self-touching, then return to this point and continue on to \mathcal{B}). For standing notation, we denote this “point” by z_0 . A variety of paths converge at z_0 : certainly there is a blue path from \mathcal{A} , denoted $B_{\mathcal{A}}$, a blue path to \mathcal{B} , denoted $B_{\mathcal{B}}$, and an additional loop starting from z_0 (or its immediate vicinity) which contains z in its interior. The loop we may view as two blue paths of comparable lengths, denoted L_z^1 and L_z^2 . However, since the lasso point was deemed to be essential, there are two additional yellow arms emanating from the immediate vicinity of z_0 . These yellow arms may themselves encircle the blue loop and/or terminate at boundary \mathcal{C} . We denote these yellow paths $Y_{\mathcal{C}}^1$ and $Y_{\mathcal{C}}^2$.

Since z_0 is the last lasso point on the blue journey from \mathcal{A} to \mathcal{B} , we automatically get that the two loop arms are *strictly* self-avoiding. Also, without loss of generality, we may take the yellow arms to be strictly self-avoiding. Further, by Lemma 4.3 of [5], we may take either the portion of the path from \mathcal{A} to z_0 to be strictly self-avoiding or the portion of the path from \mathcal{B} to z_0 to be strictly self-avoiding. To summarize, we have six paths emanating from z_0 , four blue and two yellow, with all paths disjoint except for possible sharings between $B_{\mathcal{A}}$ and $B_{\mathcal{B}}$. For simplicity, let us start with the connected component of z in $\Omega \setminus (\alpha_k \cup \beta_k \cup \gamma_k)$ where $\alpha_k, \dots, \gamma_k$ are short crosscuts defining the prime ends e_{AB}, \dots, e_{CA} , respectively. It is noted that in this restricted setting, the various portions of the boundary are at a finite (macroscopic) distance from one another. Thus, on a mesoscopic scale, we are always near only a single boundary.

The case where z_0 is close to z is handled by RSW-type bounds (see proof of Lemma 7.2 in [5]). The terms where z_0 is in the interior follow from the 5^+ arms; these arguments are the subject of Lemma 7.2 and Lemma 7.3 in [5]. We are left with the case where say z_0 is within a distance of N^λ of the boundary but outside some box of side N^{μ_2} separating e_{AB} from z .

Let $\delta > 0$. For N large enough, $\partial\Omega$ can be covered by no more than $J_\delta N^{M+\delta-\lambda}$ boxes of side N^λ . Now we take these boxes and expand by a factor of, say, two and we see that the region within N^λ of the boundary can be covered by $J_\delta N^{M+\delta-\lambda}$ boxes of side $2N^\lambda$. We surround each of these boxes by a box of side N^{μ_1} , where $\mu_2 > \mu_1 > \lambda$.

Now suppose z_0 is inside the inner box. We still have the six arms $B_{\mathcal{A}}, B_{\mathcal{B}}, L_z^1, L_z^2, Y_{\mathcal{C}}^1$ and $Y_{\mathcal{C}}^2$, but since z_0 is now close to some boundary, we expect some arm(s) to be short

(i.e. shorter than N^λ). We note that the box of side μ_1 is still away from e_{AB} , and therefore we cannot have more than one of B_A and B_B be short due to being close to the boundary. Also, since z must be a distance of order N away from the boundary, z is outside of both of these boxes and therefore both L_z^1 and L_z^2 are long. The upshot is that regardless of which boundary z_0 is close to, one and only one of the six arms will be short: If z_0 is close to \mathcal{A} (respectively \mathcal{B}), then B_A (respectively B_B) will be short, and if z_0 is close to \mathcal{C} , then a moment's reflection will show that only one of the yellow arms will be short.

What we have is then five long arms and one short arm emanating from the immediate vicinity of z_0 , and these arms either end on some boundary or the boundary of the outer box of side N^{μ_1} . For reasons which will momentarily become clear, we will now perform a color switch. Topologically, the two yellow arms separate L_z^1 and L_z^2 from B_A and B_B . Denote the outer box by B_{μ_1} and consider now the region $T \equiv \Omega \cap B_{\mu_1}$. The two yellow arms together form a “crosscut” (in the sense of Kesten [7]) of T . This crosscut separates T into two disjoint regions T_b and T_l , where T_b contains B_A and B_B and T_l contains L_z^1 and L_z^2 . We condition on the crosscut which minimizes the area of T_l . Next we apply Lemma 4.3 of [5] to reduce the blue arm adjacent to the longer of the two yellow arms – which we take to be Y_C^1 – to be strictly self-avoiding, which without loss of generality we assume to be B_A . Since B_A forms a crosscut of T_b , there is a crosscut which maximizes the region which contains B_B , which we denote Ω_B . The region Ω_B is now an unconditioned region, and we may apply Lemma 3.17 of [5] to switch the color of B_B from blue to yellow, while preserving the probability. The resulting yellow path we will denote Y_B .

We now have three blue paths and three yellow paths. The blue paths are now all strictly self-avoiding. Y_C^1 is still strictly self-avoiding, but the path Y_B may very well interact with (i.e. share hexagons with, due to the *-rules) Y_C^2 . If indeed there is sharing, then let $\hat{Y} = Y_B \cup Y_C^2$ be the geometric union of the two paths. \hat{Y} can then be reduced to be a strictly self-avoiding path, which we now denote Y . In any case, we now have (at least) five long paths emanating from z_0 , three blue and two yellow, with the yellow paths separating the blue paths, and with all paths strictly self-avoiding. The probability of such an event is certainly bounded above (possibly strictly since the boxes will most likely intersect Λ^c) by the full space event $\Pi_5(N^{\mu_1}, 2N^\lambda)$ – see Definition 3.10. The upshot of Lemma 5 of [7] is that

$$\pi_5(N^{\mu_1}, 2N^\lambda) \leq C \left(\frac{N^\lambda}{N^{\mu_1}} \right)^2, \quad (17)$$

where C is a constant. This result can, almost without apology, be taken verbatim from [7]; the proviso concerned “relocation of arms” which was discussed in the first paragraph of the proof of Lemma 7.3 in [5]. We consider (17) to be established.

If we sum over all such boxes of side $2N^\lambda$, we find that the contribution from the near boundary regions is a constant times

$$N^{M+\delta-\lambda+2\lambda-2\mu_1} = N^{M+\delta+\lambda-2\mu_1}.$$

Since $M < 2$, we may first choose δ and λ such that $M + \delta + \lambda < 2$, then we may first choose μ_2 and then μ_1 large enough so that the exponent is negative. Finally let us take care of the crosscuts. For k large, the event that a path emanates from the crosscut e.g. β_k and goes to \mathcal{B} tends to 0 as $k \rightarrow \infty$ (uniformly in N for all N sufficiently large): Indeed, a path emanating from β_k must pass through a minimal sized “bottleneck” – whose diameter, η_k , tends to 0 as $k \rightarrow \infty$. This implies the existence of a long path emanating from a small region, with probability which vanishes with some power of η_k . Similarly for the other two prime ends. All estimates are uniform in z provided z remains a fixed non-zero (Euclidean) distance from the boundary.

The proof of Lemma 3.9 for V_N and W_N are the same. \square

Corollary 7.4 of [5] concerned the difference between the blue and yellow versions of these functions (Cauchy–Riemann relations are only established for color-neutral sums). However, the argument of Corollary 7.4 in [5] reduced the difference between the two colored versions to six arm events in the bulk and five arm events near the boundary, to which the above arguments can be applied. Replacing Lemma 7.2 (and Lemma 7.3) in [5] with Lemma 3.9 gives a proof of Lemma 3.8.

Now we are in position to prove Lemma 2.2.

Proof of Lemma 2.2. By conformal invariance of the limiting functions (and an application of the Schwarz–Christoffel transform), it is a well established fact that limiting behavior of the three Carleson–Cardy functions implies Cardy’s Formula. While we actually establish convergence of the lattice functions to h_A, h_B and h_C , we do so for $z \in \Omega$. To get $C_\varepsilon \rightarrow C_0$ technically we must let $z \rightarrow \partial\Omega$ before the continuum limit has been taken, but this is not so very difficult. First we will rephrase our argument in the (Ω, a, b, c, d) language. Suppose we are interested in a (blue) crossing from the arc \overline{ab} to the arc \overline{cd} : this crossing probability can be retrieved as the value of one of the h functions as the distance between its argument and d tends to zero.

Now let Υ be a crosscut in Ω which separates d from c and the boundary arc \overline{ab} . Let Ω_0 be the connected component of $\Omega \setminus \Upsilon$ containing d at the boundary. Let $\eta > 0$ and $z \in \Omega_0$ a distance at most η from d and at least $\sqrt{\eta}$ from Υ , then we can set up of the order of $\log \eta$ annuli of fixed modulus, separating z and a point of the prime end d from Υ . The probability of at least one blue circuit (in the relevant portion) of such an annulus differs from unity by some power of η (these sorts of “Russo–Seymour–Welsh” arguments will be described in greater detail later, e.g. in the proof of Lemma 2.9). In the presence of such a circuit, the crossing event given by this “interior” point z would lead to the desired crossing. \square

4 Technical Lemmas and Proofs

4.1 Typical Explorer Paths

Here we provide proofs for the properties of the typical explorer path. Recall that μ_ε is a measure generated by the percolation explorer process on the ε -lattice scale in a domain Ω with two distinguished boundary prime ends a and c and μ' is any limit point of μ_ε in the weak*-Hausdorff topology.

Proof of Lemma 2.4. To verify the above, it is sufficient to verify the statement in the measures μ_ε for ε sufficiently small. Thus let $\delta \ll 1$ and η small as desired and then ε much smaller than the scale set by η . (We are envisioning that η/δ actually tends to zero.) For k large but of order unity, let us grid the domain Ω into pixels of scale $k^{-1}\delta$. It's not difficult to see that the event in question necessitates an easy-way crossing of a rectangle of this scale with aspect ratio of order unity. If there are two such crossings (of the same curve and therefore not intersecting) a distance – in the sup-norm – η of one another, then by explicit percolation estimates, this probability is of the order $e^{-c_3\delta/\eta}$. Since there are at most the order of δ^{-2} such boxes, the claim follows. \square

Proof of Lemma 2.6. First we observe that if the explorer process has a Δ_2 - Δ_1 double visit to the boundary, then this implies at least a 3-arm event on the scale of $\Delta_2 : \Delta_1$ near the boundary. This three-arm event can be viewed as the difference of crossing probabilities of certain conformal rectangles (see Figure ??), and is therefore conformally invariant. More precisely, let's take a uniformization map f of Ω to \mathbb{D} ; then the Δ_1 neighborhood of the boundary will be locally distorted by $|f'(z)|$ for some point z in the Δ_1 neighborhood of the boundary. The probability of such a three-arm event in the disc will be proportional to $\left(\frac{\Delta_1}{\Delta_2}\right)^2 |f'(z)|^2$. Integrating over all z which are Δ_1 close to the boundary, we see that the probability of the existence of such an event is proportional to the area of the image of the Δ_1 neighborhood of $\partial\Omega$ under the map f divided by Δ_2^2 and hence the probability of any such event tends to 0 as $\left(\frac{\Delta_1}{\Delta_2}\right) \rightarrow 0$. \square

Proof of Lemma 2.7. Let us rescale back so that the lattice spacing is of order unity and the diameter of Ω_ε is of order N . Then the five arm event in $D(\eta, l)$ is the event of five crossings between circles of radius ηN and lN . Approximating by square annular regions if necessary, we find that by [7], the probability of this event is bounded above by $\left(\frac{\eta}{l}\right)^2$. (This requires the same proviso about flower disjointness, when necessary, as eluded to in the proof of Lemma 3.9. Such issues were dispensed with in the proof of Lemma 7.3 in [5].) To bound the 6-arm event, we note that if we let A denote the event of one crossing in the annular region, then the probability of A is bounded by $\left(\frac{\eta}{l}\right)^\kappa$ by standard Russo-Seymour-Welsh arguments (of the type that will be exemplified in the proof of Lemma 2.9). Then letting

B be the event of 5 crossings in the annular region and applying Lemma 3.7 to $A \circ B$ yields the desired result. \square

4.2 Limit is Supported on Curves

Here we provide a proof of Lemma 2.8, i.e. any limit point of the μ_ε 's is supported on curves. Our proof will utilize three additional lemmas, but first we must discuss crosscuts.

As alluded to several times before, we envision Ω as the conformal image of the upper half plane via some map $\phi : \mathbb{H} \rightarrow \Omega$. The prime end a is defined in the usual fashion as the set of all limit points of sequences $\phi(z_n)$, $z_n \rightarrow z_a$, where $z_a \in \mathbb{R}$ is fixed. Alternatively, define

$$A_k = \overline{\phi(\{|z - z_a| \leq 1/k, \text{Im}z > 0\})},$$

then the prime end a can be defined as $\cap_k A_k$. We also define

$$\alpha_k = \phi(\{|z - z_a| = 1/k, \text{Im}z > 0\})$$

as the k^{th} crosscut of a . We define similar quantities for b and call them B_k and β_k , respectively. Finally let us also define $\gamma^{k,\varepsilon}$ to be the curve formed by γ^ε from the last exit from A_k to the first entrance into B_k after this last exit from A_k . We remark that for finite k , with non-zero probability, γ^ε will form multiple crossings of the region $\Omega_k \equiv \Omega \setminus (A_k \cup B_k)$, but this probability tends to zero as $k \rightarrow \infty$, as can be seen by applying Cardy's formula (or by using Russo–Seymour–Welsh type arguments, c.f. the proof of Lemma 2.9).

Lemma 4.1. *Consider the domain Ω_k and let μ'_k be a limit point of the measures on the curves $\gamma^{k,\varepsilon}$. Then the μ'_k 's are supported on Hölder continuous curves. Moreover, the weak convergence to μ'_k can be taken with respect to the topology defined by the sup-norm distance between curves.*

Proof. These claims follow from the result of [1]. We claim that on Ω_k , the curves $\{\gamma^{k,\varepsilon}\}$ satisfy hypothesis H1 of [1] namely: The probability of multiple crossings of circular shells (intersected with Ω_k) goes to zero as the multiplicity gets large. This is clear if we consider circular shells with the outer radius sufficiently small, dependent on k . Indeed, for R less than some R_k , there is no possibility of both blue and yellow boundary inside Ω_k intersected with the corresponding circular shell. Thus we must only rule out many crossings of $\gamma^{k,\varepsilon}$ of the circular shell either in the presence of no boundary or in the presence of a monochrome boundary – with the rate of decay which increases to infinity with the number of traversals. These estimates follow from straightforward repeated applications of Lemma 3.7. \square

For the next lemma, we need another definition. We say that we have a *jump* of magnitude (at least) ℓ if

$$\gamma^{k+\ell,\varepsilon} \cap (\Omega_\varepsilon \setminus (A_k \cup B_k)) \neq \gamma^\varepsilon \cap (\Omega_\varepsilon \setminus (A_k \cup B_k)).$$

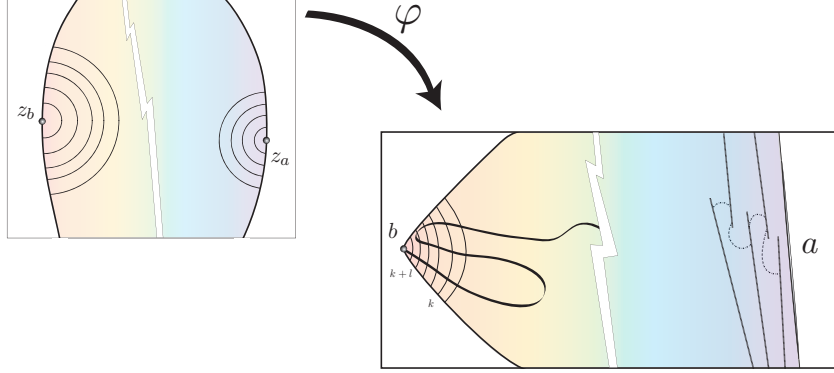


Figure 4: A jump of magnitude l occurring in the vicinity of the prime end b .

For an illustration see Figure 4.

Lemma 4.2. *For every k the magnitude of the jump stays bounded as $\varepsilon \rightarrow 0$ with probability one.*

Proof. The modulus of the conformal rectangle $(A_k \setminus A_{k+l}, \alpha_k, \alpha_{k+l})$ tends to infinity as $l \rightarrow \infty$. We observe that in the event of a jump there must be a crossing of this conformal rectangle. As $\varepsilon \rightarrow 0$, we may utilize Cardy's formula to show that the probability of such a crossing is bounded by some constant $\delta_{k,\ell}$ which tends to zero as $\ell \rightarrow \infty$, i.e. as $\varepsilon \rightarrow 0$, the probability of jumps of unbounded magnitude is zero. Analogous arguments hold for the B_k 's. \square

We are now prepared to define the **Dist** function referred to in the statement of Lemma 2.10.

Definition 4.3. Let $\lambda_\ell > 0$ be fixed numbers that satisfy $\sum_\ell \lambda_\ell = 1$, e.g. $\lambda_\ell = 2^{-\ell}$. If γ_r and γ_g are two curves in Ω from a to b , we denote, as before, γ_r^ℓ (or $\gamma_r^{\ell,\varepsilon}$) the portion of the curve in Ω_ℓ , etc. Let $d_\ell(\gamma_r, \gamma_g)$ denote the usual sup norm distances between γ_r^ℓ and γ_g^ℓ . Then we define

$$\mathbf{Dist}(\gamma_r, \gamma_g) = \sum_\ell \lambda_\ell d_\ell(\gamma_r, \gamma_g).$$

Proof of Lemma 2.8. We first establish that any limiting measure μ' is supported on curves from a to b . By Lemma 4.2, a μ' generic set intersected with $\Omega \setminus (A_k \cup B_k)$ is the same as μ'_{k+l} generic curves (these objects are curves by Lemma 4.1) intersected with $\Omega \setminus (A_k \cup B_k)$ for some l . The family of domains $\Omega \setminus (A_k \cup B_k)$ is monotone and exhaustive. μ' is concentrated

on curves by Alexandroff's Theorem. By Lemma 4.2 again, these curves are crosscuts from a to b .

To show that these are Löwner crosscuts it is enough to show that they almost surely satisfy conditions (L1) and (L2). Consider a parametrization of γ with non-vanishing speed. It is not difficult to see that a violation of (L1) implies that there exists some point z_0 which is visited at least three times if z_0 is in the bulk or twice if z_0 is on the boundary. We remind the reader that this is in the continuum; at the lattice level, our collisions could represent approaches which are microscopically large but macroscopically small e.g. a sublinear power of N . Note that each such encounter in the interior leads to 6 arms. Thus to finish we appeal to Lemma 2.7; let us divide the continuum into a grid of scale η . In order for the described event to occur in the interior of the domain, there must be some l such that the 6-arm event happened in some annulus $D(\eta, l)$. The probability of this is given in Lemma 2.7 and therefore the probability that this happens anywhere is bounded by $(\eta/l)^{2+\kappa} \left(\frac{1}{\eta^2}\right) = \frac{1}{l^2} \left(\frac{\eta}{l}\right)^\kappa$. So, ultimately, the probability of such an event is zero.

If z_0 is ε -close to the boundary, violation of (L1) implies a long double crossing below z_0 , for any ε , but this has vanishingly small probability, by Lemma 2.6. Finally, a violation of (L2) is equivalent to the existence of some doubling back, as defined in Definition 2.3 and is therefore forbidden (with probability) by Lemma 2.4. \square

4.3 Preservation of $M(\partial\Omega) < 2$

Here we show that if we start with some domain Ω with $M(\partial\Omega) < 2$, then the exploration process also yields a curve with Minkowski dimension less than 2.

Proof of Lemma 2.9. Let $z \in \text{Int}(\Omega)$ and $g_\delta(z)$ the box (rhombus) of radius δ surrounding z and $D(z)$ denote the distance between z and $\partial\Omega$. We claim that there is some $\psi > 0$ such that for all ε sufficiently small,

$$\mathbb{P}_\varepsilon(\mathbb{X}_t^\varepsilon \in g_\delta(z)) < C_2 \left(\frac{\delta}{D}\right)^\psi$$

where C_2 is a constant.

This follows from Russo–Seymour–Welsh theory, which we do here in some detail. Indeed, if $r < s$, let $A_{s,r}(z) \equiv B_s(z) \setminus B_r(z)$ denote the rhomboidal annulus centered at z , where, if necessary, the sides are approximated, within ε , by the lattice structure. Assume temporarily that $A_{s,r}(z) \subset \text{Int}(\Omega)$. Clearly, if there is both a yellow and a blue ring in $A_{s,r}$, then \mathbb{X}_t^ε cannot possibly visit $B_r(z)$ (since the yellow portion of \mathbb{X}_t^ε cannot penetrate the blue ring and similarly with yellow \leftrightarrow blue). Now by Theorem 3.10, item (iii) in [5], the probability of a blue ring in $A_{M,\lambda M}$ is bounded below uniformly in ε by a strictly positive constant that depends only on λ . Let $\eta > 0$ denote a lower bound on the probability that in

$A_{4L,3L}$ there is a blue ring and in $A_{3L,2L}$ a yellow. Now let k satisfy $2^k > \varepsilon^{-1}D > 2^{k-1}$ and similarly $2^\ell > \varepsilon^{-1}\delta > 2^{\ell-1}$. Then, give or take, there are $k - \ell$ independent annuli in which the pair of rings described can occur. The probability that all such ring pair events fail is less than $C_1(1 - \eta)^{k-\ell} \leq C_2 \left(\frac{\delta}{D}\right)^\psi$, where C_1 and C_2 are constants and $\psi > 0$ is defined via η .

Let us fix a square grid of scale δ with $\varepsilon \ll \delta \ll 1$. Let \mathcal{N}_δ denote the number of boxes of scale δ that are visited by the process. We claim that for all ε sufficiently small

$$\mathbb{E}_\varepsilon(\mathcal{N}_\delta) \leq C_{\psi'} \left(\frac{1}{\delta}\right)^{2-\psi'} = C_{\psi'} n^{2-\psi'}, \quad (18)$$

where $\psi' > 0$ is a constant and $n = n_\delta = \delta^{-1}$ represents the characteristic scale of Ω on the grid of size δ^{-1} . In particular we may take $\psi' < \min\{\psi, \theta\}$, where $\theta \in [0, 1]$ describes the roughness of the boundary: $M(\partial\Omega) = 2 - \theta$.

Let n_k denote the number of boxes a distance $k\delta$ (i.e. k boxes) from $\partial\Omega$ and

$$N_l = \sum_{k \leq l} n_k.$$

Our first claim is that for all δ ,

$$N_l < C_{\theta'} n^{2-\theta'} l^{\theta'}, \quad (19)$$

for any $\theta' < \theta$, where $C_{\theta'}$ is a constant. To see this, let us estimate the total area of boxes on a grid of size κ intersected by or within one unit of $\partial\Omega$. It is not hard to see that this is bounded by $C_{\theta'} \times \left(\frac{1}{\kappa}\right)^{2-\theta'} \times \kappa^2 = C_{\theta'} \kappa^{\theta'}$, where $C_{\theta'}$ is a constant which is uniform for a fixed $\theta' < \theta$. Taking $\kappa = l\delta$ and noting that *these* boxes contain all of the $n_1 + \dots + n_l$ boxes of scale δ (i.e. boxes within l units of $\partial\Omega$), the claim follows.

Now, clearly,

$$\mathbb{E}_\varepsilon(\mathcal{N}_\delta) \leq C_2 \sum_{k=1}^{l_{\max}} n_k \cdot \left(\frac{1}{k}\right)^\psi.$$

Let us now dispense with the sum in the display. Summing by parts, we get

$$\sum_{k=1}^{l_{\max}} n_k \left(\frac{1}{k}\right)^\psi = N_{l_{\max}} l_{\max}^{-\psi} + \sum_{k=1}^{l_{\max}-1} N_k \left(\frac{1}{k^\psi} - \frac{1}{(k+1)^\psi}\right).$$

Now if $\psi > \theta$, then $\psi > \theta'$. Using Eq.(19) and pulling out an $n^{2-\theta'}$, the sum is convergent. Meanwhile, the first term (again using the estimate in Eq.(19)) is smaller. Conversely, if $\psi \leq \theta$, then both terms are of order $n^{2-\theta'} l_{\max}^{\theta'-\psi}$ and the result follows if we take $l_{\max} = n$. It is reemphasized that the estimate in Eq.(18) is uniform in ε ; by further sacrifice of the constant, we may claim that Eq.(18) holds for all box-scales in the range $[\delta, 2\delta]$.

The remaining argument is now immediate. Letting $\delta_k = 2^{-k}$ we have that for any $\delta \in [\delta_{k+1}, \delta_k]$ and $s > 0$

$$\mathbb{P}_\varepsilon(\mathcal{N}_\delta > C_{\psi'} n_\delta^{2-\psi'+s}) \leq \frac{1}{2^{ks}}. \quad (20)$$

The result follows, for any $s > 0$, by taking $\varepsilon \rightarrow 0$ and summing over k . \square

4.4 Uniform Continuity

Here we provide a proof of Lemma 2.10. We will need a definition and an auxiliary lemma.

Proof of Lemma 2.10. We need not assume that our domain is locally connected. To simplify the exposition, we will always refer to “boundary arcs” and points on the boundary as pre-images of the corresponding objects on \mathbb{D} under the uniformizing map. It should be noted that all the difficulties associated with the corner points a, b, c, d are circumvented since we will be deleting neighborhoods of these points in the course of the proof.

We envision Ω to be rectangular, with a as the right bottom corner, and $\eta \ll \delta_2 \ll \delta_1 \ll \Delta^4 \ll 1$. Here $\Xi_{\vartheta, \Delta}$ denotes the set of curves γ emanating from a which have the following properties:

1. γ has no $\delta_2/2-3\eta$ double back;
2. There are neither $\Delta^3/2-2\Delta^4$ nor $\delta_1/2-2\delta_2$ double visits to the boundary.
3. γ does not return to the 2Δ neighborhood of a after having left a $\sqrt{\Delta}$ neighborhood of a , i.e. if

$$t_0 = \inf_t \{\text{dist}(a, \gamma(t)) > \sqrt{\Delta}\},$$

then for $t > t_0$, $\text{dist}(a, \gamma(t)) > 2\Delta$;

4. γ stays a distance 2Δ away from the corners b and d ;
5. γ never triple visits a neighborhood of radius $2\delta_2$ of any point having traversed a distance $\delta_1/2$ in between the visits.

By Lemmas 2.4, 2.6, Russo–Seymour–Welsh arguments, Lemma 2.7, respectively, we have that $\mu_\varepsilon(\Xi_{\vartheta, \Delta}) > 1 - \frac{1}{2}\vartheta$. Notice that since $\gamma_1 \in \Xi_{\vartheta, \Delta}$ and γ_2 is sup-norm close to γ_1 , both curves satisfy the properties 1–4 without the numerical factors. We emphasize that while γ_1 was supposed to be explorer process typical, this curve will not interact with the background configuration of Ω ; the above mentioned are the *only* properties of the γ 's which will be used.

We will first dispense with all cases where $\gamma_1(T)$ and $\gamma_2(T)$ are “close” to any boundary. We start with situations where $\gamma_1(T)$ and $\gamma_2(T)$ is Δ -close to a, b, c, d ; the latter three corners we need not consider by the definition of $\Xi_{\vartheta, \Delta}$, $\gamma_1(T)$ and $\gamma_2(T)$ and/or the

assumptions of the lemma. Again by the definition of $\Xi_{\vartheta, \Delta}$, $\gamma_1([0, T])$ and $\gamma_2([0, T])$ are contained in a $\sqrt{\Delta}$ -neighborhood of a , but in this case, by Russo–Seymour–Welsh, a is sealed by many rings of both colors with high probability and hence neither γ_1 nor γ_2 participate in any crossing event of interest.

Next suppose $\gamma_1(T)$ and $\gamma_2(T)$ are Δ^4 close to some boundary and not close to any corners. In particular, $\gamma_1(T)$ and $\gamma_2(T)$ are close to either a pure blue or pure yellow boundary; since the two cases are handled the same way, we consider, without loss of generality, the $[c, d]$ (blue) boundary. First draw a straight line segment joining $\gamma_1(T)$ to the arc $[c, d]$ and let us call the boundary endpoint of the segment m . Next let t_0 denote the first time γ_1 visits the segment $[m, \gamma_1(T)]$. By Russo–Seymour–Welsh, with probability close to unity, there exists a blue ring R_B inside the annulus of modulus $\Delta^3:\Delta^2$ centered at m , which we denote A . Consider the domain enclosed by $[a, d]$, $[d, m]$, $[m, \gamma(t_0)]$ and $\gamma([0, t_0])$. The γ_1 part of the outer boundary of the domain is colored blue. The first time R_B hits the boundary of the domain after it intersects the arc $[a, d]$ is on the correct side of the explorer process (before time t_0). R_B does not intersect $\gamma_1[t_0, T]$, because there are no Δ^4 – Δ^3 double visits to the boundary. So the ring R_B provides a blue crossing, which occurs with high probability.

Exactly the same argument works for γ_2 , so in this case the blue crossing probability is large for both γ_1 and γ_2 . Finally, in the case that $\gamma_1(T)$ and $\gamma_2(T)$ are close to a yellow boundary, a similar argument shows that the yellow crossing probability would be large (and hence the blue crossing probability small).

We are now ready to turn attention to the “bulk” case: i.e. $\gamma_1(T)$ and $\gamma_2(T)$ are Δ^4 away from any boundary and Δ away from any corners. We will abbreviate $C_\varepsilon(\gamma_1) = C_\varepsilon(\Omega \setminus \gamma_1([0, T]), \gamma_1(T), b, c, d)$ and similarly for $C_\varepsilon(\gamma_2)$, and the corresponding conformal rectangles will be denoted R_1 and R_2 . Let B_{δ_1} denote the ball of radius δ_1 around $\gamma_1(T)$, and notice that by an assumption stated in the lemma, B_{δ_1} is well away from c , and since we are in bulk case, it is also well away from all the other corners and the boundary.

First by Russo–Seymour–Welsh arguments, we can safely assume that any blue or yellow paths we will discuss stay away from B_{δ_1} . Let us start by assuming that there is a blue crossing for R_1 ; to prove the lemma in this case it is enough to show that with high probability there is a blue crossing for R_2 , since γ_1 and γ_2 are interchangeable. We can separate into three disjoint cases:

1. There is a blue crossing of Ω_ε which intersects neither $\gamma_1([0, T])$ nor $\gamma_2([0, T])$.
2. There is a blue crossing of R_1 which hits $\gamma_1([0, T])$ but does not hit $\gamma_2([0, T])$.
3. Any blue crossing of R_1 hits $\gamma_2([0, T])$.

In case 1, we clearly have a blue crossing of both R_1 and R_2 and there is nothing to prove.

Next let us examine case 3, which can be divided into two new subcases: either there is a crossing for R_1 which also constitutes a crossing R_2 (meaning that the “blue side” of

$\gamma_2([0, T])$ is connected in R_2 by the crossing to the boundary arc $[c, d]$ or any crossing for R_1 only connects (in R_2) the “yellow side” of $\gamma_2([0, T])$ to $[c, d]$. Clearly, again, in the former subcase there is nothing to prove. We claim that the latter subcase is equivalent to case 2, with blue and yellow (and γ_1 and γ_2) interchanged: Indeed, let us consider the highest such blue crossing for R_1 , i.e. the crossing which minimizes the domain above the crossing and γ_1 , which we call D ; each point in D next to the boundary is connected to $[b, c]$ by a yellow path inside D . Now since the blue crossing is assumed to intersect $\gamma_2([0, T])$ but fails to be a blue crossing in R_2 , there exists a point on the yellow side of $\gamma_2([0, T])$ which is connected to $[c, d]$ in the domain D by some yellow path \mathcal{P} – this is the point on the yellow side of γ_2 which is first hit by the blue crossing for R_1 . First observe that since \mathcal{P} is inside D , it cannot intersect γ_1 . Now let us show that \mathcal{P} can be selected so that it doesn’t intersect γ_2 and thus we can reduce to case 2 as promised, or there is a blue crossing for R_2 . If any yellow path in D connecting the yellow side of γ_2 with $[b, c]$ fail to be a crossing, then it must intersect the blue side of γ_2 (since this is the only obstacle), but this necessarily implies that there is a blue crossing in D for R_2 .

Finally, we are ready to treat case 2. Let us select the highest blue crossing for R_1 and denote by M the point where it hits γ_1 . We now surround M by a ball of radius δ_2 , denoted B_{δ_2} . Let s denote the last time γ_2 exits the ball B_{δ_2} . Assume first that $\gamma_2(s)$ can be connected to d by a curve Γ inside Ω not intersecting the blue crossing, B_{δ_2} or $\gamma_2([0, s])$. Let D' be the domain below $\gamma_2([0, s]) \cup \Gamma$. Now we perform the Russo–Seymour–Welsh type argument to show that with high probability there is a blue ring inside the annulus $\{\eta < |z - M| < \delta_2\}$. This blue ring surrounds the point M which is outside of the domain D' . It intersects $\gamma_2([0, s])$, and so it joins the blue crossing of R_1 to the outer boundary of D' , but does not cross Γ . Hence the ring provides a blue crossing for R_2 .

Now we show that it is highly unlikely that Γ as described doesn’t exist. The curve γ_2 has no δ_1 – η double back, by assumption. So, there exists a point $\gamma_2(t^*)$ in B_{δ_1} which is at least η away from $\gamma_2([0, s])$. Now let us observe that $\gamma_2(s)$ is connected to d as described above if and only if the same happens with $\gamma_2(t^*)$, since $\gamma_2(t^*)$ can be connected to $\gamma_2(s)$ by γ_2 itself. We work with $\gamma_2(t^*)$ from now on.

We will again need to deal with the boundary case separately, this time for the point M . Let us start with the bulk case, i.e. M is at least δ_2 away from the boundary. Under these circumstances, Γ can fail to exist for one of the following two reasons: 1) $\gamma_2(t^*)$ is separated from d by the blue crossing and B_{δ_2} or 2) $\gamma_2(t^*)$ is separated from d by the union of $\gamma_2([0, s])$ and B_{δ_2} . This describes all the cases because $\gamma_2(t^*)$ lies outside the η –neighborhood of D (recall this is the domain above the blue crossing of R_1 and γ_1). In the first of these cases, the highest blue crossing for R_1 , denoted \mathcal{B} , should have the following property: \mathcal{B} visits B_{δ_2} , goes around B_{δ_1} and revisits B_{δ_2} , since it separates $\gamma_2(t^*) \in B_{\delta_1}$ from d . Since \mathcal{B} is the *highest* crossing of R_1 , it leads to the existence of 3 blue and 2 yellow δ_1 – δ_2 long arms emanating from B_{δ_2} . The only exception is when M itself is δ_1 -close to the boundary, in which case we lose one yellow arm. Such an event has probability proportional to $(\delta_1/\delta_2)^2$

by Lemma 2.7 and Lemma 2.6. Hence such an event happens with probability tending to 0 as δ_1/δ_2 tends to 0, because the $M(\gamma_2) < 2$. In the second of these cases, the curve γ_2 itself must visit B_{δ_2} three times, going a distance δ_1 from it in between, which contradicts assumption 5.

Finally, we handle the cases where M is near the boundary. If M is close to $[a, b]$, then the argument from the bulk cases applies without any changes. If M is close to $[b, c]$, then either there is a double boundary visit by γ_2 on the scale $\delta_1 : \delta_2$, which contradicts assumption 2, or the bulk argument applies again. The same is true if M is close to $[d, a]$. The non-trivial boundary case is when M is close to $[c, d]$. Here, in addition to the bulk argument, we should notice that if $\gamma_2(s)$ is separated from d by \mathcal{B} and B_{δ_2} , then it can be connected to the arc $[c, d]$ itself and using this arc be connected to d , thus creating the curve Γ . If such a connection to $[c, d]$ is not possible, then there is a double boundary visit by γ_2 . This concludes the proof. \square

4.5 A Priori Bounds

In this paragraph we prove Lemma 2.11. Let us observe that

Lemma 4.4. *Let $\gamma(t)$ be the chordal SLE generated by $w(t)$. Then*

- $\text{Im}(\gamma(t)) \leq 2\sqrt{t}$.
- $\sup_{s \leq t} |\gamma(s)| \geq \frac{|w_t|}{4}$.

Proof. For the first statement note that $\partial_t(\text{Im}(g_t)) = -2\text{Im}(g_t)/|g_t - w_t|^2 \geq -2/\text{Im}(g_t)$, so $\partial_t(\text{Im}(g_t))^2/4 \geq -1$. Integrating, we get $(\text{Im}(g_t))^2 \geq (\text{Im}(z))^2 - 4t$. So if $\text{Im}(z) > 2\sqrt{t}$ then $\text{Im}(g_t) > 0$.

For the second part, assume that $R = \sup_{s \leq t} |\gamma_s|$. Let $[\mu_t, \lambda_t]$ be the image $g_t(\gamma[0, t])$. Note that $w_t \in [\mu_t, \lambda_t]$ and that $\mu_t \leq 0 \leq \lambda_t$, since μ_t is decreasing and λ_t is increasing and $\mu_0 = \lambda_0 = w_0 = 0$.

Now let $A(z) = z + R^2/z$ be the map of the complement of the half-disk $\{|z| \leq R\}$ to \mathbb{H} . Note that A maps the half-circle $\{|z| = R\}$ onto $[-2R, 2R]$. The map $g_t(A^{-1}(z))$ is well-defined, since $\gamma([0, t]) \subset \{|z| \leq R\}$, and maps $[-2R, 2R]$ onto a curve separating $[\mu_t, \lambda_t]$ from infinity. By the monotonicity of harmonic measure, $|w(t)| \leq \lambda_t - \mu_t \leq 4R$, which proves the second statement of the lemma. \square

Now we are in a position to prove Lemma 2.11.

Proof of Lemma 2.11. On the basis of the above lemma, $|w_t| > n$ implies that in the half plane a rectangle of aspect ratio of the order n/\sqrt{t} has been crossed by $g_0(\gamma_{[0, t]})$. But this means that $\gamma_{[0, t]}$ itself crossed a *conformal* rectangle with conformal modulus n/\sqrt{t} .

Invoking Lemma 2.2, the probability of such an event is bounded by $C_1 e^{-C_2 \frac{n}{\sqrt{t}}}$ for some $C_1, C_2 > 0$. \square

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