1. [3 marks]
A $100,000 mortgage is to be repaid over 10 years by equal monthly payments made at the end of each month; that is, the first payment is one month after the loan is made. If interest is 10% compounded semiannually, the amount of each payment is

A $1,264.45
B $1,310.34
C $1,273.96
D $1,299.72
E $1,335.10

2. [3 marks]
A $100 bond with 10 years until maturity has semiannual coupons at an annual coupon rate of 10%. If its annual yield is 9%, then its market price is

A $108.48
B $107.76
C $105.97
D $109.03
E $106.50

3. [3 marks]
If $A$ is a $4 \times 4$ matrix and $\det(A) = 5$, then $\det(3A^2) =$

A 225
B 75
C 2025
D $-75$
E 300
4. [3 marks]

\[ f(x) = (x^2 - 4)^{1/3} \]

- an absolute minimum at \( x = 0 \) and no absolute maximum
- absolute minimums at \( x = 2 \) and \( x = -2 \) and a relative maximum at \( x = 0 \)
- a relative minimum at \( x = 0 \) and absolute maximums at \( x = -2 \) and \( x = 2 \)
- a relative maximum at \( x = 0 \) and no relative minimums
- no relative maximums or minimums

5. [3 marks]

If the cost function is given by:

\[ c = 0.01q^2 + 6q + 100 \]

Then average cost \( \bar{c} \) is minimized when \( q = \)

- 300
- 0
- 1
- 100
- 1000

6. [3 marks]

\[ \lim_{x \to +\infty} \frac{\ln(1 + e^{2x})}{x} \]

- is 1
- is 4
- is 0
- is 2
- does not exist
7. [3 marks]

\[ \int_{e^4}^e \frac{1}{x(\ln x)^{1/2}} \, dx = \]

A \hspace{0.5cm} \frac{1}{e^2} \\
B \hspace{0.5cm} \frac{1}{2e^4} \\
C \hspace{0.5cm} \frac{1}{2e^4} - \frac{1}{2e} \\
D \hspace{0.5cm} 2(e - \sqrt{e}) \\
E \hspace{0.5cm} 2

8. [3 marks]

If \( \int_{1}^{x} f(t) \, dt = e^x \ln x \) when \( x > 0 \), then \( f(1) = \)

A \hspace{0.5cm} e \\
B \hspace{0.5cm} 2 \\
C \hspace{0.5cm} 1 \\
D \hspace{0.5cm} 0 \\
E \hspace{0.5cm} e^x \ln x

9. [3 marks]

\[ \int \frac{x^2 + 1}{x^2 + x} \, dx = \]

A \hspace{0.5cm} x + \ln |x| - 2 \ln |x + 1| + C \\
B \hspace{0.5cm} \ln |x| + \ln |x + 1| + C \\
C \hspace{0.5cm} x^{-1} + 2(x + 1)^{-1} + C \\
D \hspace{0.5cm} x - \ln x + 2 \ln(x + 1) + C \\
E \hspace{0.5cm} \ln \left| \frac{x}{x + 1} \right| + C
10. [3 marks]
Using a subdivision of the interval [1, 3] into 4 subintervals of equal length, the trapezoidal rule yields the following approximation for
\[
\int_1^3 \frac{1}{\ln(x + 1)} \, dx:
\]
A 1.8722  
B 2.7230  
C 1.7342  
D 0.9863  
E 1.9409

11. [3 marks]
If \( a > 0 \), \( \int_0^\infty ax e^{-ax} \, dx \)
A diverges  
B equals \( a \)  
C equals \( \frac{1}{a} \)  
D equals 1  
E equals \( e^{-a} \)

12. [3 marks]
If \( \frac{dy}{dx} = 2xy \) and \( y = e \) when \( x = 0 \) then, when \( x = 1 \), \( y = \)
A 1  
B \( e \)  
C \( e^2 \)  
D \( e^3 \)  
E \( e^4 \)
13. [3 marks]
If \( f(x, y) = e^{xy} \), then when \( x = 2 \) and \( y = 3 \), \( \frac{\partial^3 f}{\partial x \partial y^2} = \)
\[ \begin{align*}
\text{A} & \quad 24e^6 \\
\text{B} & \quad 16e^6 \\
\text{C} & \quad 30e^6 \\
\text{D} & \quad 12e^6 \\
\text{E} & \quad 18e^6 
\end{align*} \]

14. [3 marks]
Let \( x(r, s) \) and \( y(r, s) \) be functions such that:
\[ \begin{align*}
\frac{\partial x}{\partial r}(2, 1) &= -7 \\
\frac{\partial x}{\partial s}(2, 1) &= -2 \\
\frac{\partial y}{\partial r}(2, 1) &= 8 \\
\frac{\partial y}{\partial s}(2, 1) &= 4 
\end{align*} \]
If \( z = 2x^2 + xy + 3y^2 \), then when \( (r, s) = (2, 1) \), \( \frac{\partial z}{\partial s} = \)
\[ \begin{align*}
\text{A} & \quad 14 \\
\text{B} & \quad -223 \\
\text{C} & \quad 1 \\
\text{D} & \quad -86 \\
\text{E} & \quad -47 
\end{align*} \]

15. [3 marks]
For \( p_A > 0 \) and \( p_B > 0 \), products A and B have joint demand functions
\[ q_A(p_A, p_B) = 10 - 4p_A - 2p_B - 3p_A^2 + p_B^2 \]
and
\[ q_B(p_A, p_B) = 7 + 4p_A - p_B - p_A^2 - 5p_B^2. \]
For which \( p_A \) and \( p_B \) are the two products complementary?
\[ \begin{align*}
\text{A} & \quad p_A < 2 \text{ and } p_B > 1 \\
\text{B} & \quad p_A < 1 \text{ and } p_B > 2 \\
\text{C} & \quad p_A > 2 \text{ and } p_B < 1 \\
\text{D} & \quad p_A < 2 \text{ and } p_B < 1 \\
\text{E} & \quad p_A > 1 \text{ and } p_B < 2 
\end{align*} \]
PART B. WRITTEN-ANSWER QUESTIONS

B1.

(a) [7 marks]

Use the method of Lagrange multipliers to minimize $f(x, y) = x^2y$ subject to the constraint $\frac{1}{x} + \frac{1}{y} = 1$.

[No need to justify that you are indeed at a minimum.]

(b) [5 marks]

Given that $x, y$ and $z$ satisfy

$$xy + yz + z^3x = 14$$

find $\frac{\partial z}{\partial x}$ when $(x, y, z) = (4, 2, 1)$

B2.

(a) [7 marks]

Find and classify all critical points of the function

$$f(x, y) = x^2 - 12y^2 + 4y^3 + 3y^4$$

(b) [5 marks]

Evaluate the following integral:

$$\int_0^1 \int_z^{2y} \int_0^{z^2} 30xyz \, dx \, dy \, dz$$

B3. Consider the function

$$f(x) = \begin{cases} x \ln |x| - x & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

(a) [2 marks]

Find all points where $f$ is not continuous (if any), showing that your answer is correct.

(b) [9 marks]

Sketch the graph of $y = f(x)$. A complete answer includes an explanation of all standard features of the graph.

(c) [6 marks]

Find the area bounded by the graph of $y = f(x)$ and the $x$ axis, where

$$f(x) = \begin{cases} x \ln |x| - x & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$
B4.

(a)  [8 marks]
Mr. Abbott owes Mr. Costello two debts:
- $300 due in 5 years
- $100 plus interest at 7% compounded annually, due in 3 years.

They have agreed that the combined debt is to be settled with 3 payments:
- the first payment to be made now
- the second payment to be twice the amount of the first, to be made in 2 years
- the third payment to be twice the amount of the second, to be made in 4 years.

If money is worth 8% compounded quarterly, what is the amount (to the nearest cent) of the first payment?

(b)  [6 marks]
Assume $A$ is a square matrix such that

\[
A \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 6 \\ 3 & 0 & 9 \end{bmatrix}.
\]

Find $A^{-1}$. 

Solutions to April 2003 Exam, MAT133Y

PART A

1. ANSWER: ③

\[(1 + i)^{12} = (1.05)^2\]

\[100,000 = Ra_{12|i}\]

\[R = \frac{100,000i}{1 - (1 + i)^{-120}} = \frac{100,000[(1.05)^{\frac{i}{12}} - 1]}{1 - (1.05)^{-20}}\]

\[R = \$1310.34\]

2. ANSWER: ⑤

\[P = 100(1.045)^{-20} + 5a_{\overline{20}|.045}\]

\[P = \$106.50\]

3. ANSWER: ②

If \(A\) is \(n \times n\), \(\text{det}(kA) = k^n \text{det}A\)

\[\text{det}(3A^2) = 3^4 \text{ det}(A^2) = 3^4 \times 5^2 \quad \text{since} \quad \text{det}(A^2) = (\text{det}A)(\text{det}A)\]

\[\text{det}(3A^2) = 2025\]

4. ANSWER: ①

\[f'(x) = \frac{1}{3}(x^2 - 4)^{-\frac{2}{3}} \cdot 2x = \frac{2x}{3(x^2-4)^{\frac{2}{3}}}\]

with critical points at \(x = -2, 0, 2\);

but \(f' < 0\) when \(x < 0\)

and \(f' > 0\) when \(x > 0\)

so there is an absolute minimum at \(x = 0\) and no other extrema of any kind.

5. ANSWER: ③

\[\bar{c} = \frac{c}{q} = .01q + 6 + \frac{100}{q}\]

\[\frac{dc}{dq} = .01 - \frac{100}{q^2} = 0 \quad \text{when} \quad q^2 = 10,000\]

\[q = 100\]

and \(\frac{dc}{dq} = \frac{0.1}{q^2}(q^2 - 10,000)\) \(\left\{\begin{array}{ll}
< 0 & 0 < q < 100 \\
> 0 & 100 < q
\end{array}\right\}\]

6. ANSWER: ③

\[\lim_{x \to \infty} \frac{\ln(1 + e^{2x})}{x} = \lim_{x \to \infty} \frac{2e^{2x}}{1 + e^{2x}}\]

\[\lim_{x \to \infty} \frac{4e^{2x}}{2e^{2x}} = 2\]
7. ANSWER: \(E\) 
Let \(u = \ln x; \ du = \frac{dx}{x}\) and \(u = 1\) when \(x = e, \ u = 4\) when \(x = e^4\).

\[
\int_e^{e^4} \frac{1}{x(\ln x)^{\frac{3}{2}}} \ dx = \int_1^{e^4} \frac{du}{u^{\frac{3}{2}}} = 2u^{\frac{1}{2}} \bigg|_1^4 = 2(2 - 1) = 2
\]

8. ANSWER: \(A\) 
By the Fundamental Theorem of Calculus

\[
f(x) = (e^x \ln x) = e^x \ln x + \frac{e^x}{x}
\]

\[
f(1) = e \cdot 0 + \frac{e}{1} = e
\]

9. ANSWER: \(A\) 

\[
\frac{x^2 + 1}{x^2 + x} = 1 + \frac{-x + 1}{x(x + 1)}
\]

\[
\frac{-x + 1}{x(x + 1)} = \frac{A}{x} + \frac{B}{x + 1}
\]

\[
A(x + 1) + Bx = -x + 1
\]

\[
x = 0 \Rightarrow A = 1
\]

\[
x = -1 \Rightarrow -B = 2 \Rightarrow B = -2
\]

\[
\int \frac{x^2 + 1}{x^2 + x} \ dx = \int \left(1 + \frac{1}{x} - \frac{2}{x + 1}\right) \ dx
\]

\[
= \left[ x + \ln |x| - 2 \ln |x + 1| + C \right]
\]

10. ANSWER: \(E\) 

\[
T_4 = \frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + 2y_3 + y_4), \quad \Delta x = \frac{3 - 1}{4} = \frac{1}{2}
\]

\[
= \frac{1}{4} \left( \frac{1}{\ln 2} + \frac{2}{\ln 2.5} + \frac{2}{\ln 3} + \frac{2}{\ln 3.5} + \frac{1}{\ln 4} \right)
\]

\[
= 1.9409
\]

11. ANSWER: \(C\) 

\[
\int_0^\infty ax e^{-ax} \ dx = \lim_{R \to \infty} \int_0^R ax e^{-ax} \ dx
\]

Let \(u = x, \ dv = ae^{-ax} \ dx\)

\[
du = dx, \ v = -e^{-ax}
\]
\[= \lim_{R \to \infty} \left[ -xe^{-ax} \bigg|_0^R + \int_0^R e^{-ax} \, dx \right] \]
\[= \lim_{R \to \infty} -Re^{-aR} - \lim_{R \to \infty} \frac{1}{a} e^{-ax} \bigg|_0^R \]
\[= 0 - \lim_{R \to \infty} \frac{1}{a} (e^{-aR} - 1) \]
\[= \frac{1}{a} \]

12. ANSWER: C

\[\frac{dy}{y} = 2x \, dx \]
\[\ln y = x^2 + C \quad \ln e = C \text{ so } C = 1 \]
\[\ln y = x^2 + 1 \quad \text{so when } x = 1, \ln y = 2 \]
\[y = e^2 \]

13. ANSWER: B

\[f_y = xe^{xy} \]
\[f_{yy} = x^2 e^{xy} \]
\[f_{yyx} = 2xe^{xy} + x^2 e^{xy} = (2x + x^2 y)e^{xy} \]
\[= (4 + 12)e^6 = 16e^6 \quad \text{at } x = 2, y = 3. \]

14. ANSWER: D

\[\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (4x + y) \frac{\partial x}{\partial s} + (x + 6y) \frac{\partial y}{\partial s} \]
\[= (4 \cdot 5 - 3)(-2) + (5 - 6 \cdot 3)(4) \quad \text{when } (r, s) = (2, 1) \]
\[= -86 \]

15. ANSWER: C

\[\frac{\partial q_A}{\partial p_B} = -2 + 2p_B = 2(p_B - 1); \quad \frac{\partial q_B}{\partial p_A} = 4 - 2p_A = 2(2 - p_A) \]
\[\frac{\partial q_A}{\partial p_B} < 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} < 0 \quad \text{when } p_A > 2 \quad \text{and} \quad p_B < 1 \]
**PART B**

**B1.**

(a) 

\[ \mathcal{L} = x^2 y - \lambda \left( \frac{1}{x} + \frac{1}{y} - 1 \right) \]

\[ \mathcal{L}_x = 2xy + \frac{\lambda}{x^2} = 0 \quad \mathcal{L}_y = -\left( \frac{1}{x} + \frac{1}{y} - 1 \right) = 0 \]

\[ \mathcal{L}_y = x^2 + \frac{\lambda}{y^2} = 0 \]

\[ x^2 y^2 = -\lambda = 2x^3 y \]

and because \( \frac{1}{x} + \frac{1}{y} = 1, x \neq 0 \) and \( y \neq 0 \). Dividing by \( x^2 y \), 
\( y = 2x \).

\[
\begin{align*}
\frac{1}{x} + \frac{1}{2x} &= 1 \\
\frac{3}{2x} &= 1 \Rightarrow x = \frac{3}{2} \\
\frac{1}{x} + \frac{1}{y} &= 1 \Rightarrow \frac{2}{3} + \frac{1}{y} = 1 \Rightarrow y = 3
\end{align*}
\]

so \( x = \frac{3}{2}, \ y = 3 \) \( \lambda = -\frac{81}{4} \).

(b)

\[
y + y \frac{\partial z}{\partial x} + z^3 + 3z^2 x \frac{\partial z}{\partial x} = 0
\]

\[
2 + 2 \frac{\partial z}{\partial x} + 1 + 12 \frac{\partial z}{\partial x} = 0.
\]

At \( (4, 2, 1) = (x, y, z) \)

\[ 3 + 14 \frac{\partial z}{\partial x} = 0 \]

\[
\frac{\partial z}{\partial x} = \frac{-3}{14}
\]

Alternatively \( \frac{\partial z}{\partial x} = -\frac{y + z^3}{y + 3z^2 x} = \frac{-3}{14} \) at \( (4, 2, 1) \).

**B2.**

(a)

\[ f_x = 2x \]

\[ f_y = -24y + 12y^2 + 12y^3 = 12y(y^2 + y - 2) = 12y(y + 2)(y - 1) \]

\[ f_x = 0 \Rightarrow x = 0 \]
\[ f_y = 0 \Rightarrow y = 0, 1, \text{ or } -2 \]

**Crit pts: \((0, 0), (0, 1), (0, -2)\)**

\[ D = f_{xx}f_{yy} - f_{xy}^2 = 2(-24 + 24y + 36y^2) - 0^2 \]

\[ D = 24(3y^2 + 2y - 2) \]

| \(D(0, 0)\) | \(-48 < 0\) | no extremum (or saddle pt) |
| \(D(0, 1)\) | \(72 > 0\) | \(f_{xx} = 2 > 0\) local min |
| \(D(0, -2)\) | \(144 > 0\) | \(f_{xx} = 2 > 0\) local min |

(b)

\[
\int_0^1 \int_z^2 \int_0^{2y} 30xyz \, dx \, dy \, dz = \int_0^1 \int_z^2 \int_0^{2y} 15x^2 \bigg|_{x=0}^{x=2y} \, yz \, dy \, dz
\]

\[ = \int_0^1 \int_z^2 15 \cdot 4y^2 \cdot yz \, dy \, dz \]

\[ = \int_0^1 \int_z^2 60y^3z \, dy \, dz \]

\[ = \int_0^1 15y^4 \bigg|_{y=z}^{y=2} \, z \, dz \]

\[ = \int_0^1 15(z^8 - z^4) \, dz \]

\[ = \int_0^1 15(z^9 - z^5) \, dz \]

\[ = 15 \left( \frac{z^{10}}{10} - \frac{z^6}{6} \right) \bigg|_0^1 \]

\[ = 15 \left( \frac{1}{10} - \frac{1}{6} \right) = 15 \left( \frac{-2}{30} \right) = -1 \]
B3.

(a) The only difficulty is at $x = 0$:

$$\lim_{x \to 0} x \ln |x| = \lim_{x \to 0} \frac{\ln |x|}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{-x^2} = \lim_{x \to 0} -x = 0$$

So $\lim_{x \to 0} f(x) = \lim_{x \to 0} (x \ln |x| - x) = 0 = f(0)$

$f$ is continuous everywhere.

(b) $f'(x) = \ln |x|

crit pts at $x = \pm 1$ and 0

<table>
<thead>
<tr>
<th>$f'$</th>
<th>$f''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, -1)$</td>
<td>+ inc</td>
</tr>
<tr>
<td>$(-1, 0)$</td>
<td>- dec</td>
</tr>
<tr>
<td>$(0, 1)$</td>
<td>- dec</td>
</tr>
<tr>
<td>$(1, \infty)$</td>
<td>+ inc</td>
</tr>
</tbody>
</table>

$x = -1$ local max, $x = 1$ local min

$f''(x) = \frac{1}{x}$

<table>
<thead>
<tr>
<th>$f''$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>- conc down</td>
</tr>
<tr>
<td>$(0, \infty)$</td>
<td>+ conc up</td>
</tr>
</tbody>
</table>

$x = 0$ pt. of inflection

$$\lim_{x \to \infty} x \ln |x| - x = \lim_{x \to \infty} x [\ln |x| - 1] = \infty$$

$$\lim_{x \to -\infty} x [\ln |x| - 1] = -\infty$$

no H.A. no V.A.
B3. cont.

(c) \[
\text{Area} = \int_{-e}^{e} (x \ln |x| - x) \, dx - \int_{0}^{e} (x \ln |x| - x) \, dx \\
= -2 \int_{0}^{e} (x \ln |x| - x) \, dx \\
\]

\[
u = \ln |x| - 1 \quad \text{dv} = x \, dx \quad \\du = \frac{1}{x} \, dx \quad v = \frac{x^2}{2}
\]

\[
\text{Area} = -2 \left[ \frac{x^2}{2} (\ln |x| - 1) \right]^{e}_{0} - \frac{1}{2} \int_{0}^{e} x \, dx \]

\[
= 0 + \int_{0}^{e} x \, dx = \frac{e^2}{2}
\]

Note that we have used \( \lim_{x \to 0^+} x^2 \ln |x| = 0 \) in evaluating the integral.

B4.

(a) \[
\begin{array}{c|c|c|c|c}
0 & 8 & 12 & 16 & 20 \\
\hline
x & 2x & 4x & \$300
\end{array}
\]

Let \( x \) be the first payment

\[
x + 2x(1.02)^{-8} + 4x(1.02)^{-16} = 100(1.07)^3(1.02)^{-12} + 300(1.02)^{-20}
\]

\[
x = \frac{100(1.07)^3(1.02)^{-12} + 300(1.02)^{-20}}{1 + 2(1.02)^{-8} + 4(1.02)^{-16}}
\]

\[
x = \$53.10
\]

(b) \[
\begin{pmatrix}
2 & -1 & 0 \\
1 & 0 & 3
\end{pmatrix}
= A^{-1}
\begin{pmatrix}
0 & 1 & 6 \\
3 & 0 & 9
\end{pmatrix}
\]

Let \( A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \)

\[
\begin{pmatrix}
2 & -1 & 0 \\
1 & 0 & 3
\end{pmatrix}
= \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\begin{pmatrix}
0 & 1 & 6 \\
3 & 0 & 9
\end{pmatrix}
= \begin{pmatrix} 3b & a & 6a + 9b \\ 3d & c & 6c + 9d \end{pmatrix}
\]

Then \( 3b = 2 \quad -1 = a \quad \text{and} \quad 6a + 9b = 0 \)

\[
\begin{align*}
3d &= 1 \\
0 &= c \\
6c + 9d &= 3
\end{align*}
\]

\[
a = -1 \quad b = \frac{2}{3} \quad c = 0 \quad d = \frac{1}{3}
\]

\[
A^{-1} = \begin{pmatrix}
-1 & \frac{2}{3} \\
0 & \frac{1}{3}
\end{pmatrix}
\]