6. L'Hôpital's Rule

Everyone knows that $0/1 = 0$. What do we mean when we say that $1/0 = \infty$ or $-\infty$, or "does not exist"? e.g., $\lim\limits_{x\to 1} \dfrac{x}{x-1}$ is infinite or does not exist. We can't actually divide by zero; we mean something like the example above, that is, if $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = 0$ then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ is infinite or "does not exist".

This much you more or less knew already, but what is $\lim\limits_{x\to a} f(x)/g(x)$ if $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$? We call this a $0/0$ form.

e.g.

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

or

$$\lim_{x\to 2} \frac{(x+2)^{\frac{1}{2}} - 2}{(x+6)^{\frac{1}{3}} - 2} = 3$$

How do we find these limits? There is a useful procedure known as L'Hôpital's Rule.

L'Hôpital's Rule

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ and

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L$$

as well.

(There are additional assumptions on f and g, but these are commonly satisfied by the functions we deal with in this course, so we shall skip the details.)

In other words, if you are trying to evaluate $\lim_{x\to a} f(x)/g(x)$ and it is of the form $0/0$, then try $\lim_{x\to a} f'(x)/g'(x)$. If you get an answer, the **same** answer will work for $\lim_{x\to a} f(x)/g(x)$.

In our examples, $\frac{e^x - 1}{x}$ is in $0/0$ form at $x = 0$. Also $\frac{(e^x - 1)'}{(x)'} = \frac{e^x}{1}$ and $\lim_{x \to 0} \frac{e^x}{1} = e^0 = 1$ Therefore $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$.

$\frac{(x+2)^{\frac{1}{2}} - 2}{(x+6)^{\frac{1}{3}} - 2}$ is in $0/0$ form at $x = 2$. Also

$$\frac{((x+2)^{\frac{1}{2}} - 2)'}{((x+6)^{\frac{1}{3}} - 2)'} = \frac{\frac{1}{2}(x+2)^{-\frac{1}{2}}}{\frac{1}{3}(x+6)^{-\frac{2}{3}}}$$

and

$$\lim_{x \to 2} \frac{\frac{1}{2}(x+2)^{-\frac{1}{2}}}{\frac{1}{3}(x+6)^{-\frac{2}{3}}} = \frac{\frac{1}{2}\frac{1}{2}}{\frac{1}{3}\frac{1}{4}} = 3$$

Therefore

$$\lim_{x \to 2} \frac{(x+2)^{\frac{1}{2}} - 2}{(x+6)^{\frac{1}{3}} - 2} = 3$$

Why does this rule work? Notice that if $f(a) = 0$ and $g(a) = 0$ then

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}}$$

So

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} = \frac{f'(a)}{g'(a)}$$

If $\frac{f'(a)}{g'(a)}$ makes sense, and $\frac{f(a)}{g(a)}$ is in $0/0$ form, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

This is called L'Hôpital's Rule. (The foregoing calculation has a number of technical shortcomings. Nevertheless, it does embody the central idea of a rigorous proof.)

Notice that L'Hôpital's rule doesn't work if $\lim_{x \to a} f(x) \neq 0$ or $\lim_{x \to a} g(x) \neq 0$. e.g.

$$\lim_{x \to 1} \frac{x^2}{x} = 1 \neq \lim_{x \to 1} \frac{(x^2)'}{x'} = \lim_{x \to 1} \frac{2x}{1} = 2$$

Sometimes, L'Hôpital's Rule needs to be applied more than once; e.g., checking that we still have a 0/0 form each time before we apply the derivative to both numerator and denominator,

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

L'Hôpital's Rule works in another case besides 0/0 forms. It works on expressions of the form $\pm\infty/\pm\infty$; e.g.,

$\lim_{x\to\infty} \frac{e^x}{x}$ is of the form ∞/∞ and $\frac{(e^x)'}{(x)'} = \frac{e^x}{1}$. Since $\lim_{x\to\infty} \frac{e^x}{1} = \infty$, it follows that $\lim_{x\to\infty} \frac{e^x}{x} = \infty$.

Another example: find $\lim_{x\to 0^+} x \ln x$.

(This is of the form $0 \cdot (-\infty)$. In case you think that $0 \cdot \infty$ is always zero or maybe infinity, notice that $\lim_{x\to 0} x \cdot \frac{1}{x} = 1$.)

First, turn the expression into a $\pm\infty/\pm\infty$ form.

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$$

(this is of the form $-\infty/\infty$)

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

It would have been more correct to omit the last = sign and to say instead: therefore $\lim_{x\to 0^+} x \ln x = 0$; but the circumlocution gets tiresome after a while.

Why does L'Hôpital's Rule work in these "infinite" cases? The argument is a little involved, and not so transparent, hence we won't present it here; but see Problem 11 below.

e.g.

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

In fact, any power of x over e^{ax} will go to zero as x goes to $+\infty$ as long as $a > 0$. e.g.

$$\lim_{x\to\infty} \frac{x^{100}}{e^{.00001x}} \qquad (\infty/\infty)$$

$$= \lim_{x\to\infty} \frac{100x^{99}}{.00001e^{.00001x}} \qquad \text{still } (\infty/\infty)$$

$=$ ... 100 applications of L'Hôpital's Rule later

$$= \lim_{x\to\infty} \frac{100!}{(.00001)^{100}e^{.00001x}} = 0$$

since the numerator, though enormous, does not change, while the denominator, though it looks small for all reasonable values of x, still goes to ∞ as x goes to ∞. To appreciate how powerful this method is, notice that if you try substituting some numbers to guess the limit:

$x = 2$ gives approximately $1.27 \cdot 10^{30}$, while $x = 10$ gives approximately 10^{100}.

$x^{100}/e^{.0001x}$ hardly seems to be approaching 0 as x gets large; but it does!

L'Hôpital's rule can be used on other kinds of limits if they can be manipulated so as to require the evaluation of a $0/0$ or ∞/∞ limit.

e.g., find

$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x \qquad (1^\infty)$$

Let $y = \left(1 + \frac{a}{x}\right)^x$. Then

$$\ln y = x \ln\left(1 + \frac{a}{x}\right) = \frac{\ln\left(1 + \frac{a}{x}\right)}{\frac{1}{x}}$$

which is in $0/0$ form as $x \to \infty$. Hence,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{a}{x}}\left(-\frac{a}{x^2}\right)}{-\frac{1}{x^2}}$$

simplifying algebraically

$$= \lim_{x\to\infty} \frac{a}{1+\dfrac{a}{x}} \;=\; a$$

So $\ln y \to a$ as $x \to \infty$. $y = e^{\ln y} \to e^a$ as $x \to \infty$; that is,

$$\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x \;=\; e^a$$

(Remember continuous compounding? $\left(1+\frac{r}{n}\right)^{nt} \to e^{rt}$ as n $\to \infty$.)

Exercises

1. $\displaystyle\lim_{x\to 0} \frac{a^x - 1}{x}$ $\quad a > 0$
2. $\displaystyle\lim_{x\to 0^+} \frac{1-e^x}{\sqrt{x}}$
3. What's wrong with the following calculation?

$$\lim_{x\to 1} \frac{x^3+x-2}{x^2-3x+2} = \lim_{x\to 1} \frac{3x^2+1}{2x-3} = \lim_{x\to 1} \frac{6x}{2} = 3$$

   (The answer is really -4.)

4. $\displaystyle\lim_{x\to 1^+} \frac{(\ln x)^2}{(x-1)^2}$
5. $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$
6. $\displaystyle\lim_{x\to 0} (1+3x)^{\frac{1}{2x}}$
7. $\displaystyle\lim_{x\to 1} x^{\frac{2}{x-1}}$
8. $\displaystyle\lim_{x\to 1^+} (x-1)^{\ln x}$
9. $\displaystyle\lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1}$
10. $\displaystyle\lim_{x\to 0} \left(\frac{1}{e^x-1} - \frac{1}{x}\right)$
11. If the limits of $f(x)$ and $g(x)$ are both infinite as $x \to a$, then the limits of $1/f(x)$ and $1/g(x)$ are both 0 as $x \to a$.

$\dfrac{f(x)}{g(x)} = \dfrac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$ which is in 0/0 form. Apply L'Hôpital's rule to this second expression and "solve" for $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ to get an idea of why the rule works for $\pm\infty/\pm\infty$.