Solutions to Supplementary Questions for HP Chapter 14

1. (a)

i) $\dfrac{d}{dx}\big(\ln(1+x)-\ln(2+x)\big) = \dfrac{1}{1+x} - \dfrac{1}{2+x}$

ii) $\dfrac{d}{dx}e^{x^2} = e^{x^2}\big(\dfrac{d}{dx}(x^2)\big) = 2xe^{x^2}$

iii) $\dfrac{d}{dx}(1+x^2)^{15} = 15(1+x^2)^{14}\big(\dfrac{d}{dx}(1+x^2)\big) = 30x(1+x^2)^{14}$

(b)

i) Since $\dfrac{d}{dx}e^{x^2} = 2xe^{x^2}$, then $\dfrac{d}{dx}e^{\frac{x^2}{2}} = xe^{\frac{x^2}{2}}$, hence $\displaystyle\int xe^{\frac{x^2}{2}}dx = e^{\frac{x^2}{2}} + c$

ii) The answer in (iii) is 30 times larger than the integrand, so try $\dfrac{1}{30}(1+x^2)^{15}$. Here $\dfrac{d}{dx}\dfrac{1}{30}(1+x^2)^{15} = x(1+x^2)^{14}$, hence $\displaystyle\int x(1+x^2)dx = \dfrac{1}{30}(1+x^2)^{15}$.

iii) Let A, B be constants such that

$$\frac{A}{1+x} + \frac{B}{2+x} = \frac{1}{(1+x)(2+x)} \Rightarrow \frac{(2+x)A+(1+x)B}{(1+x)(2+x)} = \frac{(A+B)x+(2A+B)}{(1+x)(2+x)} = \frac{1}{(1+x)(2+x)}\ .$$

It now follows that:

(1) $A+B=0$ (since the coefficient of x is zero on the R.H.S.) (2) $2A+B=1$

Solving, we get $A=1$, $B=-1$, and so $\dfrac{1}{(1+x)(2+x)} = \dfrac{1}{1+x} - \dfrac{1}{2+x}$.

Hence $\displaystyle\int \frac{1}{(1+x)(2+x)}dx = \int \Big(\frac{1}{(1+x)} - \frac{1}{(2+x)}\Big)dx$, and from part (a) i), we see that this is $\ln(1+x) - \ln(2+x) + c$.

iv) Noting a pattern from question (ii), we try $(1+x^2)^{13}$: $\dfrac{d}{dx}(1+x^2)^{13} = 26x(1+x^2)^{12}$. This is 26 times too big, so: $\displaystyle\int x(1+x^2)^{12}dx = \frac{1}{26}(1+x^2)^{13}$.

v) Using the same strategy as (iii), set

$$\frac{1}{(1+x)(3+x)} = \frac{A}{(1+x)} + \frac{B}{(3+x)} = \frac{(3+x)A+(1+x)B}{(1+x)(3+x)} = \frac{(A+B)x+(3A+B)}{(1+x)(3+x)}\ ,$$

so:

(1) $A+B=0$ (2) $3A+B=1$.
Solving, we get $A=\frac{1}{2}$ and $B=-\frac{1}{2}$ so

$$\begin{aligned}\int \frac{1}{(1+x)(3+x)}dx &= \int (\frac{1/2}{(1+x)} - \frac{1/2}{(3+x)})dx \\ &= \frac{1}{2}\int \frac{1}{1+x}dx - \frac{1}{2}\int \frac{1}{3+x}dx \\ &= \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(3+x) + c \\ &= \ln\sqrt{1+x} + \ln\frac{1}{\sqrt{3+x}} + c \\ &= \ln\sqrt{\frac{1+x}{3+x}} + c\end{aligned}$$

2.

$$\begin{aligned}x^7+x^6-x-1 &= (x+1)(x^6-1) \\ &= (x+1)(x-1)(x^5+x^4+x^3+x^2+x+1) \\ &= (x^2-1)(x^5+x^4+x^3+x^2+x+1)\end{aligned}$$

Also, $x^4-x^2 = x^2(x^2-1)$ so

$$\begin{aligned}\int \frac{x^7+x^6-x-1}{x^4-x^2}dx &= \int \frac{(x^2-1)(x^5+x^4+x^3+x^2+x+1)}{(x^2-1)x^2}dx \\ &= \int \frac{x^5}{x^2}dx + \int \frac{x^4}{x^2}dx + \int \frac{x^3}{x^2}dx + \int \frac{x^2}{x^2}dx + \int \frac{x}{x^2}dx + \int \frac{1}{x^2}dx \\ &= \int x^3dx + \int x^2dx + \int xdx + \int dx + \int \frac{1}{x}dx + \int \frac{1}{x^2}dx \\ &= \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x| - \frac{1}{x} + c\end{aligned}$$

3. First, we calculate $v(t)$ and $p(t)$:

1) $v(t) = \int a(t)dt = 6(\frac{1}{2}t^2) - 9t + c = 3t^2 - 9t + c$. But $v(1)=0$, so $3(1)^2 - 9(1) + c = 0 \Rightarrow c = 6$, so $v(t) = 3t^2 - 9t + 6 \quad (t \geq 0)$

2. $p(t) = \int v(t)dt = 3(\frac{1}{3}t^3) - 9(\frac{1}{2}t^2) + 6t + c' = t^3 - \frac{9}{2}t^2 + 6t + c'$, where $p(0) = 1$, and so $(0)^3 - \frac{9}{2}(0)^2 + 6(0) + c' = 1 \Rightarrow c' = 1$, so $p(t) = t^3 - \frac{9}{2}t^2 + 6t + 1 \quad (t \geq 0)$

Now we answer the questions:

(a) $v(5) = 3(5)^2 - 9(5) + 6 = 75 - 45 + 6 = 36$m/s.

(b) $p(2) = 2^3 - \frac{9}{2}(2)^2 + 6(2) + 1 = 8 - 18 + 12 + 1 = 3$m.

(c) $v(\frac{5}{4}) = 3(\frac{5}{4})^2 - 9(\frac{5}{4}) + 6 = \frac{75}{16} - \frac{180}{16} + \frac{96}{16} = -\frac{9}{16}$ m/s
$a(\frac{5}{4}) = 6(\frac{5}{4}) - 9 = \frac{15}{2} - \frac{18}{2} = -\frac{3}{2}$ m/s^2
The object has negative velocity at this time, and it is getting even more negative, so the object is speeding up.

(d) We check for critical points:

$$\begin{aligned} v(t) = 0 &\Rightarrow 3t^2 - 9(t) + 6 = 0 \\ &\Rightarrow t^2 - 3t + 2 = 0 \\ &\Rightarrow (t-2)(t-1) = 0 \\ &\Rightarrow t = 1 \quad \text{or } t = 2 \end{aligned}$$

1) Now, $p(1) = (1)^3 - \frac{9}{2}(1)^2 + 6(1) + 1 = \frac{7}{2}$m., and this is a maximum, since $a(1) = 6(1) - 9 = -3\,\text{m/s}^2$ is negative.

2) $p(2) = 3$m. (from (b) above), and this is a minimum, since $a(2) = 6(2) - 9 = 3\,\text{m/s}^2$ is positive.
We also check endpoints, so $p(0) = 1$m., and $\lim_{t\to\infty} t^3 - \frac{9}{2}t^2 + 6t + 1 = \lim_{t\to\infty} t^3 = \infty$.

So the object starts at $x = 1$m., moves right to $x = \frac{7}{2}$m., then left to $x = 3$m., and then moves right from then on. So the closest the object is to the origin is at the beginning, when $x = 1$m.

4.

$$\frac{df}{dx} = \frac{1}{x^2} \Rightarrow f(x) = \begin{cases} -\frac{1}{x} + c_1 & x > 0 \\ -\frac{1}{x} + c_2 & x < 0 \end{cases}$$

Now $f(1) = 1$, and since $x > 0$, we have $-\frac{1}{1} + c_1 = 1 \Rightarrow c_1 = 2$. But also $f(-1) = 2$, and since $x < 0$, we have $\frac{-1}{-1} + c_2 = 2 \Rightarrow c_2 = 1$. So

$$f(x) = \begin{cases} -\frac{1}{x} + 2 & x > 0 \\ -\frac{1}{x} + 1 & x < 0 \end{cases},$$

and $\frac{d}{dx}f(x) = \frac{1}{x^2} \quad (x \neq 0)$.

5. (a) Since $\frac{d}{dx}e^{2x} = 2e^{2x}$, then $\frac{d}{dx}\frac{e^{2x}}{2} = e^{2x}$, hence $\int e^{2x} = \frac{e^{2x}}{2} + c$.

(b) Similarly, $\int 2^x dx = \frac{2^x}{\ln 2} + c$ since $\frac{d}{dx}\frac{2^x}{\ln 2} = \frac{\ln 2}{\ln 2}2^x$.

(c) Since $\frac{d}{dx}a^{bx} = b\ln a(a^{bx})$, then $\int a^{bx} = \frac{a^{bx}}{b\ln a} + c$.

6. Let $u = 1 + x^2$. Then $du = 2xdx$, so $xdx = \frac{du}{2}$. Also, $x^2 = u - 1$, so $x^4 = (u-1)^2$:

$$\begin{aligned}
\int \sqrt{1+x^2}x^5 dx &= \int \sqrt{1+x^2}x^4 \cdot xdx \\
&= \int \sqrt{u}(u-1)^2 \frac{du}{2} \\
&= \frac{1}{2}\int \sqrt{u}(u^2 - 2u + 1)du \\
&= \frac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})du \\
&= \frac{1}{2}(\frac{2}{7}u^{7/2} - 2\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}) + c \\
&= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + c
\end{aligned}$$

7. (a)

$$\begin{aligned}
\frac{d}{dx}(\frac{x^2}{2(1-x^2)} + c) &= \frac{2x\big(2(1-x^2)\big) - (-4x)(x^2)}{[2(1-x^2)]^2} \\
&= \frac{4x - 4x^3 + 4x^3}{4(1-x^2)^2} \\
&= \frac{x}{(1-x^2)^2}\ .
\end{aligned}$$

So $\int \frac{x}{(1-x^2)^2}dx = \frac{x^2}{2(1-x^2)} + c$.

(b) Let $u = 1 - x^2$. Then $du = -2xdx$. So

$$\begin{aligned}
\int \frac{x}{(1-x^2)^2}dx &= -\frac{1}{2}\int \frac{-2x}{(1-x^2)^2}dx \\
&= -\frac{1}{2}\int \frac{1}{u^2}du \\
&= -\frac{1}{2}(-\frac{1}{u}) + c \\
&= \frac{1}{2u} + c \\
&= \frac{1}{2(1-x^2)} + c
\end{aligned}$$

Although in both parts (a) and (b), we write "c" for the constant of integration, this is sloppy and we should write (for example) C_1 for part (a) and C_2 for part (b). Now,

$$\Big(\frac{x^2}{2(1-x^2)}+C_1\Big)-\Big(\frac{1}{2(1-x^2)}+C_2\Big)=\frac{x^2-1}{2(1-x^2)}+C_1-C_2=-\frac{1}{2}+C_1-C_2\ ,$$

and so the two competing solutions differ by a constant. In fact, if we let $C_2 = C_1-\frac{1}{2}$, then $-\frac{1}{2}+C_1-C_2=0$, and hence $\Big(\frac{x^2}{2(1-x^2)}+C_1\Big)=\Big(\frac{1}{2(1-x^2)}+C_2\Big)$.

8.

$$\begin{aligned}\int[f(x)g''(x)-g(x)f''(x)]dx &= \int[f(x)g''(x)+f'(x)g'(x)]-[g(x)f''(x)+g'(x)f'(x)]\\ &= \int[f(x)g''(x)+f'(x)g'(x)]dx-\int[g(x)f''(x)+g'(x)f'(x)]dx\\ &= f(x)g'(x)+C_1-[g(x)f'(x)+C_2] \quad (\text{ Let } C_3=C_1-C_2)\\ &= f(x)g'(x)-g(x)f'(x)+C_3\end{aligned}$$

9.

SOLUTION A.

$\displaystyle\int\frac{e^x-1}{e^x+1}dx=\int\frac{e^x}{e^x+1}dx-\int\frac{1}{e^x+1}dx$

i) $\displaystyle\int\frac{e^x}{e^x+1}dx$: Let $u=e^x+1$. Then $du=e^x dx$, and

$$\int\frac{e^x}{e^x+1}dx=\int\frac{1}{u}=\ln|u|+C_1=\ln|e^x+1|+C_1=\ln(e^x+1)+C_1$$

ii) $\displaystyle\int\frac{1}{e^x+1}dx$. Multiply through by $\frac{e^{-x}}{e^{-x}}$: $\displaystyle\int\frac{1}{e^x+1}dx=\int\frac{e^{-x}}{1+e^{-x}}dx.$
Now let $u=1+e^{-x}$. Then $du=-e^{-x}dx$, so

$$\begin{aligned}\int\frac{e^{-x}}{1+e^{-x}}dx &= -\int\frac{-e^{-x}}{1+e^{-x}}dx=-\int\frac{1}{u}du=-\ln|u|+C_2=-\ln|1+e^{-x}|+C_2\\ &= -\ln(1+e^{-x})+C_2\end{aligned}$$

$$\begin{aligned}\therefore\ \int\frac{e^x-1}{e^x+1}dx &= \big(\ln(e^x+1)+C_1\big)-\big(-\ln(1+e^{-x})+C_2\big) \quad (\text{ Let } C_3=C_1-C_2)\\ &= \ln\big((e^x+1)(e^{-x}+1)\big)+C_3\\ &= \ln(2+e^x+e^{-x})+C_3\end{aligned}$$

OR alternatively,

SOLUTION B.

Multiply through by $\dfrac{(e^{-x}+1)}{(e^{-x}+1)}$ to get $\displaystyle\int \frac{(e^x-1)(e^{-x}+1)}{(e^x+1)(e^{-x}+1)}dx = \int \frac{e^x-e^{-x}}{2+e^x+e^{-x}}dx$. Now let $u = 2+e^x+e^{-x}$. Then $du = (e^x - e^{-x})dx$, and so:

$$\begin{aligned}
\int \frac{e^x-1}{e^x+1}dx &= \int \frac{e^x-e^{-x}}{2+e^x+e^{-x}}dx \\
&= \int \frac{1}{u}du \\
&= \ln|u| + C \\
&= \ln|2+e^x+e^{-x}| + C \\
&= \ln(2+e^x+e^{-x}) + C
\end{aligned}$$

10.

$$\begin{aligned}
C'(x) = 4,000 + \frac{(2x-1)(5x+3)}{x-5} \Rightarrow C(x) &= \int (4,000 + \frac{(2x-1)(5x+3)}{x-5})dx \\
&= 4,000x + C_1 + \int \frac{(2x-1)(5x+3)}{x-5})dx
\end{aligned}$$

Now,

$$\begin{aligned}
\int \frac{(2x-1)(5x+3)}{x-5})dx &= \int \frac{10x^2+x-3}{x-5}dx \\
&= \int ((10x+51) + \frac{252}{x-5})dx \\
&= \int 10x dx + \int 51 dx + \int (\frac{252}{x-5})dx \\
&= 5x^2 + 51x + 252\ln|x-5| + C_2
\end{aligned}$$

$$\begin{aligned}
\therefore\ C(x) &= 4,000x + 5x^2 + 51x + 252\ln|x-5| + K \quad (\text{ where } K = C_1 + C_2) \\
&= 5x^2 + 4,051x + 252\ln|x-5| + K
\end{aligned}$$

Since the fixed costs are 1,000,000 Riyal, $C(0) = 1,000,000$.

Hence $5(0)^2 + 4,051(0) + 252\ln|0-5| + C = 1,000,000$ Riyal,

$\Rightarrow 252\ln 5 + C = 1,000,000$

$\Rightarrow C = 1,000,000 - 252\ln 5 \quad (\ln 5 \approx 1.609)$

$\approx 999,594$ Riyal

So $C(x) = x^2 + 4,051x + 252\ln|x-5| + 999,594$ Riyal.

11. (a)

$$\begin{aligned}\sum_{i=1}^{n}(2i-1) &= \sum_{i=1}^{n}2i+\sum_{i=1}^{n}-1\\ &= 2\sum_{i=1}^{n}i-\sum_{i=1}^{n}1\\ &= 2\Big(\frac{n(n+1)}{2}\Big)-n\\ &= n(n+1)-n\\ &= n^2\end{aligned}$$

(b)

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)^2 &= \sum_{i=1}^{n}(4i^2-4i+1)\\ &= 4\sum_{i=1}^{n}i^2-4\sum_{i=1}^{n}i+\sum_{i=1}^{n}1\\ &= 4\Big(\frac{n(n+1)(2n+1)}{6}\Big)-4\Big(\frac{n(n+1)}{2}\Big)+n\\ &= \frac{2n(n+1)(2n+1)-6n(n+1)+3n}{3}\\ &= \frac{2n(n+1)(2n+1)-3n\big((2n+2)-1\big)}{3}\\ &= \frac{2n(2n+1)(n+1)-3n(2n+1)}{3}\\ &= \frac{n(2n+1)[(2n+2)-3]}{3}\\ &= \frac{n(2n+1)(2n-1)}{3}\end{aligned}$$

12.

SOLUTION 1. As was stated, the length of a side of the square is:

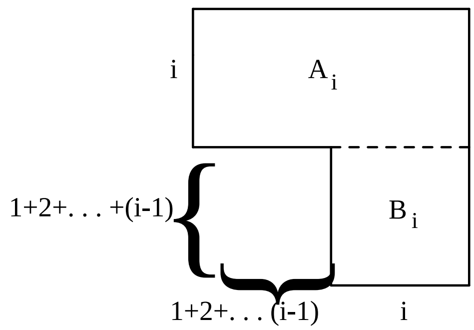
$$1+2+3+\cdots+n=\sum_{i=1}^{n}=\frac{n(n+1)}{2}\ ,$$

and hence the area of the square is $[\frac{n(n+1)}{2}]^2$.
That is one way to calculate the area of the square.

SOLUTION 2. Another way to calculate the area of the square is to find the area of each gnomon G_i, and then sum these areas.

So consider an arbitrary gnomon $G_i (1 \le i \le n)$. (In particular the following *does* work for $i = 1$ as well).

G_i looks as follows:



Divide the gnomon into two pieces A_i and B_i by extending the "middle" line.

The area of A_i is $\quad i\big(\frac{i(i+1)}{2}\big)$.

The area of B_i is $\quad i\big(1+2+3+\cdots+(i-1)\big)$.

$$= i\Big(\frac{(i-1)\big((i-1)+1\big)}{2}\Big)$$

$$= i\Big(\frac{(i-1)i}{2}\Big) \text{ (note if } i=1 \text{ then this is zero as it should be)}$$

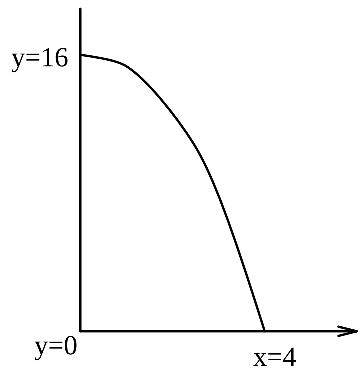
So the area of G_i is:

$$\begin{aligned} G_i = A_i + B_i &= i\Big(\frac{i(i+1)}{2}\Big) + i\Big(\frac{i(i-1)}{2}\Big) \\ &= \frac{i^2}{2}\big((i+1)+(i-1)\big) \\ &= \frac{i^2}{2}(2i) = i^3 \end{aligned}$$

So the area of the square, which is the sum of the areas of the gnomons, is $\sum_{i=1}^{n} i^3$.

We conclude that $\sum_{i=1}^{n} i^3 = \big[\frac{n(n+1)}{2}\big]^2$ because Solution 1 and Solution 2 both give the area of the same square.

13. (a) In short, $f'(x) = -2x$, $f''(x) = -2$ is concave down everywhere. Also $f'(x) = 0 \Rightarrow x = 0$, and lastly $f(0) = 16$ and $f(4) = 0$. We sketch:



(b)

i) $\overline{S}_4 = f(1) + f(2) + f(3) + f(4)$
$= 15 + 12 + 7 + 0$
$= 34$

ii) $\underline{S}_4 = f(0) + f(1) + f(2) + f(3)$
$= 16 + 15 + 12 + 7$
$= 50$

iii) $\not{S}_4 = f(\frac{1}{2}) + f(\frac{3}{2}) + f(\frac{5}{2}) + f(\frac{7}{2})$
$= \frac{63}{4} + \frac{55}{4} + \frac{39}{4} + \frac{15}{4} = \frac{172}{4}$
$= 43$

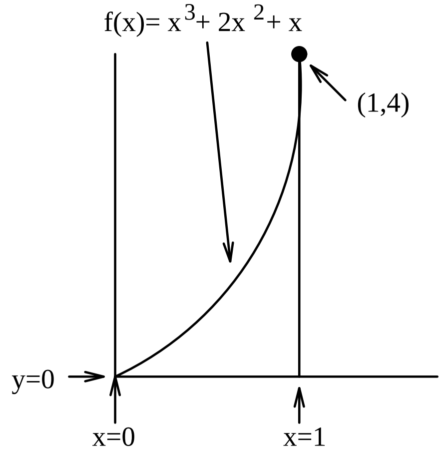
(c)

$$\begin{aligned}
\lim_{n\to\infty} \overline{S}_n &= \lim_{n\to\infty} \Big[\frac{4}{n}\big(16 - (\frac{4}{n})^2\big) + \frac{4}{n}\big(16 - (\frac{8}{n})^2\big) + \cdots + \frac{4}{n}\big(16 - (\frac{4n}{n})^2\big)\Big] \\
&= \lim_{n\to\infty} \Big[\frac{4}{n}\big(16n - \frac{4^2}{n^2}(\sum_{i=1}^{n} i^2)\big)\Big] \\
&= \lim_{n\to\infty} \Big[64 - \frac{64}{n^3}\big(\frac{n(n+1)(2n+1)}{6}\big)\Big] \\
&= 64 - \lim_{n\to\infty} \frac{32}{3}(1 + \frac{1}{n})(2 + \frac{1}{n}) \\
&= 64 - \frac{64}{3} \\
&= \frac{128}{3} \\
&= 42\frac{2}{3}
\end{aligned}$$

14. (a) The only problematic curve is $f(x) = x^3 + 2x^2 + x$.
In short, $f'(x) = 3x^2 + 4x + 1$ and $f''(x) = 6x + 4$.

Letting $f'(x) = 0$, we get $3x^2 + 4x + 1 = 0 \Rightarrow x = -1$ or $x = -\frac{1}{3}$. There is an inflection point at $x = -\frac{2}{3}$, but we are only concerned with x in $[0, 1]$, so everywhere in $[0, 1]$, the curve is concave up (since $f''(0) = 6(0) + 4 = 4$, for instance).

Noting that $f(0) = 0$ and $f(1) = 1 + 2 + 1 = 4$, we graph:



(b) First, we calculate S_n: Since the distance from $x = 1$ to $x = 0$ is one, the length of each subinterval is just $\frac{1}{n}$, so:

$$
\begin{aligned}
S_n &= \frac{1}{n}f(\frac{1}{n}) + \frac{1}{n}f\big(2(\frac{1}{n})\big) + \frac{1}{n}f\big(3(\frac{1}{n})\big) + \cdots + \frac{1}{n}\big(f\big(n(\frac{1}{n})\big)\big) \\
&= \frac{1}{n}\big(f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n}{n})\big) \\
&= \frac{1}{n}\big([(\frac{1}{n})^3 + 2(\frac{1}{n})^2 + (\frac{1}{n})] + [(\frac{2}{n})^3 + 2(\frac{2}{n})^2 + (\frac{2}{n})] + \cdots + [(\frac{n}{n})^3 + 2(\frac{n}{n})^2 + (\frac{n}{n})]\big) \\
&= \frac{1}{n}\big([(\frac{1}{n})^3 + (\frac{2}{n})^3 + \cdots + (\frac{n}{n})^3] + \big[2(\frac{1}{n})^2 + 2(\frac{2}{n})^2 + \cdots + 2(\frac{n}{n})^2\big] \\
&\quad + [(\frac{1}{n}) + (\frac{2}{n}) + \cdots + (\frac{n}{n})]\big) \\
&= \frac{1}{n}\big(\frac{1}{n^3}(\sum_{i=1}^{n} i^3) + \frac{2}{n^2}(\sum_{i=1}^{n} i^2) + \frac{1}{n}(\sum_{i=1}^{n} i)\big) \\
&= \frac{1}{n}\big(\frac{1}{n^3}(\frac{n(n+1)}{2})^2 + \frac{2}{n^2}(\frac{n(n+1)(2n+1)}{6}) + \frac{1}{n}(\frac{n(n+1)}{2})\big) \\
&= \frac{1}{4}\frac{n^2(n+1)^2}{n^4} + \frac{1}{3}\frac{n(n+1)(2n+1)}{n^3} + \frac{1}{2}\frac{n(n+1)}{n^2}
\end{aligned}
$$

Now,
$$
\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\big(\frac{1}{4}(1 + \frac{1}{n})^2 + \frac{1}{3}(1 + \frac{1}{n})(2 + \frac{1}{n}) + \frac{1}{2}(1 + \frac{1}{n})\big) \\
&= \frac{1}{4} + \frac{2}{3} + \frac{1}{2} = \frac{17}{12}
\end{aligned}
$$

This is the area of the region.

15. (a) Using the hint, we guess that there is an interval of length one, each subinterval being of length $\frac{1}{n}$. By doing this, we get $\frac{i^4}{n^4}$ remaining within the $\sum$ sign, which is leading us towards $f(x) = x^4$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i^4}{n^4} \ .$$

Hence, $S_n = \frac{1}{n} \sum_{i=1}^{n} (\frac{i}{n})^4$
$= \frac{1}{n}[(\frac{1}{n})^4 + (\frac{2}{n})^4 + (\frac{3}{n})^4 + \cdots + (\frac{n}{n})^4]$
Letting $f(x) = x^4$ as per the hint, we get:
$= \frac{1}{n}f(\frac{1}{n}) + \frac{1}{n}f(\frac{2}{n}) + \frac{1}{n}f(\frac{3}{n}) + \cdots + \frac{1}{n}f(\frac{n}{n})$ This is just summing the areas of the n rectangles found under the function $f(x)$ from $x = 0$ to $x = 1$.
Hence, $\lim_{n\to\infty} (\sum_{i=1}^{n} \frac{i^4}{n^5}) = \int_0^1 x^4 dx$

(b) $S_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + (\frac{i}{n})^2}$. Let $f(x) = \frac{1}{1+x^2}$

$\Rightarrow S_n = \frac{1}{n}f(\frac{1}{n}) + \frac{1}{n}f(\frac{2}{n}) + \cdots + \frac{1}{n}f(\frac{n}{n})$

$\Rightarrow \lim_{n\to\infty} S_n = \int_0^1 \frac{1}{1+x^2}dx$

(c) $S_n = \sum_{i=1}^{n} [3(1 + \frac{2i}{n})^5 - 6]\frac{2}{n}$
$= \frac{2}{n} \sum_{i=1}^{n} 3(1 + \frac{2i}{n})^5 - 6$. Let $f(x) = 3(1+x)^5 - 6$
$= \frac{2}{n} \sum_{i=1}^{n} f(\frac{2i}{n})$
$= \frac{2}{n}f(\frac{2}{n}) + \frac{2}{n}f(\frac{4}{n}) + \frac{2}{n}f(\frac{6}{n}) + \cdots + \frac{2}{n}f(\frac{2n}{n})$
Hence, $\lim_{n\to\infty} S_n = \int_0^2 \big(3(1+x)^5 - 6\big)dx$.

16. First, $\int_{-a}^{a} x^2 dx = \frac{x^3}{3}\Big|_{-a}^{a} = \frac{a^3}{3} - \frac{(-a)^3}{3} = \frac{2a^3}{3}$. Now,

(a) $\int_0^{2a} (u+a)^2 du = \frac{(u+a)^3}{3}\Big|_0^{2a} = \frac{(3a)^3}{3} - \frac{a^3}{3} = \frac{26a^3}{3}$

(b) $2\int_0^a x^2 dx = 2\left(\frac{x^3}{3}\Big|_0^a\right) = 2\left(\frac{a^3}{3} - 0\right) = \frac{2a^3}{3}$

(c) $-\int_a^{-a} x^2 dx = \int_{-a}^{a} x^2 dx \left(= \frac{2a^3}{3}\right)$

(d) $\int_{-a}^{a} (x^2 + x)dx = \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{-a}^{a} = \left(\frac{a^3}{3} + \frac{a^2}{2}\right) - \left(\frac{(-a^3)}{3} - \frac{(-a)^2}{2}\right)$
$= \left(\frac{a^3}{3} - \frac{-a^3}{3}\right) + \left(\frac{a^2}{2} - \frac{a^2}{2}\right) = \frac{2a^3}{3}$

(e) $\int_0^{2a} (u-a)^2 du = \frac{(u-a)^3}{3}\Big|_0^{2a} = \frac{(2a-a)^3}{3} - \frac{(0-a)^3}{3} = \frac{a^3}{3} - \frac{-a^3}{3} = \frac{2a^3}{3}.$

So the answer is (a).

17. First, calculate $\int_{g(x)}^{h(x)} f(t)dt$:
By the Fundamental Theorem, this is

$$F\big(h(x)\big) - F\big(g(x)\big) \ , \quad \text{where } F \text{ is any antiderivative of } f.$$

Now,

$$\begin{aligned}\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)dt\right) &= \frac{d}{dx}\big(F(h(x)) - F(g(x))\big)\\ &= \frac{d}{dx}F(h(x)) - \frac{d}{dx}F(g(x))\\ &= f(h(x))h'(x) - f(g(x))g'(x) \ \blacksquare\end{aligned}$$

18. (a)

$$\begin{aligned}x^2 - 1 \geq 0 &\Leftrightarrow (x-1)(x+1) \geq 0\\ &\Leftrightarrow (x-1) = 0, x+1 = 0, [(x-1) > 0 \text{ and } (x+1) > 0] \text{ or } [(x-1) < 0\\ &\qquad \text{and } (x+1) < 0]\\ &\Leftrightarrow x = 1, x = -1, (x > 1 \text{ and } x > -1) \text{ or } (x < 1 \text{ and } x < -1)\\ &\Leftrightarrow x \geq 1 \text{ or } x \leq -1\end{aligned}$$

Then $x^2 - 1 < 0 \Leftrightarrow -1 < x < 1.$

Now

$$\begin{aligned}
\int_{-2}^{3}|x^2-1|dx &= \int_{-2}^{-1}|x^2-1|dx + \int_{-1}^{1}|x^2-1| + \int_{1}^{3}|x^2-1|dx \\
&= \int_{-2}^{-1}(x^2-1)dx + \int_{-1}^{1}-(x^2-1)dx + \int_{1}^{3}(x^2-1)dx \\
&= (\frac{x^3}{3}-x)\Big|_{-2}^{-1} + (\frac{-x^3}{3}+x)\Big|_{-1}^{1} + (\frac{x^3}{3}-x)\Big|_{1}^{3} \\
&= \big([\frac{(-1)^3}{3}-(-1)] - [\frac{(-2)^3}{3}-(-2)]\big) + \big([\frac{-(1)^3}{3}+1] - [\frac{-(-1)^3}{3}+(-1)]\big) \\
&\quad + \big([\frac{(3)^3}{3}-(3)] - [\frac{(1)^3}{3}-(1)]\big) \\
&= [\frac{2}{3}-(-\frac{2}{3})] + [\frac{2}{3}-(-\frac{2}{3})] + [6-(-\frac{2}{3})] \\
&= \frac{4}{3}+\frac{4}{3}+\frac{20}{3} = \frac{28}{3}
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{-1}^{2}(x-2|x|)dx &= \int_{-1}^{0}(x-2|x|)dx + \int_{0}^{2}(x-2|x|)dx \\
&= \int_{-1}^{0}\big(x-2(-x)\big)dx + \int_{0}^{2}(x-2x)dx \\
&= \int_{-1}^{0}3xdx + \int_{0}^{2}-xdx \\
&= \frac{3}{2}x^2\Big|_{-1}^{0} - \frac{x^2}{2}\Big|_{0}^{2} \\
&= \big(0-\frac{3}{2}(-1)^2\big) - \big(\frac{2^2}{2}-0\big) \\
&= -\frac{3}{2}-\frac{4}{2} = -\frac{7}{2}
\end{aligned}$$

(c) First, $x-1\geq 0 \Leftrightarrow x\geq 1$. Now

$$\begin{aligned}
\int_{0}^{2}(x^2-|x-1|)dx &= \int_{0}^{1}(x^2-|x-1|)dx + \int_{1}^{2}(x^2-|x-1|)dx \\
&= \int_{0}^{1}\big(x^2-(-(x-1))\big)dx + \int_{1}^{2}\big(x^2-(x-1)\big)dx \\
&= \int_{0}^{1}(x^2+x-1)dx + \int_{1}^{2}(x^2-x+1)dx \\
&= (\frac{x^3}{3}+\frac{x^2}{2}-x)\Big|_{0}^{1} + (\frac{x^3}{3}-\frac{x^2}{2}+x)\Big|_{1}^{2} \\
&= (-\frac{1}{6}-0) + (\frac{16}{6}-\frac{5}{6}) = \frac{5}{3}
\end{aligned}$$

19. (a) Although we could use the strategy suggested above, since we know $\int \frac{1}{x}dx$, we compute directly: $\int \frac{1}{x}dx = \ln|x| + C$.

i) $F(2) = 0 \Rightarrow \ln|2| + C = 0 \Rightarrow C = -\ln|2|$, so $F(x) = \ln|x| - \ln|2| = \ln|\frac{x}{2}|$.

ii) $F(2) = -3$. Here, we just need a function that is (uniformly) three less than the answer to (a), so we obtain: $F(x) = \ln|\frac{x}{2}| - 3$.

(b) Since we presently do not know how to obtain $\int \sqrt{1+x^3}dx$, we use the strategy suggested above. By the Preamble, $\int_a^x (\sqrt{1+t^3})dt$ is an antiderivative of $\sqrt{1+x^3}$ for any constant a.

i) $F(3) = 0$. Now we must choose an appropriate a. We need: $\int_a^3 (\sqrt{1+t^3})dt = 0$. Now it becomes obvious that we can choose $a = 3$, and now $F(x) = \int_3^x (\sqrt{1+t^3})dt$.

ii) $F(3) = 1$. This function is just (uniformly) one more than the function obtained in part (a): $F(x) = (\int_3^x \sqrt{1+t^3}dt) + 1$.

20. (a) $\int_0^2 4x^3 dx = x^4\Big|_0^2 = 16$

For Simpson's rule, $n = 4$ so $h = \frac{2-0}{4} = \frac{1}{2}$ and

$$\begin{aligned}\int_0^2 F(x)dx &\approx \frac{1}{6}[4(4)(\frac{1}{2})^3 + 2(4)(1)^3 + 4(4)(\frac{3}{2})^3 + (4)(2)^3] \\ &= \frac{1}{6}(4)[\frac{1}{2} + 2 + \frac{27}{2} + 8] \\ &= \frac{2}{3}[\frac{48}{2}] \\ &= 16, \text{ so the error is zero.}\end{aligned}$$

(b) Utilizing the same strategy as question 22 of Chapter 16, we find $\int_{-h}^{h} f(x)dx$ for the cubic polynomial $ax^3 + bx^2 + cx + d$. Let $g_1(x) = ax^3$ and let $g_2(x) = bx^2 + cx + d$.

Now,

$$\begin{aligned}
\int_{-h}^{h} f(x)dx &= \int_{-h}^{h} g_1(x)dx + \int_{-h}^{h} g_2(x)dx \\
&= \int_{-h}^{h} ax^3 dx + \frac{h}{3}[g_2(-h) + 4g_2(0) + g_2(h)] \\
&\quad \text{(this is from the results of question 22, since } g_2 \text{ is a quadratic function)} \\
&= \frac{ax^4}{4}\Big|_{-h}^{h} + \frac{h}{3}[g_2(-h) + 4g_2(0) + g_2(h)] \\
&= \frac{ah^4}{4} - \frac{a(-h)^4}{4} + \frac{h}{3}[g_2(-h) + 4g_2(0) + g_2(h) + \big(a(-h)^3 + 4a(0)^3 + a(h)^3\big)] \\
&= 0 + \frac{h}{3}[\big(g_1(-h) + g_2(-h)\big) + \big(4g_1(0) + 4g_2(0)\big) + \big(g_1(h) + g_2(h)\big)] \\
&= \frac{h}{3}[f(-h) + 4f(0) + f(h)]
\end{aligned}$$

The rest of the proof is exactly the same as that of question 22.

21. (a)

$$\begin{aligned}
T_n &= \frac{h}{2}[f(a) + 2f(a+h) + 2f(a+2h) + \ldots + 2f(a+(n-1)h) + f(b)] \\
4T_n &= h[2f(a) + 4f(a+h) + 4f(a+2h) + \ldots + 4f(a+(n-1)h) + 2f(b)]
\end{aligned}$$

Using the same notation for $T_{n/2}$, with half the number of subintervals each interval will have twice the length, so we have

$$T_{n/2} = h[f(a) + 2f(a+2h) + 2f(a+4h) + \ldots + 2f(a+(n-1)2h) + f(b)]$$

so

$$\begin{aligned}
4T_n - T_{n/2} &= h[2f(a) - f(a)) + 4f(a+h) + (4f(a+2h) - 2f(a+2h)) + \cdots + 4f(a+(n-1)h) \\
&\quad + (2f(b) - f(b))] \\
&= h[f(a) + 4f(a+h) + 2f(a+2h) + \cdots + 4f(a+(n-1)h) + f(b)]
\end{aligned}$$

$$\begin{aligned}
\frac{1}{3}(4T_n - T_{n/2}) &= \frac{h}{3}[f(a) + 4f(a+h) + 2f(a+2h) + \cdots + 4f(a+(n-1)h) + f(b)] \\
&= S_n
\end{aligned}$$

(b) $\int_0^2 e^x dx = e^x\Big|_0^2 = e^2 - 1 \approx 6.389056099$

$$T_2 = \frac{1}{2}(e^0 + 2e^1 + e^2) \approx 6.912809878$$
$$T_4 = \frac{1}{2}(e^0 + 2e^{1/2} + 2e + 2e^{3/2} + e^2) \approx 6.52161011$$
$$S_4 = \frac{1}{3}(4T_4 - T_2) \approx \frac{1}{3}(4(6.52161011) - 6.912809878)$$
$$\approx 6.391210187$$

The errors are:
$T_2 : -0.523753779$
$T_4 : -0.132554011$
$S_4 : -0.002154088$
Simpson's rule with 4 subintervals is most accurate.

22. $\int_0^1 x^{5/2} dx = \frac{2}{7}x^{7/2}\Big|_0^1 = \frac{2}{7}$ For each rule, $h = \frac{(b-a)}{n} = \frac{1}{4}$. For the corrected trapezoidal rule, $f'(x) = \frac{5}{2}x^{3/2}$
Trapezoidal rule: $\frac{1}{8}[2(\frac{1}{4})^{5/2} + 2(\frac{1}{2})^{5/2} + 2(\frac{3}{4})^{5/2} + 1^{5/2}]$
Corrected trap. rule: $\frac{1}{8}[2(\frac{1}{4})^{5/2} + 2(\frac{1}{2})^{5/2} + 2(\frac{3}{4})^{5/2} + 1^{5/2}] - \frac{1}{192}[\frac{5}{2}1^{3/2}]$
Simpson's rule: $\frac{1}{12}[4(\frac{1}{4})^{5/2} + 2(\frac{1}{2})^{5/2} + 4(\frac{3}{4})^{5/2} + 1^{5/2}]$ The results are:

	Area	Error
Trapezoidal rule:	0.298791496	−0.013077211
Corrected trap. rule:	0.285770663	−0.000056377
Simpson's rule:	0.285592546	0.000121740

The Corrected trapezoidal rule provides the best approximation, with Simpson's rule second.

23. (a)

(i) For $f(x) = \sqrt{x}$, $f''(x) = -\frac{1}{4x^{3/2}}$
For x in the interval $(1, 4)$, $|f''(x)| < \frac{1}{4}$ which means $|E|\Big|\frac{(b-a)^3}{12n^2}f''(c)\Big| < \Big|\frac{3^3}{12n^2}\frac{1}{4}\Big|$.
We need $\Big|\frac{3^3}{12n^2}\frac{1}{4}\Big| < 0.01$ so $n^2 > \frac{3^3}{12}\frac{1}{4}\frac{1}{0.01} = 56.25$, $n > 7.5$ or $n > 7$.

(ii) Simpson's rule $f^{(3)}(x) = \frac{3}{8x^{5/2}}$ and $f^{(4)}(x) = -\frac{15}{16x^{7/2}}$

For x in the interval $(1,4)$, $|f^{(4)}(x)| < \frac{15}{16}$ so $|E| = \left|\frac{(b-a)^4 f^{(4)}(c)}{180n^5}\right| < \frac{3^4}{180n^5}\frac{15}{16}$ for all c in the interval $(1,4)$. We need $\frac{3^4}{180n^5}\frac{15}{16} < 0.01$
so

$$n^5 > \frac{3^4}{180}\frac{15}{16}\frac{1}{0.01} = 42.1875$$
$$n > 2.11 \text{ or just } n > 2.$$

23. (b)

(i) $f(x) = x^5$, $f'(x) = 5x^4$, $f''(x) = 20x^3$
For x in the interval $(1,3)$, $|f''(x)| < 540$ so $|E| = \left|\frac{(b-a)^3 f''(c)}{12n^2}\right| < \frac{2^3}{12n^2}(540)$ for all c in the interval $(1,3)$. Therefore we need $\frac{2^3}{12n^2}(540) < 0.01$,

$$n^2 > \frac{2^3}{12}\frac{(540)}{0.01} = 36000$$
$$n > 189.7 \text{ or just} n > 189.$$

(ii) $f^{(3)}(x) = 60x^2$, $f^{(4)}(x) = 120x$
For x in the interval $(1,3)$, $|f^{(4)}(x)| < 360$
so $|E| = \left|\frac{(b-a)^4}{180n^5} f''(c)\right| < \frac{2^4}{180n^5}(360)$ for all c in the interval $(1,3)$.
Therefore we need $\frac{2^4}{180n^5}(360) < 0.01$,

$$n^5 > \frac{2^4(360)}{180(0.01)}$$
$$n^5 > 3200$$
$$n > 5$$

24. We find the points of intersection of $f(x)$ and the x-axis, i.e. we find the roots of $f(x)$:

$$f(x) = x^4 + x^3 - 4x^2 - 4x = x(x^3 + x^2 - 4x - 4)\ .$$

Since the coefficients of x^3 and x^2 are identical, and likewise the coefficients of x and x^0 are identical, we try $(x+1)$:

$$(x^3 + x^2 - 4x - 4) = (x^2 - 4)(x + 1)\ .$$

And since $(x^2 - 4) = (x - 2)(x + 2)$, we get :

$$f(x) = (x+2)(x+1)x(x-2)\ , \quad \text{so the roots are } x = -2, -1, 0, 2\ .$$

Now, if x is in $(-2,-1)$, then $f(x)$ is negative.

If x is in $(-1,0)$, then $f(x)$ is positive.

If x is in $(0,2)$, then $f(x)$ is negative.

So, we find:

$$\int_{-2}^{-1} -(x^4+x^3-4x^2-4x)dx + \int_{-1}^{0}(x^4+x^3-4x^2-4x)dx + \int_{0}^{2} -(x^4+x^3-4x^2-4x)dx$$

$$= -\left(\frac{x^5}{5}+\frac{x^4}{4}-\frac{4}{3}x^3-2x^2\right)\Big|_{-2}^{-1} + \left(\frac{x^5}{5}+\frac{x^4}{4}-\frac{4}{3}x^3-2x^2\right)\Big|_{-1}^{0} - \left(\frac{x^5}{5}+\frac{x^4}{4}-\frac{4}{3}x^3-2x^2\right)\Big|_{0}^{2}$$

$$= -\left[\left(-\frac{1}{5}+\frac{1}{4}+\frac{4}{3}-2\right)-\left(-\frac{32}{5}+4+\frac{32}{3}-8\right)\right]$$

$$+\left[0-\left(-\frac{1}{5}+\frac{1}{4}+\frac{4}{3}-2\right)\right]-\left[\left(\frac{32}{5}+4-\frac{32}{3}-8\right)-0\right]$$

$$= -\left[-\frac{37}{60}-\frac{4}{15}\right]+\left[\frac{37}{60}\right]-\left[-\frac{124}{15}\right]$$

$$= \frac{37}{60}+\frac{37}{60}+\frac{16}{60}+\frac{496}{60}=\frac{586}{60}=\frac{293}{30}=9\frac{23}{30}$$

25. We know that $\int_1^b f(x)dx = \sqrt{b^2+1}-\sqrt{2}$.

Remembering the Fundamental Theorem (which states that $\int_a^b f(x) = F(b)-F(a)$ where F is an antiderivative of f), then it is a "safe bet" that $F(b)=\sqrt{b^2+1}$ and $F(1)=\sqrt{2}$, since

$$\int_1^b f(x)dx = F(b)-F(1) = \sqrt{b^2+1}-\sqrt{2} \quad (\text{ where } F'(x)=f(x))\ .$$

So a wise guess for $F(x)$ is $F(x)=\sqrt{x^2+1}$, where $F'(x)=f(x)$. So to find $f(x)$, we just calculate:

$$\frac{d}{dx}(\sqrt{x^2+1}) = \frac{d}{dx}(x^2+1)^{1/2} = \frac{2x}{2(x^2+1)^{1/2}} = \frac{x}{(x^2+1)^{1/2}}\ .$$

Now, we double-check:

$$\int_1^b \frac{x}{(x^2+1)^{1/2}}dx = \int_1^b \frac{1}{2}\frac{2x}{(x^2+1)^{1/2}}dx = \int_*^* \frac{1}{2u^{1/2}}du = u^{1/2}\Big|_*^*$$

$$= (x^2+1)^{1/2}\Big|_1^b = \sqrt{b^2+1}-\sqrt{2}\ .\ \checkmark$$

So $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

26. (a) As per the hint, we translate everything m units to the left. This obviously does not change the area that we are looking for. Now, we find the area:

$$\begin{aligned}
A &= \int_{-h}^{h} (px^2 + qx + r)dx \\
&= (\frac{px^3}{3} + \frac{qx^2}{2} + rx)\Big|_{-h}^{h} \\
&= (\frac{p(h)^3}{3} + \frac{q(h)^2}{2} + rh) - (\frac{p(-h)^3}{3} + \frac{q(-h)^2}{2} + r(-h)) \\
&= \frac{2}{3}p(h)^3 + 2rh \\
&= \frac{1}{3}h(2ph^2 + 6r) \\
&= \frac{1}{3}h[(p(-h)^2 + r) + 4r + (p(h)^2 + r) - qh + qh] \\
&= \frac{1}{3}h[(p(-h)^2 + q(-h) + r) + (4p(0)^2 + 4q(0) + 4r) + (p(h)^2 + q(h) + r)] \\
&= \frac{1}{3}h[f(-h) + 4f(0) + f(h)]
\end{aligned}$$

Substituting back, we get $\frac{1}{3}h[f(a) + 4f(m) + f(b)]$.

(b) Since n is even, we can divide $[a, b]$ into the $\frac{n}{2}$ different intervals:

$[a, a + 2h], [a + 2h, a + 4h], \ldots, [a + 2(k-1)h, a + 2kh], \ldots, [a + (n-4)h, a + (n-2)h]$, $[a + (n-2)h, a + nh(= b)]$.

We now apply part (a) to each interval. So for $1 \leq k \leq \frac{n}{2}$, we consider $[a + 2(k-1)h, a + 2kh]$: By part (a), if we let $h' = \frac{1}{2}\big((a + 2kh) - (a + 2(k-1)h)\big) = \frac{1}{2}(2h) = h$, and if we note $m = \frac{\big((a + 2(k-1)h) + (a + 2kh)\big)}{2} = a + (2k-1)h$, then the area of f under this region is:

$$\frac{h'}{3}[f(a + (2k-2)h) + 4f(a + (2k-1)h) + f(a + 2kh)] \ .$$

Now summing all of these regions for $1 \leq k \leq \frac{n}{2}$, we get (since $h' = h$):

$$\begin{aligned}
&\frac{h}{3}[f(a) + 4f(a+h) + f(a+2h)] + \frac{h}{3}[f(a+2h) \\
&+ 4f(a+3h) + f(a+4h)] + \cdots + \frac{h}{3}[f(a + (2k-2)h) + 4f(a + (2k-1)h) \\
&+ f(a+2kh)] + \cdots + \frac{h}{3}[f(a + (n-2)h) + 4f(a + (n-1)h) + f(b)] \\
&= \frac{h}{3}[f(a) + 4f(a+h) + 2f(a+2h) + \cdots + 4f(a + (n-1)h) + f(b)]
\end{aligned}$$

27. (a) $\dfrac{dy}{dx} = \dfrac{3}{2}(x-1)^{1/2}$ so
$L = \int_2^{10} \sqrt{1 + \frac{9}{4}(x-1)}dx = \int_2^{10} \sqrt{\frac{9}{4}x - \frac{5}{4}}dx.$
Let $u = \frac{9}{4}x - \frac{5}{4}$, $du = \frac{9}{4}dx$

$$\begin{aligned} L &= \int_{x=2}^{x=10} \frac{4}{9}\sqrt{u}du \\ &= \frac{8}{27}u^{3/2}\Big|_{x=2}^{x=10} \\ &= \frac{8}{27}(\frac{9}{4}x - \frac{5}{4})^{3/2}\Big|_{x=2}^{x=10} \\ &= \frac{8}{27}[(\frac{85}{4})^{3/2} - (\frac{13}{4})^{3/2}] \\ &= \frac{85^{3/2} - 13^{3/2}}{27} \end{aligned}$$

(b) $8x^2 = y^3$ so $y = 2x^{2/3}$ and $\dfrac{dy}{dx} = \dfrac{4}{3}x^{-1/3}$

$$\begin{aligned} L &= \int_1^{2\sqrt{2}} \sqrt{1 + \frac{16}{9}x^{-2/3}}dx = \int_1^{2\sqrt{2}} \sqrt{\frac{x^{2/3} + 16/9}{x^{2/3}}}dx \\ &= \int_1^{2\sqrt{2}} \frac{\sqrt{x^{2/3} + 16/9}}{x^{1/3}}dx \end{aligned}$$

Let

$$\begin{aligned} u &= x^{2/3} + \frac{16}{9} \\ du &= \frac{2}{3x^{1/3}}dx \\ \frac{3}{2}du &= \frac{1}{x^{1/3}}dx \end{aligned}$$

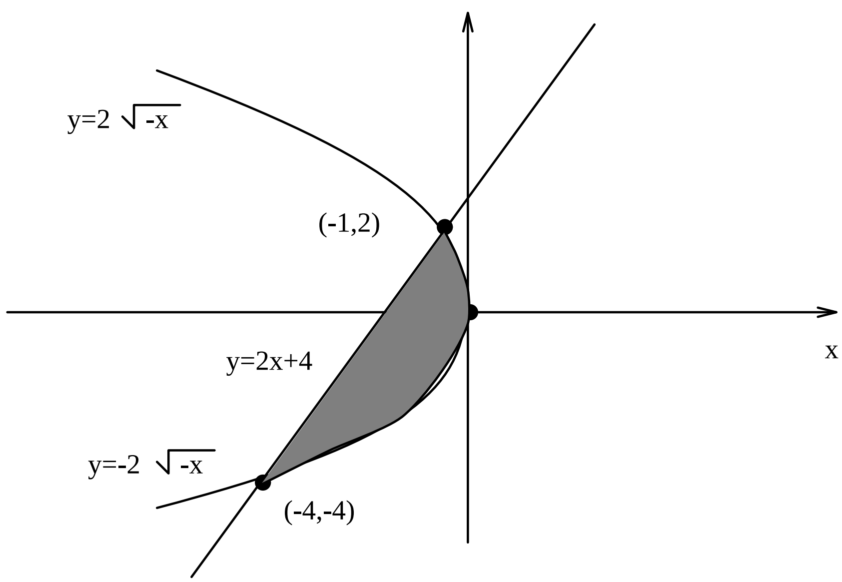
so

$$\begin{aligned} L &= \int_{x=1}^{x=2\sqrt{2}} \frac{3}{2}\sqrt{u}du = u^{3/2}\Big|_{x=1}^{x=2\sqrt{2}} \\ &= (x^{2/3} + \frac{16}{9})^{3/2}\Big|_1^{2\sqrt{2}} = (x^{2/3} + \frac{16}{9})^{3/2}\Big|_1^{2^{3/2}} \\ &= (2 + \frac{16}{9})^{3/2} - (1 + \frac{16}{9})^{3/2} = (\frac{34}{9})^{3/2} - (\frac{25}{9})^{3/2} \\ &= \frac{34\sqrt{34} - 125}{27} \end{aligned}$$

(c) $24xy = x^4 + 48$ so $y = \dfrac{x^4+48}{24x} = \dfrac{x^3}{24} + \dfrac{2}{x}$, $\dfrac{dy}{dx} = \dfrac{x^2}{8} - \dfrac{2}{x^2}$

$$\begin{aligned}
L &= \int_2^3 \sqrt{1 + (\frac{x^2}{8} - \frac{2}{x^2})^2}dx = \int_2^3 \sqrt{1 + (\frac{x^4}{64} - \frac{1}{2} + \frac{4}{x^4})}dx \\
&= \int_2^3 \sqrt{\frac{1}{64x^4}(x^8 + 32x^4 + 256)}dx \\
&= \int_2^3 \frac{1}{8x^2}\sqrt{(x^4+16)^2}dx = \int_2^3 \frac{x^4+16}{8x^2}dx \\
&= \int_2^3 \frac{x^2}{8} + \frac{2}{x^2}dx = (\frac{x^3}{24} - \frac{2}{x})\Big|_2^3 = \frac{27}{24} - \frac{2}{3} - \frac{8}{24} + 1 \\
&= \frac{27 - 16 - 8 + 24}{24} = \frac{27}{24} = \frac{9}{8}
\end{aligned}$$

28. (a) The region bounded is shown in the graph:



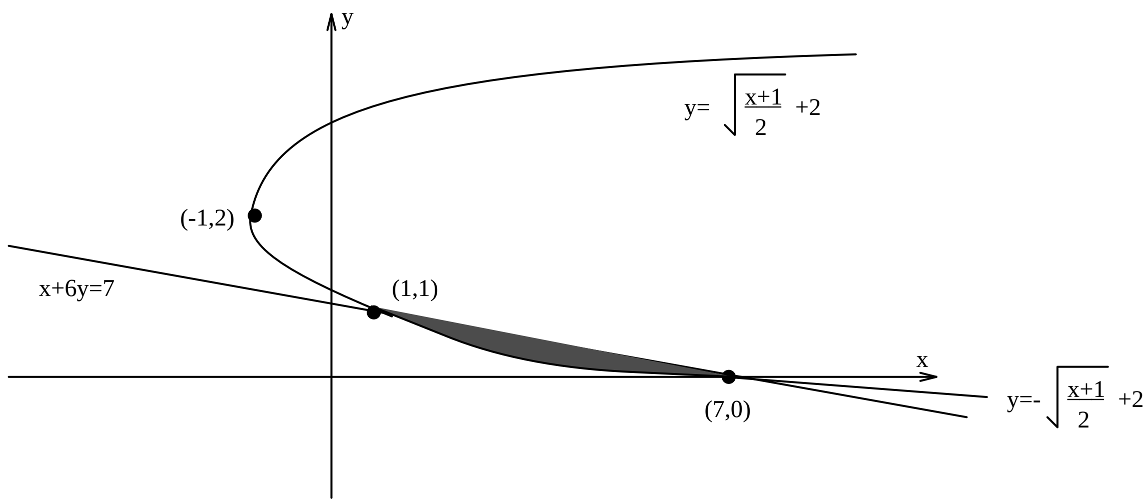
(i) Integrating with respect to x, the area is given by

$$\begin{aligned}
&\int_{-1}^{0} (2\sqrt{-x}) - (-2\sqrt{-x})dx + \int_{-4}^{-1} (2x+4) - (-2\sqrt{-x})dx \\
&= \int_{-1}^{0} 4\sqrt{-x}dx + \int_{-4}^{-1} 2x + 4 + 2\sqrt{-x}dx \\
&= -\frac{8}{3}(-x)^{3/2}\Big|_{-1}^{0} + x^2 + 4x - \frac{4}{3}(-x)^{3/2}\Big|_{-4}^{-1} \\
&= \frac{8}{3} + 1 - 4 - \frac{4}{3} - 16 + 16 + \frac{4}{3}(8) \\
&= \frac{36}{3} - 3 = 9
\end{aligned}$$

(ii) Integrating with respect to y, the area is

$$\int_{-4}^{2}(\frac{-y^2}{4}) - (\frac{y-4}{2})dy = \frac{1}{2}\int_{-4}^{2} -\frac{1}{2}y^2 - y + 4dy$$
$$= \frac{1}{2}[-\frac{1}{6}y^3 - \frac{1}{2}y^2 + 4y\Big|_{-4}^{2}]$$
$$= \frac{1}{2}[-\frac{4}{3} - 2 + 8 - \frac{32}{3} + 8 + 16] = \frac{1}{2}(-\frac{36}{3} + 30)$$
$$= 9$$

(b) The region bounded is shown:



(i) Integrating with respect to x, the area is

$$\int_{1}^{7}(-\frac{1}{6}x + \frac{7}{6}) - (-\sqrt{\frac{x+1}{2}} + 2)dx = \int_{1}^{7} -\frac{1}{6}x - \frac{5}{6} + \sqrt{\frac{x+1}{2}}dx$$
$$= -\frac{1}{12}x^2 - \frac{5}{6}x + \frac{4}{3}(\frac{x+1}{2})^{3/2}\Big|_{1}^{7} = (-\frac{49}{12} - \frac{35}{6} + \frac{32}{3} + \frac{1}{12} + \frac{5}{6} - \frac{4}{3})$$
$$= \frac{-49 - 70 + 128 + 1 + 10 - 16}{12} = \frac{4}{12} = \frac{1}{3}$$

(ii) and with respect to y,

$$\int_{0}^{1}(7-6y) - (2(y-2)^2 - 1)dy = \int_{0}^{1} 7 - 6y + 1 - 2y^2 + 8y - 8dy$$
$$= \int_{0}^{1} -2y^2 + 2ydy2\int_{0}^{1} -y^2 + ydy = 2[-\frac{1}{3}y^3 + \frac{1}{2}y^2\Big|_{0}^{1}]$$
$$= 2[-\frac{1}{3} + \frac{1}{2}] = 2(\frac{1}{6}) = \frac{1}{3}$$

29. The slope of $y = x + 2$ is one. So we find the point on $y = x^2$ with slope one:

$$y' = 2x\ ,\ y' = 1 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}\ , \quad \text{and the point is } C = (\frac{1}{2}, \frac{1}{4})\ .$$

We now determine the distance from C to the line: From basic mathematics, we take the line perpendicular to $y = x + 2$ that goes through C, and find out where it intersects with $y = x + 2$ (there are more sophisticated methods that are shorter, but this works).

The line we are looking for has slope $\frac{-1}{1} = -1$ and passes through $C = (\frac{1}{2}, \frac{1}{4})$:

Using $y - y_1 = m(x - x_1)$, we get $y - \frac{1}{4} = -1(x - \frac{1}{2}) \Rightarrow y = -x + \frac{1}{2} + \frac{1}{4} = -x + \frac{3}{4}$

Now, we find the point of intersection between $y = x + 2$ and $y = -x + \frac{3}{4}$:

$$x + 2 = -x + \frac{3}{4} \Rightarrow 2x = -\frac{5}{4} \Rightarrow x = -\frac{5}{8}\ .$$

So the point of intersection P is $(-\frac{5}{8}, 1\frac{3}{8})$. And the distance from P to C is

$$\sqrt{(\frac{1}{2} - (-\frac{5}{8})^2 + (\frac{1}{4} - \frac{11}{8})^2} = \sqrt{\frac{81}{64} + \frac{81}{64}} = \frac{9\sqrt{2}}{8}\ .$$

Now we find A and B (the intersection points of $y = x^2$ and $y = x + 2$)

$$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2, -1$$

So $A = (-1, 1)$ and $B = (2, 4)$. The distance from A to B is: $\sqrt{(4 - 1)^2 + (2 - (-1))^2} = 3\sqrt{2}$.

And now the area of triangle $ABC = \frac{1}{2}\overline{AB}\ \overline{PC} = \frac{1}{2}(3\sqrt{2})(\frac{9\sqrt{2}}{8}) = \frac{27}{8}$

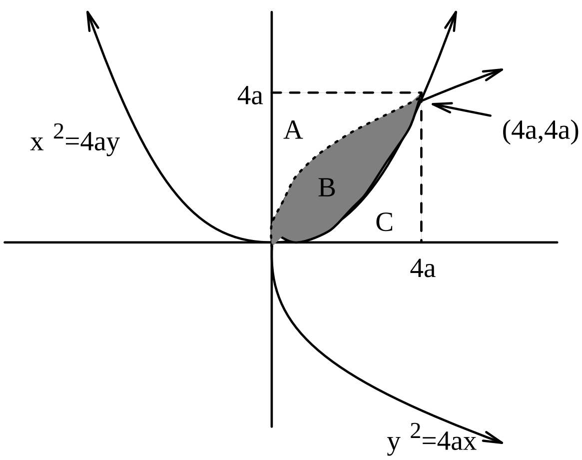
The area between the line and parabola (integrate w.r.t.x):

$$\int_{-1}^{2} ((x + 2) - x^2)dx = \int_{-1}^{2} xdx + 2\int_{-1}^{2} dx - \int_{-1}^{2} x^2 dx = \frac{x^2}{2}\Big|_{-1}^{2} + 2(x)\Big|_{-1}^{2} - (\frac{x^3}{3})\Big|_{-1}^{2}$$
$$= (2 - \frac{1}{2}) + 2(2 - (-1)) - (\frac{8}{3} - \frac{-1}{3}) = \frac{3}{2} + 6 - 3$$
$$= \frac{9}{2}$$
$$= \frac{4}{3}(\frac{27}{8}) = \frac{4}{3} \quad (\text{ Area of triangle } ABC)\ .$$

30. Find the points of intersection:

$$x^2 = 4ay \Rightarrow y = \frac{x^2}{4a}$$
$$\Rightarrow y^2 = 4ax \Rightarrow (\frac{x^2}{4a})^2 = 4ax$$
$$\Rightarrow x^4 - (4a)^3 x = 0$$
$$\Rightarrow x(x^3 - (4a)^3) = 0$$
$$\Rightarrow x(x-4a)(x^2 + 4ax + (4a)^2) = 0$$

But note that since $a > 0$, then $x^2 + 4ax + (4a)^2 \neq 0$ for any $x \in \mathbb{R}$, so $x = 0, 4a$.
So the points of intersection are $(0,0)$ and $(4a, 4a)$.
The area looks as follows:



Now, $x^2 - 4ay \Rightarrow y = \frac{x^2}{4a}$.
Also, $y^2 = 4ax$
$\Rightarrow y = +\sqrt{4ax}$
(we need the upper half of the parabola)

We find the value of

$$\int_0^{4a} (\sqrt{4ax} - \frac{x^2}{4a})dx = 2\sqrt{a}\int_0^{4a} x^{1/2}dx - \frac{1}{4a}\int_0^{4a} x^2 dx$$
$$= 2\sqrt{a}(\frac{2}{3}x^{3/2})\Big|_0^{4a} - \frac{1}{4a}(\frac{x^3}{3})\Big|_0^{4a}$$
$$= \frac{4}{3}a^{1/2}(4a)^{3/2} - \frac{(4a)^3}{12a}$$
$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$
$$= \frac{(4a)^2}{3}$$

NOTE: A MUCH QUICKER METHOD:

We find the area of C:

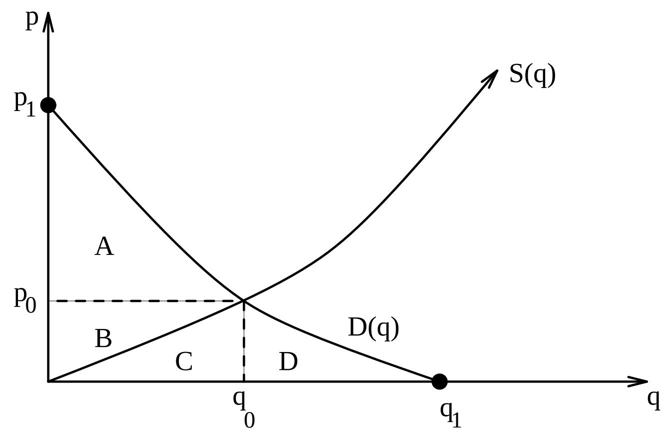
$$\int_0^{4a} \frac{x^2}{4a}dx = \frac{1}{4a}(\frac{x^3}{3})\Big|_0^{4a} = \frac{(4a)^2}{3}\ .$$

By symmetry, the area of A is also $\dfrac{(4a)^2}{3}$. Now the area of B is the area of the $4a \times 4a$ square less the area of A and C

$$= (4a)^2 - \left(\frac{(4a)^2}{3} + \frac{(4a)^2}{3}\right)$$
$$= \frac{(4a)^2}{3}$$

31. (a) p_1: The maximum price that any consumer is willing to pay.

(b) q_1: The maximum quantity that any consumer needs/wants.



(c) $\displaystyle\int_0^{q_0} S(q)dq$: This is region C.

It represents the accumulated revenue obtained by all producers who are willing to sell the product for a price of $\leq p_0$ *if* each of these producers sold their product at the minimum price that each was willing to sell it for.

(d) $\displaystyle\int_0^{q_0} D(q)dq$: This is regions A, B and C.

It represents the accumulated spending of all consumers who are willing to buy the product for a price of $\geq p_0$ *if* each of these consumers bought their product at the maximum price that each was willing to buy it for.

(e) p_0q_0 : This is regions B and C.

It represents the total revenue of the producers (who are willing to sell the product for a price of $\leq p_0$) *if* each of these producers sold their product at the equilibrium price p_0.

(f) $\displaystyle\int_0^{q_0} (D(q) - S(q))dq$: This is regions A and B.

This represents the total gain enjoyed by the system as a whole, for producers and consumers alike (who are willing to sell/buy for a price of $\leq / \geq p_0$). This is obvious since the region is the sum of the consumers' surplus and producers' surplus.