1. [3 marks]
   The market price of a $100 bond with an annual coupon rate of 4%, paying semi-annually, which has 15 semi-annual coupons left, and which has an annual yield to maturity of 5%, is
   \(\text{A} \quad \$66.86\)
   \(\text{B} \quad \$76.81\)
   \(\text{C} \quad \$89.87\)
   \(\text{D} \quad \$93.81\)
   \(\text{E} \quad \$97.85\)

2. [3 marks]
   If \(A = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\) and \(B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\), then \(B^T \cdot AB =\)
   \(\text{A} \quad [-1]\)
   \(\text{B} \quad [0]\)
   \(\text{C} \quad [2]\)
   \(\text{D} \quad [-2]\)
   \(\text{E} \quad [1]\)
3. [3 marks]

The solution of the system

\[ \begin{align*}
2x + 3y - 5z &= 4, \\
4x + 5y - 7z &= 10.
\end{align*} \]

in terms of the parameter \( z \), has \( x = \)

A. \(-z + 4\)
B. \(3z - 2\)
C. \(-3z + 1\)
D. \(2z - 3\)
E. \(-2z + 5\)

4. [3 marks]

The solution to the inequality \( \frac{\ln x}{x - 2} \geq 0 \) is given by

A. \(0 \leq x \leq 1\) or \(x \geq 2\)
B. \(x < 1\) or \(x > 2\)
C. \(0 < x \leq 1\) or \(x > 2\)
D. \(x \leq 0\) or \(x > 2\)
E. \(x \leq 1\) or \(x > 2\)

5. [3 marks]

The function \( f(x) = x^2e^{-x} \)

A. increases on \((0, \infty)\)
B. has an absolute maximum at \(x = 2\)
C. increases on \((-\infty, 0)\)
D. increases on \((0, 2)\)
E. has no horizontal asymptote
6. [3 marks]
The line which is tangent to the graph of \( y = (2x)^2 \) at the point \( \left( \frac{1}{2}, 1 \right) \) has equation:

A. \( 8y - 2x = 7 \)
B. \( 4y - 2x = 3 \)
C. \( y = 2x \)
D. \( 2y - 2x = 1 \)
E. \( 4y + 2x = 5 \)

7. [3 marks]
The average value, on \([-4, 2]\), of \( f(x) = |x^3| \) is

A. \( \frac{34}{3} \)
B. 10
C. -10
D. 68
E. -60

8. [3 marks]
For 10 years, cash flows into an account at the constant rate of $1,000 per year. To the nearest dollar, the present value of the cash flow, if the account earns 4% compounded continuously, is

A. $9,138
B. $6,097
C. $7,515
D. $8,242
E. $8,836
9. \textbf{[3 marks]}

The demand functions for the two products toves (T) and mome raths (M) are as follows:

\[ q_T = \frac{1}{p_T + p_M} + c_1 p_M \]
\[ q_M = \frac{1}{p_T + p_M} + c_2 p_T \]

where \( p_T \) is the price of one tove and \( p_M \) is the price of one mome rath and \( c_1 \) and \( c_2 \) are constants.

At the moment \( p_T = \$2000 \) and \( p_M = \$1000 \). For which of the following pairs of constants \( c_1 \) and \( c_2 \) are toves and mome raths competitive products (i.e., substitutes) at these prices?

\[ \text{A} \quad c_1 = 1, \ c_2 = 2 \]
\[ \text{B} \quad c_1 = -0.0001, \ c_2 = -0.0002 \]
\[ \text{C} \quad c_1 = 0.0000000001, \ c_2 = 0.0000000001 \]
\[ \text{D} \quad c_1 = -1, \ c_2 = -1.5 \]
\[ \text{E} \quad c_1 = -1, \ c_2 = 0.5 \]

10. \textbf{[3 marks]}

The partial derivative \( \frac{\partial f}{\partial x} \) of \( f(x, y, z) = e^{x^2-yzx} \) is

\[ \text{A} \quad e^{x^2-zx} \]
\[ \text{B} \quad e^{2x-yz} \]
\[ \text{C} \quad e^{x^2-zx} \ln(zx-yz) \]
\[ \text{D} \quad e^{x^2-yzx}(2x-yz) \]
\[ \text{E} \quad e^{2x-yz}(2x-yz) \]

11. \textbf{[3 marks]}

If \( f(x, y, z) = 2x^3 - 3y^2z + 8xy^3z^4 - 6xyz + 3xy^2 \) then \( f_{xyz}(1,1,1) = \)

\[ \text{A} \quad 90 \]
\[ \text{B} \quad 102 \]
\[ \text{C} \quad 24 \]
\[ \text{D} \quad 18 \]
\[ \text{E} \quad -4 \]
12. [3 marks]
If \( z = a^2 + 2b^2 - 4ac \) where \( a = 2t, b = 4s - 3t, c = 2st^2 \) then, when \( s = t = 1 \),
\[
\frac{\partial z}{\partial t} =
\]
(A) \(-36\)
(B) 12
(C) \(-52\)
(D) \(-28\)
(E) 42

13. [3 marks]
If \( z = (x^2 + y^2)^{10} \) where \( x = 4r^2s^3 \) and \( y = e^{2r^2 + 3s - 3} \) then when \( r = 0 \) and \( s = 1 \),
\[
\frac{\partial z}{\partial r} =
\]
(A) 40
(B) 20
(C) 0
(D) 10
(E) 1

14. [3 marks]
If \( x^2 + xy + yz + z^2 = 6 \) defines \( z \) implicitly as a function of \( x \) and \( y \), then
\[
\frac{\partial z}{\partial y} =
\]
(A) \(-\frac{y + 3z}{y}\)
(B) \(-\frac{x + 2z}{y}\)
(C) \(-\frac{x + 2z}{x + 2y}\)
(D) \(-\frac{x + z}{2z}\)
(E) \(-\frac{x + z}{y + 2z}\)
15. [3 marks]

\[ \int_{-1}^{1} \int_{y}^{2} (2x + 3y) \, dx \, dy = \]

A \[ \frac{-11}{3} \]
B \[ \frac{-34}{15} \]
C \[ \frac{-9}{16} \]
D \[ \frac{-4}{3} \]
E \[ \frac{-2}{5} \]
PART B. WRITTEN-ANSWER QUESTIONS

B1. [10 marks]
You wish to purchase a mine which will produce an annual return of $32,000 per year for 12 years, after which the mine will have no value. At the end of each year, you plan to place money in a sinking fund earning 6.2% compounded annually, so as to replace the purchase price exactly after 12 years. If you want to earn 8% annually on this investment, what purchase price should you pay?

[Hint: Remember to include the replaced purchase price as part of the total return.]

B2. (a) (i) [2 marks]
Find \( a \) so that the following limit exists
\[
\lim_{x \to 0} \frac{e^{2x+1} - e - ax}{x^2}
\]

(ii) [5 marks]
Find \( a \) and \( b \) so that the following limit exists, then find the limit.
\[
\lim_{x \to 0} \frac{e^{2x+1} - e - ax - bx^2}{x^3}
\]

B2. (b) [6 marks]
Solve the initial value problem
\[
y(2) = 1
\]
\[
(x - 1)y' = 1
\]
then find \( y(3) \).
B3. \([12 \text{ marks}]\)
Find the following integrals.

(a) \([6 \text{ marks}]\)
\[
\int_0^\infty xe^{-3x} \, dx
\]

(b) \([6 \text{ marks}]\)
\[
\int \frac{x - 14}{x^2 - 4} \, dx
\]

B4. \([10 \text{ marks}]\)
Find and classify the critical points of the function given by:
\[
f(x, y) = -x^3 + 3xy^2 + 12y^2 + 4y^3
\]

B5. \([10 \text{ marks}]\)
A consumer has $600 to spend on two products which cost $20 per unit and $30 per unit respectively. The utility derived by the consumer from \(x\) units of the first product and \(y\) units of the second product is given by the utility function
\[
U(x, y) = 10x^{0.6}y^{0.4}
\]
[Note: the utility function measures the total satisfaction received by the consumer.]

Use the method of Lagrange multipliers (no marks for any other method) to find how many units of each product the consumer should buy to maximize his/her utility.

[You do not have to show that your answer does in fact give a maximum.]
Solutions to April 2005 Exam, MAT133Y

PART A

1. ANSWER: 

\[ r = 0.02 \quad i = 0.025 \quad n = 15 \]
\[ p = 100(1.025)^{-15} + 2d_{15,0.025} \]
\[ p = 93.809. \]

2. ANSWER: 

\[ B^T AB = [1, 1] \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
\[ = [1, 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [2] \]

3. ANSWER: 

\[
\begin{bmatrix}
2 & 3 & -5 & | & 4 \\
4 & 5 & -7 & | & 10
\end{bmatrix}
\]
\[ R_1 \rightarrow \frac{1}{2}R_1 \]
\[ R_2 \rightarrow R_2 - 4R_1 \]
\[ R_2 \rightarrow -R_2 \]
\[ R_1 \rightarrow R_1 - \frac{3}{2}R_2 \]
\[ x + 2z = 5 \]
\[ x = -2z + 5 \]

More simply
\[ 10x + 15y - 25z = 20 \]
\[ 12x + 15y - 21z = 30 \]
\[ 2x + 4z = 10 \implies 2x = 10 - 4z \]
\[ x = 5 - 2z \]
4. ANSWER: ∅

\[
\begin{array}{c|cccccc}
\ln x & - & + & + \\
0 & - & 1 & - 2 & + \\
\end{array}
\]

(0, 1) \cup (2, \infty) give > 0 and 1 gives 0. So (0, 1] \cup (2, \infty)

5. ANSWER: ∅

\[
f'(x) = 2xe^{-x} - x^2e^{-x} = x(2 - x)e^{-x}
\]

\[
\begin{array}{c|c|c}
& f' & f \\
(-\infty, 0) & - & \text{dec} \\
(0, 2) & + & \text{inc} \\
(2, \infty) & - & \text{dec} \\
\end{array}
\]

Note that the max at \( x = 2 \) is only local (\( \lim_{x \to -\infty} f(x) = \infty \)).
And \( \lim_{x \to \infty} x^2e^{-x} = 0 \) so there is a horizontal asymptote.

6. ANSWER: ∅

\[
\ln y = x^2 \ln 2x
\]

\[
\frac{1}{y} y' = 2x \ln 2x + \frac{x^2}{2x} \cdot 2
\]

at \((\frac{1}{2}, 1)\), 
\[
y' = 1.0 + \frac{1}{2} = \frac{1}{2}
\]

\[
y - 1 = \frac{1}{2}(x - \frac{1}{2}) = \frac{1}{2}x - \frac{1}{4}
\]

\[
y = \frac{1}{2}x + \frac{3}{4}
\]

\[
4y = 2x + 3
\]

\[
4y - 2x = 3
\]
7. ANSWER: \(\Box\)

\[
\tilde{f} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

\[
= \frac{1}{6} \int_{-4}^2 |x^3| \, dx
\]

\[
= \frac{1}{6} \left[ \int_{-4}^0 -x^3 \, dx + \int_0^2 x^3 \, dx \right]
\]

\[
= \frac{1}{6} \left[ -\frac{x^4}{4} \bigg|_{-4}^0 + \frac{x^4}{4} \bigg|_0^2 \right]
\]

\[
= \frac{1}{6} [64 + 4] = \frac{68}{6} = \frac{34}{3}
\]

8. ANSWER: \(\Box\)

P.V. = \(\int_0^{10} 1000 e^{-0.04t} \, dt\)

\[
= -25,000 e^{-0.04t} \bigg|_0^{10}
\]

\[
= 25,000 [1 - e^{-4}]
\]

\[
\approx 8242
\]

9. ANSWER: \(\Box\)

\[
\frac{\partial q_T}{\partial p_M} = -\frac{1}{(p_T + p_M)^2} + c_1
\]

\[
\frac{\partial q_T}{\partial p_M} = -\frac{1}{(3000)^2} + c_1
\]

\[
\frac{\partial q_M}{\partial p_T} = -\frac{1}{(3000)^2} + c_2
\]

We need both partials \(> 0\), so \(c_1\) and \(c_2 > 0\). In fact

\[
c_1 > \frac{1}{(3000)^2} = \frac{1}{9 \times 10^6}
\]

\[
c_2 > \frac{1}{9 \times 10^6}
\]

is what we need. Only \(\Box\) does this.
10. **ANSWER: ①**

\[
\frac{\partial f}{\partial x} = e^{x^2 - yzx}(2x - yz)
\]

11. **ANSWER: ④**

\[
f_z = -3y^2 + 32xy^3z^3 - 6xy
\]

\[
f_{zx} = 32y^3z^3 - 6y
\]

\[
f_{zxy} = 96y^2z^3 - 6 = 90 \text{ at } (1, 1, 1)
\]

12. **ANSWER: ③**

\[
\begin{align*}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial a} \frac{da}{dt} + \frac{\partial z}{\partial b} \frac{db}{dt} + \frac{\partial z}{\partial c} \frac{dc}{dt} \\
&= (2a - 4c)2 + 4b(-3) - 4c \cdot 4st
\end{align*}
\]

At \(s = t = 1\)

\[
a = 2 \quad \Rightarrow \quad (4 - 8) \cdot 2 - 12 = -8.4
\]

\[
b = 1 \quad \Rightarrow \quad -(4 + 3) - 8 = -4.4
\]

\[
c = 2 \quad \Rightarrow \quad -52
\]
13. ANSWER: A

\[
\begin{align*}
  r &= 0, \quad s = 1 \\
x &= 0, \quad y = 1 \\
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} \\
&= 10(x^2 + y^2)^9 \cdot 2 \cdot 8rs^3 + 10(x^2 + y^2)^9 \cdot 2y \cdot e^{2r+3s-3} \cdot 2 \\
&= 10 \cdot 2 \cdot 2 = 40
\end{align*}
\]

14. ANSWER: B

\[
\begin{align*}
x + z + y \frac{\partial z}{\partial y} + 2z \frac{\partial z}{\partial y} &= 0 \\
\frac{\partial z}{\partial y} &= -\frac{x + z}{y + 2z}
\end{align*}
\]

15. ANSWER: B

\[
\begin{align*}
\int_{-1}^{1} \int_{y}^{y^2} (2x + 3y) \, dx \, dy &= \\
&= \int_{-1}^{1} \left[ x^2 + 3xy \right]_{x=y}^{x=y^2} \, dy \\
&= \int_{-1}^{1} \left[ (y^4 + 3y^3) - (y^2 + 3y^2) \right] \, dy \\
&= \left[ \frac{y^5}{5} + y^4 - \frac{4y^3}{3} \right]_{-1}^{1} \\
&= \frac{2}{5} - \frac{8}{3} = -\frac{34}{15}
\end{align*}
\]
PART B

B1.

Let $X$ be the purchase price.
Let $R$ be the annual payment to the sinking fund.

\[
\begin{array}{cccccccc}
0 & 1 & 2 & \ldots & \ldots & \ldots & 12 \\
32,000 & 32,000 & 32,000 & \ldots & \ldots & 32,000 \\
R & R & R & \ldots & \ldots & X \\
\end{array}
\]

\[R_{s_{12},0.02} = X\]

Analogous to a bond with face value $X$ coupon $32,000 - R$ and yield to maturity .08 and price $X$.

\[X = X (1.08)^{-12} + (32,000 - R) a_{12|0.08}\]

Substituting $R = \frac{X}{s_{12|0.062}}$, this equation can be solved for $X$. More cleverly

\[X - X (1.08)^{-12} = (32,000 - R) a_{12|0.08}\]
\[X (1 - (1.08)^{-12}) = (32,000 - R) a_{12|0.08}\]
\[.08 X a_{12|0.08} = (32,000 - R) a_{12|0.08}\]

$.08X = 32,000 - R$: we could have started here. Since the purchase price is always being replaced, the 8% of purchase price need only cover the net annual income.

\[.08X + \frac{X}{s_{12|0.062}} = 32,000\]
\[X = \frac{32,000}{.08 + \frac{1}{s_{12|0.062}}}\]

\[X \approx \$230,900\]
B2.

(a) (i)

\[
\lim_{x \to 0} \frac{e^{2x+1} - e - ax}{x^2} = 0 \quad \text{no matter what } a \text{ is}
\]

\[
= \lim_{x \to 0} \frac{2e^{2x+1} - a}{2x}
\]

can only have a limit if \( \lim_{x \to 0} 2e^{2x+1} = a \) i.e. if

\[
a = 2e
\]

and then

\[
= \lim_{x \to 0} \frac{4e^{2x+1}}{2} = 2e
\]

so the limit does exist when \( a = 2e \).

(a) (ii)

\[
\lim_{x \to 0} \frac{e^{2x+1} - e - ax - bx^2}{x^3} = 0 \quad \text{no matter what } a \text{ and } b \text{ are}
\]

\[
= \lim_{x \to 0} \frac{2e^{2x+1} - a - 2bx}{3x^2}
\]

can only have a limit if

\[
\lim_{x \to 0} (2e^{2x+1} - a - 2bx) = 0
\]

i.e. \( a = 2e \) as before

and then

\[
= \lim_{x \to 0} \frac{4e^{2x+1} - 2b}{6x}
\]

can only have a limit if

\[
\lim_{x \to 0} (4e^{2x+1} = 2b)
\]

i.e. \( b = 2e \)

and then

\[
= \lim_{x \to 0} \frac{8e^{2x+1}}{6}
\]

\[
= \frac{4e}{3}
\]
B2.
(b) 
\[(x - 1)y \frac{dy}{dx} = 1\]
\[ y \ dy = \frac{dx}{x - 1} \]
\[ y^2 \ = \ \ln |x - 1| + C \]
\[ \frac{y^2}{2} = \ln 1 + C \Rightarrow C = \frac{1}{2} \]
\[ y^2 \ = \ \ln |x - 1| + \frac{1}{2} \]
\[ y^2 = 2 \ln |x - 1| + 1 \]
\[ y = \sqrt{\ln(x - 1)^2 + 1} \]
(Note that) \( y = -\sqrt{\ln(x - 1)^2 + 1} \) gives \( y(2) = -1 \).
\[ y = \sqrt{2 \ln |x - 1| + 1} \]
\[ y(3) = \sqrt{2 \ln 2 + 1} \]
\[ y(3) \approx 1.5448 \]

B3.
(a) Let
\[
= \lim_{R \to \infty} \int_{0}^{R} xe^{-3x} \, dx
\]
\[ u = x \quad dv = e^{-3x} \, dx \]
\[ du = dx \quad v = -\frac{e^{-3x}}{3} \]
\[
= \lim_{R \to \infty} \left[ -\frac{xe^{-3x}}{3} \bigg|_{0}^{R} + \frac{1}{3} \int_{0}^{R} e^{-3x} \, dx \right]
\]
\[ = \lim_{R \to \infty} \left[ -\frac{Re^{-3R}}{3} - \frac{1}{9} e^{-3x} \bigg|_{0}^{R} \right] \]
\[ = \lim_{R \to \infty} \left[ -\frac{Re^{-3R}}{3} - \frac{e^{-3R}}{9} + \frac{1}{9} \right] \]
But by L'Hop or otherwise
\[
\lim_{R \to \infty} Re^{-3R} = 0; \ e^{-3R} \to 0
\]
as well.

So \[ \lim_{R \to \infty} = \frac{1}{9} \]

D313
B3.

\[
\frac{x - 14}{x^2 - 4} = \frac{A}{x - 2} + \frac{B}{x + 2} = \frac{A(x + 2) + B(x - 2)}{x^2 - 4}
\]

\[
x - 14 = A(x + 2) + B(x - 2)
\]

\[
x = 2 \Rightarrow -12 = 4A \Rightarrow A = -3
\]

\[
x = -2 \Rightarrow -16 = -4B \Rightarrow B = 4
\]

Alternatively solve

\[
1 = A + B
\]

\[
-14 = 2A - 2B
\]

to get

\[
A = -3
\]

\[
B = 4
\]

\[
\int = \int \left[ -\frac{3}{x - 2} + \frac{4}{4 + 2} \right] dx = \begin{array}{c}
-3 \ln |x - 2| + 4 \ln |x + 2| + C
\end{array}
\]

B4.

\[
f_x = -3x^2 + 3y^2
\]

\[
f_y = 6xy + 24y + 12y^2
\]

setting these to 0

\[
y^2 = x^2 \Rightarrow y = x \quad \text{or} \quad y = -x
\]

and

\[
y(x + 4 + 2y) = 0
\]

If \( y = x \), \( y(3y + 4) = 0 \), hence \( y = 0 \) or \( y = -\frac{4}{3} \) Critical points are

\[
(0,0) \quad \text{and} \quad \left( -\frac{4}{3}, -\frac{4}{3} \right)
\]

If \( y = -x \), \( y(4 + y) = 0 \) \( y = 0 \), or \( y = -4 \). Critical points are \((0,0)\) which we already have and

\[
(4, -4)
\]
\[
\begin{array}{c|ccc}
\text{Point} & f_{xx} & f_{yy} = 6x + 24 + 24y & f_{xy} = 6y \\
\hline
(0, 0) & -6x & 24 & 0 \\
(4, -4) & -24 & -48 & -24 \\
\left(-\frac{4}{3}, \ -\frac{4}{3}\right) & 8 & -16 & -8 \\
\end{array}
\]

\[D(x, y) = f_{xx}f_{yy} - f_{xy}^2\]

\[
D(0, 0) = 0 \quad \text{[no info.]} \\
D(4, -4) = 24 \cdot 48 - 24^2 = 24^2 = 576 > 0 \text{ extremum and } f_{xx} = -24 < 0 \quad \text{local max.} \\
D\left(-\frac{4}{3}, -\frac{4}{3}\right) = 8(-16) - 8^2 = -64 < 0 \quad \text{no local extremum.}
\]

B5.

\[20x + 30y = 600\]
\[2x + 3y = 60\]

\[L = 10x^{0.6}y^{0.4} - \lambda(2x + 3y - 60)\]

\[L_x = 6x^{-0.4}y^{0.4} - 2\lambda = 0 \Rightarrow \lambda = 3\left(\frac{y}{x}\right)^{0.4}\]

\[L_y = 4x^{0.6}y^{-0.6} - 3\lambda = 0 \Rightarrow \lambda = \frac{4}{3}\left(\frac{x}{y}\right)^{0.6}\]

\[L_y = -(2x + 3y - 60) = 0 \Rightarrow 2x + 3y = 60\]

\[3\left(\frac{y}{x}\right)^{0.4} = \frac{4}{3}\left(\frac{x}{y}\right)^{0.6} \Rightarrow 3y = \frac{4}{3}x\]

So

\[2x + \frac{4}{3}x = 60 \Rightarrow \frac{10x}{3} = 60 \Rightarrow x = 18\]

\[3y = \frac{4}{3} \cdot 18 \Rightarrow y = 8\]

So

\[x = 18, \ y = 8\]