

FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2004

MAT 133Y1Y

Calculus and Linear Algebra for Commerce

PART A. MULTIPLE CHOICE

1. [3 marks]

If $f(x) = \frac{(x^2 + e^{2x})^3 e^{-2x}}{(1 + x - x^2)^{2/3}}$, then $f'(0) =$

- A. 0
- B. 1
- C. $e^{\frac{10}{3}}$
- D. $\frac{10}{3}$
- E. $e^{\frac{16}{3}}$

2. [3 marks]

Let $f(x) = \begin{cases} e^{-x} & x \leq 0 \\ ax + b & x > 0 \end{cases}$ where a and b are constants.

If f is differentiable at $x = 0$, then $f(x) = 0$ when $x =$

- A. 1
- B. 2
- C. $\ln 2$
- D. $\ln 3$
- E. e

3. [3 marks]

The graph of $y = (1 - \ln x)^2$ has

- A. no inflection points
- B. an inflection point at $x = 1$
- C. an inflection point at $x = 2$
- D. an inflection point at $x = \sqrt{e}$
- E. an inflection point at $x = e^2$

4. [3 marks]

The largest value attained by $y = x \ln x$ if $0 < x \leq e$ is

- A. e
- B. $\frac{1}{e}$
- C. 0
- D. 1
- E. 2

5. [3 marks]

$\lim_{x \rightarrow 1} x^{\frac{1}{1-x}} =$

- A. e
- B. $\frac{1}{e}$
- C. 1
- D. 0
- E. undefined

6. [3 marks]

If $F(x) = \int_x^7 e^t \ln |t| dt$, then $F''(1) =$

- A. $-e$
- B. e
- C. 0
- D. $-2e$
- E. $2e$

7. [3 marks]

$$\int_0^1 \frac{x}{x^2 + 1} dx =$$

- A. $= \ln \sqrt{2}$
- B. diverges
- C. $= e$
- D. $= 0$
- E. $= \ln 2 - 1$

8. [3 marks]

The area between $f(x) = \frac{e^x}{1 + e^x}$ and the x -axis from $x = 0$ to $x = \ln 5$ is

- A. $\ln 3$
- B. $\frac{1}{3}$
- C. $\ln 4$
- D. $\frac{4}{\ln 5}$
- E. $\frac{5}{6}$

9. [3 marks]

The area bounded by the graphs of $y = x^{\frac{2}{3}}$ and $y = x^{\frac{3}{2}}$ is

- A. $\frac{1}{4}$
- B. $\frac{1}{20}$
- C. $\frac{1}{5}$
- D. $\frac{1}{10}$
- E. $\frac{3}{20}$

10. [3 marks]

The determinant of

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}^T \begin{pmatrix} 2 & 2 \\ 5 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 5 \\ 1 & 1 \end{pmatrix}$$

is

- A. -24
- B. $\frac{3}{2}$
- C. 24
- D. $-\frac{5}{2}$
- E. $-\frac{3}{2}$

11. [3 marks]

To approximate $\int_0^1 \sqrt{1+x^3} dx$ dividing the interval $[0, 1]$ into 4 equal parts and using the Trapezoidal Rule, $T_4 =$

- A. $\frac{1}{4} \left[1 + \sqrt{1 + \frac{1}{64}} + \sqrt{1 + \frac{1}{8}} + \sqrt{1 + \frac{27}{64}} \right]$
- B. $\frac{1}{4} \left[1 + \sqrt{1 + \frac{1}{64}} + \sqrt{1 + \frac{1}{8}} + \sqrt{1 + \frac{27}{64}} + \sqrt{2} \right]$
- C. $\frac{1}{4} \left[1 + \sqrt{1 + \frac{1}{64}} + \sqrt{1 + \frac{1}{8}} + 2\sqrt{1 + \frac{27}{64}} + \sqrt{2} \right]$
- D. $\frac{1}{12} \left[1 + 2\sqrt{1 + \frac{1}{64}} + 4\sqrt{1 + \frac{1}{8}} + 2\sqrt{1 + \frac{27}{64}} + \sqrt{2} \right]$
- E. $\frac{1}{8} \left[1 + 2\sqrt{1 + \frac{1}{64}} + 2\sqrt{1 + \frac{1}{8}} + 2\sqrt{1 + \frac{27}{64}} + \sqrt{2} \right]$

12. [3 marks]

If $\frac{dy}{dx} = xy$ and $y = 1$ when $x = 1$, then when $x = 0$, $y =$

- A. 0
- B. $\frac{1}{2}$
- C. \sqrt{e}
- D. $\frac{1}{\sqrt{e}}$
- E. 2

13. [3 marks]

If $f(x, y) = 3x^3 - 2x^2 + 3y - y^3$ then the number of critical points of f is

- A. 0
- B. 1
- C. 2
- D. 3
- E. 4

14. [3 marks]

The graph of $z = y^2 - x^3 + x$ has

- A. one relative maximum and one critical point which is not an extremum
- B. one relative minimum and one critical point which is not an extremum
- C. one relative maximum and one relative minimum
- D. two relative maxima (maximums for those who don't know Latin)
- E. two critical points which are not extrema

15. [3 marks]

$$\int_{-1}^1 \int_x^{x^2} (6x - 12xy) dy dx =$$

- A. -6
- B. -4
- C. 12
- D. 4
- E. 0

PART B. WRITTEN-ANSWER QUESTIONS

B1. [10 marks]

An investor has a choice of two different investments.

- an ordinary annuity which costs \$50,000 and has 16 semiannual payments at interest rate 4% compounded semiannually.
- 47 \$1,000 bonds all of the same issue. Each bond matures in 8 years and has semiannual coupons worth \$25 each. The total price of the 47 bonds is \$50,000.

(a) [4 marks]

What is the amount of each payment of the annuity?

(b) [6] marks

Which investment has the higher yield?

Show your work.

B2. [12 marks]

A manufacturer has been selling 1000 bicycles a week at \$450 each. A market survey indicates that for each \$10 rebate offered to the buyer, the number of bicycles sold will increase by 100 per week.

[You do not have to do (a) to do (b) and (c).]

(a) [3 marks]

If p is the price of one bicycle, and q is the number of bicycles sold in a week, what is the demand function?

(b) [5 marks]

How large a rebate should the company offer the buyer to maximize its revenue?

(c) [4 marks]

If its weekly cost function is $C = 68,000 + 150q$, where C is in dollars, how large a rebate should the company offer to maximize profit?

B3. [12 marks]

(a) [6 marks]

Evaluate $\int_1^4 x^2 \ln(3x) dx$

(b) [6 marks]

Evaluate $\int_1^\infty \frac{3}{x^2(x+3)} dx$ or show that the integral diverges.

B4. [11 marks]

(a) [6 marks]

Given: $x^2 = y^2 + z^2 - 2xy$.

Find: $\frac{\partial^2 z}{\partial y \partial x}$ and show that it can be written in the form: $\frac{2xy}{z^3}$

(b) [5 marks]

If $w = x^2 + 2xy + 3y^2$, $x = e^r$, and $y = \ln(r + s)$ then find $\frac{\partial w}{\partial s}$ when $r = 0$ and $s = e$.

B5. [10 marks]

Use the method of Lagrange multipliers to find the minimum value of

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraint

$$3x + 2y + z = 6$$

[You do **not** have to show that the value you find is in fact a minimum; but you **do** have to use Lagrange multipliers.]

Solutions to April 2004 Exam, MAT133Y

PART A

1. ANSWER: D.

$$\begin{aligned} \ln f(x) &= 3 \ln(x^2 + e^{2x}) - 2x - \frac{2}{3} \ln(1 + x - x^2) \\ \frac{1}{f(x)} f'(x) &= \frac{3(2x + 2e^{2x})}{x^2 + e^{2x}} - 2 - \frac{2(1 - 2x)}{3(1 + x - x^2)} \\ \frac{f'(0)}{f(0)} &= \frac{3 \cdot 2}{1} - 2 - \frac{2}{3} = \frac{10}{3} \\ f(0) &= 1 \\ f'(0) &= \boxed{\frac{10}{3}} \end{aligned}$$

2. ANSWER: A.

For f to be diff. at $x = 0$, f must be cont. at $x = 0$ so $e^0 = a \cdot 0 + b$ i.e. $b = 1$ and $f'(0)$ must be the same from both sides, so

$$\begin{aligned} -e^0 &= a, \quad \text{i.e. } a = -1 \\ f(x) &= \begin{cases} e^{-x} & x \leq 0 \\ -x + 1 & x > 0 \end{cases} = 0 \quad \text{only if } \boxed{x = 1} \end{aligned}$$

3. ANSWER: E.

$$\begin{aligned} f'(x) &= 2(1 - \ln x)\left(-\frac{1}{x}\right) = \frac{2(\ln x - 1)}{x} \\ f''(x) &= 2\left[\frac{1 - (\ln x - 1)}{x^2}\right] = \frac{2(2 - \ln x)}{x^2} \end{aligned}$$

which changes sign at $\ln x = 2$, i.e. $\boxed{x = e^2}$

4. ANSWER: A.

$$y' = 1 + \ln x = 0 \quad \text{when } \ln x = -1, \text{ i.e. } x = \frac{1}{e}$$

	y'	y
$(0, \frac{1}{e})$	-	dec
$(\frac{1}{e}, e)$	+	inc

but $x = \frac{1}{e}$ is the min.

$\lim_{x \rightarrow 0} x \ln x = 0$ and $f(e) = e \ln e = e$ so $x = e$ $\boxed{f(e) = e}$ is the max.

5. ANSWER: B.

$$y = x^{\frac{1}{1-x}} \quad \ln y = \frac{\ln x}{1-x}$$

$$\lim_{x \rightarrow 1} \ln y = \lim_{x \rightarrow 1} \frac{\ln x}{1-x} = \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{-1}$$

because

$$\frac{0}{0} = -1$$

so

$$\lim_{x \rightarrow 1} y = e^{-1} = \boxed{\frac{1}{e}}$$

6. ANSWER: A.

$$F(x) = - \int_7^x e^t \ln |t| dt$$

$$F'(x) = -e^x \ln |x|$$

$$F''(x) = -e^x \ln |x| - \frac{e^x}{x} \quad \text{so}$$

$$F''(1) = \boxed{-e}$$

7. ANSWER: A.

$$\begin{aligned} \int_0^1 \frac{x}{x^2+1} dx &= \frac{1}{2} \int_0^1 \frac{2x}{x^2+1} dx \\ &= \frac{1}{2} \ln(x^2+1) \Big|_0^1 \\ &= \frac{1}{2} \ln 2 = \boxed{\ln \sqrt{2}} \end{aligned}$$

8. ANSWER: A.

$f(x) > 0$ always, so

$$\begin{aligned} A &= \int_0^{\ln 5} \frac{e^x}{1+e^x} dx \\ &= \ln(1+e^x) \Big|_0^5 \\ &= \ln(1+5) - \ln(1+1) \\ &= \ln\left(\frac{6}{2}\right) \\ &= \boxed{\ln 3} \end{aligned}$$

9. ANSWER: C.

$$x^{\frac{2}{3}} = x^{\frac{3}{2}} \quad \text{when } x = 0 \quad \text{and} \quad x = 1 \quad [0 = x^{\frac{3}{2}} - x^{\frac{2}{3}} = x^{\frac{2}{3}}(x^{\frac{5}{6}} - 1)].$$

At $x = \frac{1}{2}$ $x^{\frac{2}{3}} = 4^{\frac{1}{3}}$ and $x^{\frac{3}{2}} = \frac{1}{\sqrt{8}}$ so $x^{\frac{2}{3}} > x^{\frac{3}{2}}$ $0 < x < 1$.

$$A = \int_0^1 (x^{\frac{2}{3}} - x^{\frac{3}{2}}) dx = \left[\frac{3}{5}x^{\frac{5}{3}} - \frac{2}{5}x^{\frac{5}{2}} \right]_0^1 = \boxed{\frac{1}{5}}$$

10. ANSWER: D.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 6 & 8 \end{pmatrix}$$

whose $\det = -2$.

$$\det \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}^T = -5$$

and

$$\det \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}^{-1} = \frac{1}{\det \begin{pmatrix} 2 & 2 \\ 5 & 3 \end{pmatrix}} = \frac{1}{-4}.$$

So

$$\det A = (-5)\left(-\frac{1}{4}\right)(-2) = \boxed{-\frac{5}{2}}$$

11. ANSWER: E.

$$\frac{b-a}{n} = \frac{1}{4} = \Delta x \quad T_x = \frac{\Delta x}{2} [y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}, \quad x_4 = 1,$$
$$y_0 = 1, \quad y_1 = \sqrt{1 + \frac{1}{64}}, \quad y_2 = \sqrt{1 + \frac{1}{8}}, \quad y_3 = \sqrt{1 + \frac{27}{64}}, \quad y_4 = \sqrt{2}$$

By inspection of the formula for T_4 , E is the answer.

12. ANSWER: D.

$$\int \frac{dy}{y} = \int x dx \quad \ln |y| = \frac{x^2}{2} + C$$
$$y = Ae^{\frac{x^2}{2}} \quad 1 = Ae^{\frac{1}{2}}$$

so

$$A = e^{-\frac{1}{2}} \quad y = \frac{e^{\frac{x^2}{2}}}{\sqrt{e}}$$

so

$$y(0) = \boxed{\frac{1}{\sqrt{e}}}$$

13. ANSWER: E.

$$\frac{\partial f}{\partial x} = 9x^2 - 4x = x(9x - 4) = 0$$
$$\frac{\partial f}{\partial y} = 3 - 3y^2 = 3(1 - y)(1 + y) = 0$$

crit pts : $(0, 1), (0, -1), (\frac{4}{9}, 1), (\frac{4}{9}, -1)$: there are $\boxed{4}$

14. ANSWER: B.

$$\frac{\partial z}{\partial x} = -3x^2 + 1$$

$$\frac{\partial z}{\partial y} = 2y = 0$$

$$\text{crit pts } (\frac{1}{\sqrt{3}}, 0), (-\frac{1}{\sqrt{3}}, 0)$$

$$z_{xx} = -6x \quad z_{yy} = 2 \quad z_{xy} = 0 \quad D = -12x$$

$$D(\frac{1}{\sqrt{3}}, 0) = -\frac{12}{\sqrt{3}} < 0 \quad \boxed{\text{not an extremum}}$$

$$D(-\frac{1}{\sqrt{3}}, 0) = \frac{12}{\sqrt{3}} > 0 \quad \boxed{\text{an extremum}}$$

$$\text{and } z_{yy} = 2 > 0 \text{ so } \boxed{1 \text{ local min.}}$$

15. ANSWER: B.

$$\int_{-1}^1 \int_x^{x^2} (6x - 12xy) dy dx$$
$$= \int_{-1}^1 \left[6xy - 6xy^2 \right]_{y=x}^{x^2} dx = \int_{-1}^1 (6x^3 - 6x^5 - 6x^2 + 6x^3) dx$$
$$= \int_{-1}^1 (12x^3 - 6x^5 - 6x^2) dx = -12 \int_0^1 x^2 dx = \frac{-12}{3} = \boxed{-4}$$

PART B

B1.

(a)

$$50,000 = Ra_{\overline{16}|.02}$$
$$R = \frac{50,000}{a_{\overline{16}|.02}} = \frac{50,000 \times .02}{1 - (1.02)^{-16}} = \boxed{\$3,682.51}$$

(b)

$$50,000 = 47,000(1+i)^{-16} + 47.25a_{\overline{16}|i}$$

Since the yield on the annuity is 2% per half-year, the question is how i compares to 2%.

$$\text{If } i = .02, \quad P = 47,000(1.02)^{-16} + 47.25a_{\overline{16}|.02}$$

$$P = 50,190.76 > 50,000 \text{ the actual price}$$

At $i = .02$ the price comes out too high, so the yield is too low. The actual yield of the bond $> 2\%$.

The bond has the higher yield.

B2. Let x = no. of \$10 rebates in the price.

$$p = 450 - 10x \quad q = 1000 + 100x$$

(a)

$$x = \frac{450 - p}{10} \quad \frac{q - 1000}{100} = x$$

$$\text{So } \frac{450 - p}{10} = \frac{q - 1000}{100}$$

$$\boxed{p = 550 - \frac{q}{10} \quad \text{or} \quad q = 5500 - 10p}$$

(b)

$$\text{revenue} = R = pq = p(5500 - 10p)$$

$$\frac{dR}{dp} = 5500 - 20p = 0 \quad \text{when } \boxed{p = \$275}$$

which is a rebate of 175

Could also use

$$R = (450 - 10x)(1000 + 100x)$$
$$\frac{dR}{dx} = 35,000 - 2,000x = 0 \quad \text{when } x = 17.5$$

and the rebate is again $10 \times 17.5 = \$175$.

In both ways of doing the problem the answer is a max. because R is an inverted parabola which has max. at the only point where $R' = 0$.

(c)

$$\text{Profit} = \Pi = R - C = R(x) - [68,000 + 150(1000 + 100x)]$$
$$\frac{dC}{dx} = \frac{dR}{dx} - 15000 = 20,000 - 2000x = 0 \quad \text{when } x = 10$$

when rebate $\boxed{= \$100}$.

This is a max. for the same reason as in (b).

B3.

(a) Let

$$u = \ln 3x \quad dv = x^2 dx$$
$$du = \frac{1}{x} dx \quad v = \frac{x^3}{3}$$

$$\int_1^4 x^2 \ln 3x dx = \frac{x^3}{3} \ln 3x \Big|_1^4 - \int_1^4 \frac{x^2}{3} dx$$
$$= \frac{64}{3} \ln 12 - \frac{1}{3} \ln 3 - \frac{x^3}{9} \Big|_1^4$$
$$= \boxed{\frac{64}{3} \ln 12 - \frac{1}{3} \ln 3 - \left(\frac{64}{9} - \frac{1}{9}\right)}$$
$$= \boxed{21 \ln 3 + \frac{128}{3} \ln 2 - 7 \approx 45.6}$$

(b)

$$\frac{3}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$

$$Ax(x+3) + B(x+3) + Cx^2 = 3$$

$$x = 0 \quad \text{gives} \quad 3B = 3 \quad \text{or} \quad B = 1$$

$$x = -3 \quad \text{gives} \quad 9C = 3 \quad \text{or} \quad C = \frac{1}{3}$$

Since the coeff of x^2 is $A + C = 0$, $A = -C = -\frac{1}{3}$

$$\begin{aligned}
 \int_1^{\infty} \frac{3}{x^2(x+3)} dx &= \lim_{R \rightarrow \infty} \int_1^R \left[-\frac{1}{3} \cdot \frac{1}{x} + \frac{1}{3} \cdot \frac{1}{x+3} + \frac{1}{x^2} dx \right] \\
 &= \lim_{R \rightarrow \infty} \left[-\frac{1}{3} \ln|x| + \frac{1}{3} \ln|x+3| - \frac{1}{x} \right]_1^R \\
 &= \lim_{R \rightarrow \infty} \left[\frac{1}{3} \ln \left| \frac{x+3}{x} \right| - \frac{1}{x} \right]_1^R \\
 &= \lim_{R \rightarrow \infty} \left[\left(\frac{1}{3} \ln \frac{R+3}{R} - \frac{1}{R} \right) - \left(\frac{1}{3} \ln 4 - 4 \right) \right] \\
 &= \boxed{1 - \frac{1}{3} \ln 4 \approx .538}
 \end{aligned}$$

B4.

(a) Differentiating by x , considering z as a fcu. of x and y

$$2x = 2z \frac{\partial z}{\partial x} - 2y \quad \left(\frac{\partial z}{\partial x} = \frac{x+y}{z} \right)$$

Now differentiating by y ,

$$0 = 2 \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} + 2z \frac{\partial^2 z}{\partial y \partial x} - 2$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1 - \frac{\partial z}{\partial y} \frac{\partial z}{\partial x}}{z} \quad \text{we need } \frac{\partial z}{\partial y}.$$

$$0 = 2y + 2z \frac{\partial z}{\partial y} - 2x \quad \text{so } \frac{\partial z}{\partial y} = \frac{x-y}{z}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1 - \frac{(x-y)(x+y)}{z^2}}{z} = \frac{z^2 - (x^2 - y^2)}{z^3}$$

But the original equation says $z^2 + y^2 - x^2 = 2xy$, so $\frac{\partial^2 z}{\partial y \partial x} = \frac{2xy}{z^3}$

One can also take

$$\frac{\partial z}{\partial x} = \frac{x+y}{z}$$

so

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{z - (x+y) \frac{\partial z}{\partial y}}{y^2} = \frac{z - (x+y) \frac{(x-y)}{z}}{z^2}$$

if we find $\frac{\partial z}{\partial y}$ as above. This lead to the same place.

(b)

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \\ &= (2x + 2y) \cdot 0 + (2x + 6y) \cdot \frac{1}{r + s}\end{aligned}$$

When $r = 0$ and $s = e$, $x = 1$ and $y = \ln(0 + e) = 1$ so

$$\frac{\partial w}{\partial s} = \frac{2 + 6}{0 + e} = \boxed{\frac{8}{e}}$$

B5.

$$L = x^2 + y^2 + z^2 - \lambda(3x + 2y + z - 6)$$

$$L_x = 2x - 3\lambda = 0 \quad x = \frac{3\lambda}{2}$$

$$L_y = 2y - 2\lambda = 0 \quad y = \lambda$$

$$L_z = 2z - \lambda = 0 \quad z = \frac{\lambda}{2}$$

$$6 = 3x + 2y + z = \frac{9\lambda}{2} + 2\lambda + \frac{\lambda}{2} = 7\lambda \quad \text{so} \quad \lambda = \frac{6}{7}$$

$$\boxed{x = \frac{3\lambda}{2} = \frac{9}{7}, \quad y = \lambda = \frac{6}{7}, \quad z = \frac{\lambda}{2} = \frac{3}{7}}$$

Point is $\boxed{\left(\frac{9}{7}, \frac{6}{7}, \frac{3}{7}\right)}$

and $f = x^2 + y^2 + z^2 - \frac{81 + 36 + 9}{49} = \frac{126}{49} = \boxed{\frac{18}{7}} \approx 2.57$