

1. Mortgages

Mortgage loans are commonly quoted with a nominal rate compounded semi-annually; but the payments are monthly. To find the monthly payments in this case one finds the effective *monthly* rate of interest.

Let r be the nominal rate compounded semi-annually; let i be the effective monthly rate of interest. To find i in terms of r we equate the effective annual rate of compounding semi-annually with the effective annual rate of compounding monthly. Hence

$$\begin{aligned}(1+i)^{12} - 1 &= \left(1 + \frac{r}{2}\right)^2 - 1 \\ \Rightarrow (1+i)^{12} &= \left(1 + \frac{r}{2}\right)^2 \\ \Rightarrow 1+i &= \left(1 + \frac{r}{2}\right)^{\frac{1}{6}} \\ \Rightarrow i &= \left(1 + \frac{r}{2}\right)^{\frac{1}{6}} - 1.\end{aligned}$$

Example 1. What is the effective monthly rate for a mortgage if the nominal rate is 12% compounded semi-annually?

Solution. Take $r = 12$, so

$$\begin{aligned}i &= (1.06)^{\frac{1}{6}} - 1 \\ &= 0.00975879\dots\end{aligned}$$

Note that the effective monthly rate of 0.98% is less than 1%, which is simply the nominal rate divided by twelve.

Example 2. Find the monthly payments for a 25-year, \$50,000 mortgage at 12% compounded (a) semi-annually, (b) monthly.

Solution. The monthly payment for a mortgage is given by

$$R = \frac{A}{a_{\overline{n}|i}}$$

Take $A = 50,000$, $n = 25 \times 12 = 300$

(a) Take $i = 0.00975879\dots$ (See Example 1)

$$\begin{aligned}\text{Then } R &= \frac{50,000}{a_{\overline{300}|0.00975879}} \\ &= 515.95\end{aligned}$$

(b) Take $i = 1\%$.

$$\text{Then } R = \frac{50,000}{a_{\overline{300}|0.01}} = 526.61.$$

Note that the monthly payment is slightly more if interest is compounded monthly than if it is compounded semi-annually. This difference can be quite substantial if the total cost of the mortgage is considered. For the mortgage(s) above the total cost is

(a) $\$515.95 \times 300 = \$154,785.$

(b) $\$526.61 \times 300 = \$157,983.$ This represents a significant difference of \$3,198.

Example 3. Find the monthly payment and total cost of a 25-year, \$30,000 mortgage at

(a) 11% compounded semi-annually

(b) 17.25% compounded semi-annually

(c) 20% compounded semi-annually.

Solution. Take $n = 300$, $A = 30,000$. The effective monthly rates are:

(a) $i = (1.055)^{\frac{1}{6}} - 1 = 0.00896$

(b) $i = (1.08625)^{\frac{1}{6}} - 1 = 0.01388$

(c) $i = (1.1)^{\frac{1}{6}} - 1 = 0.01601$

The monthly payments are

(a) $R = \frac{30,000}{a_{\overline{300}|0.00896}} = 288.67$

(b) $R = \frac{30,000}{a_{\overline{300}|0.01388}} = 423.17$

(c) $R = \frac{30,000}{a_{\overline{300}|0.01601}} = 464.43$

And the total costs of these mortgages are

(a) $\$288.67 \times 300 = \$86,601$

(b) $\$423.17 \times 300 = \$126,951$

(c) $\$464.43 \times 300 = \$139,329$

The data in Example 3 were not chosen at random. The interest rates are those for mortgages current in April 1979, April 1981, and July 1981, respectively. The example illustrates the increased cost to the consumer as a result of rising interest rates.

**A NOTE ON HOW TO CALCULATE INTEREST AND
PRINCIPAL PAID IN EACH PERIODIC PAYMENT:**

In an amortization schedule, each periodic payment is broken down into two amounts: the **interest** and the **principal** paid during that period. To calculate these amounts for an arbitrary period it is only necessary to remember that *the outstanding principal just after a payment has been made is the present value of all payments yet to be made.*

So for instance, consider the \$50,000 mortgage of Example 2b, in which the monthly payment is $R = 526.61$, the monthly interest rate is $i = 1\%$ and the total number of payments is $n = 300$. Consider the 53^{rd} payment:

Immediately after the 52^{nd} payment, the outstanding balance is

$$Ra_{\overline{248}|0.01} = 48,196.53$$

During the 53^{rd} month the interest rate is 1% . So the interest paid in the 53^{rd} month is

$$1\% \text{ of } 48,196.53 = 481.96$$

What remains, namely $526.61 - 481.96 = 44.65$, is the principal paid in the 53^{rd} payment.

Exercises

1. Find the monthly payment for the following loans:
 - (a) An automobile loan of \$8,500 amortized over 48 months at an interest rate of
 - (i) 13.2% compounded monthly
 - (ii) 13.2% compounded semi-annually.
 - (b) A five-year loan of \$7,000 if interest is at
 - (i) 12.12% compounded monthly
 - (ii) 12.12% compounded semi-annually.

2. What are the monthly payments of a 30-year, \$40,000 mortgage if interest is at
 - (a) 16% compounded semi-annually?

- (b) 19% compounded semi-annually?
3. Show that for an amount A , amortized over n years, with (uniform) payments p times per year, at a nominal rate r compounded m times per year, the payments are

$$A \cdot \frac{(1 + \frac{r}{m})^{\frac{m}{p}} - 1}{1 - (1 + \frac{r}{m})^{-mn}} .$$

4. Fill out the following chart. (The first row has already been calculated in Example 3.)

COST OF A MORTGAGE			
25-year mortgage	April/79 11%	April/81 17.25%	July/81 20%
\$30,000	\$86,601	\$126,951	\$145,329
\$40,000			
\$50,000			
\$60,000			

5. Consider a 1-year, \$7,000 mortgage for which interest is 12% compounded semi-annually.
- (a) What is monthly payment, to the nearest dollar? Prepare an amortization schedule, showing interest and principal paid each month.
- (b) Using the monthly payment from part (a), which is properly \$621, prepare an amortization schedule (for the above mortgage) if the interest were compounded monthly, that is, if the effective monthly rate were 1%. What is the outstanding balance at the end of the year?
6. [Ref. *Mortgage Basics*, by Lloyd Lindsay C.A., Toronto Real Estate Board Newspaper, 7/11/80.]

There is another method, known as the legal method, for preparing amortization schedules that avoids compounding interest monthly. Consider the 1-year, \$7000 mortgage at 12% compounded semi-annually of question 5. In the legal method, the monthly interest rate on the outstanding balance is taken to be 1% (12% divided by 12), but the interest is

applied only every six months. So, for example, the first six months would be recorded as:

LEGAL METHOD				
Payment Number	Payment	Interest	Balance Reduction	Balance
				\$7000
1	\$621	\$70	\$621	6379
2	621	64	621	5758
3	621	58	621	5137
4	621	51	621	4516
5	621	45	621	3895
6	621	39	621	3274
Interest added on:		327	(327)	327
				3601

- (a) Complete the schedule for the next six months. Is the final balance zero?
- (b) Find the proper monthly payment for a 1-year, \$10,000 mortgage at 16% compounded quarterly. Prepare two amortization schedules, one using the correct method, and one using the legal method. Does the balance of each agree every 3 months? What is the total interest paid according to the correct method? According to the legal method.
- (c) [Analysis of the Legal Method.] Consider a 1-year, \$A mortgage at a nominal rate of r compounded semi-annually. Let the proper monthly payment be $R = A/a_{\overline{12}|i}$, where i is the effective monthly rate. Show that the interest paid according to the correct method is $12R - A$; according to the legal method, $rA(1 + \frac{r}{4}) - \frac{rR}{2}(11 + \frac{5r}{4})$. Equate these two expressions and obtain

$$R = \frac{2A(2 + r)^2}{96 + 44r + 5r^2}.$$

In terms of r and A , what is R actually equal to? What approximation for the effective monthly rate does the legal method use? Show that this approximation agrees with the actual effective monthly rate for $r = 0$, that their derivatives and their second derivatives also agree at $r = 0$.