Solutions to Supplementary Questions for HP Chapter 10

1.

$$\begin{aligned}
&\lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\
&= \lim_{h\to 0}\frac{(x+h-x)\left((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x^{n-1}\right)}{h}\\
&= \lim_{h\to 0}\left((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x^{n-1}\right)\\
&= x^{n-1} + x^{n-1} + \cdots + x^{n-1} \qquad \text{(n times)}\\
&= nx^{n-1}
\end{aligned}$$

2. (a) Let $a = 0$, $f(x) = \frac{1}{x}$, and $g(x) = -\frac{1}{x}$.

(b) Let $a = 0$, $f(x) = \operatorname{sgn}(x)$, and $g(x) = \operatorname{sgn}(x)$, where

$$\operatorname{sgn}(x) := \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}.$$

3. (a) If $c = 0$ then $\lim_{x\to 0}\frac{\sqrt[3]{1+cx}-1}{x} = \lim_{x\to 0}\frac{0}{x} = 0 = \frac{c}{3}$. Otherwise,

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt[3]{1+cx}-1}{x} &= \lim_{x\to 0}\frac{c\left(\sqrt[3]{1+cx}-1\right)}{(1+cx)-1}\\
&= \lim_{x\to 0}\frac{c\left(\sqrt[3]{1+cx}-1\right)}{\left(\sqrt[3]{1+cx}-1\right)\left(\sqrt[3]{(1+cx)^2}+\sqrt[3]{1+cx}+1\right)}\\
&= \lim_{x\to 0}\frac{c}{\left(\sqrt[3]{(1+cx)^2}+\sqrt[3]{1+cx}+1\right)} = \frac{c}{3}
\end{aligned}$$

(b)

$$\begin{aligned}
&\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}\\
&= \lim_{x\to 1}\frac{(\sqrt[3]{x}-1)(\sqrt{x}+1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)}\\
&\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(x-1)(\sqrt[3]{x^2}+\sqrt[3]{x}+1)} = \frac{2}{3}
\end{aligned}$$

4. (a) Not possible to find with the given information.

(b) L

(c) Not possible to find with the given information.

(d) Not possible to find, $-L$, not possible to find.

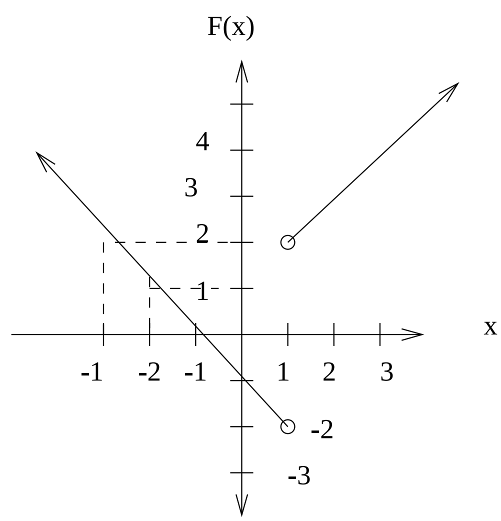
5. (a)

i) When $x > 1$, $|x-1| = x-1$, and so $\lim_{x\to 1^+} F(x) = \lim_{x\to 1^+} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1^+} x+1 = 2$

ii) When $x < 1$, $|x-1| = -(x-1)$, and so $\lim_{x\to 1^-} F(x) = \lim_{x\to 1^-} \frac{(x-1)(x+1)}{-(x-1)} = \lim_{x\to 1^-} -(x+1) = -2$

(b) No. $\lim_{x\to 1^+} F(x) = 2 \neq -2 = \lim_{x\to 1^-} F(x)$, so the left-hand limit is not the same as the right-hand limit and thus $\lim_{x\to 1} F(x)$ does not exist.

(c) (i) When $x > 1$, $F(x) = x+1$ (from (a) i)), and (ii) When $x < 1$, $F(x) = -(x+1)$ (from (a) ii)), (iii) When $x = 1$, $F(x)$ does not exist, so we have:



6. (a) The slope of the line is $\frac{\Delta y}{\Delta x} = \frac{\ln(1+\frac{r}{n})-\ln(1)}{(1+\frac{r}{n})-1} = \frac{\ln(1+\frac{r}{n})}{\frac{r}{n}} = \frac{n}{r}\ln(1+\frac{r}{n})$.

(b) Writing $h = \frac{r}{n}$, we have $h \to 0$ as $n \to \infty$ so

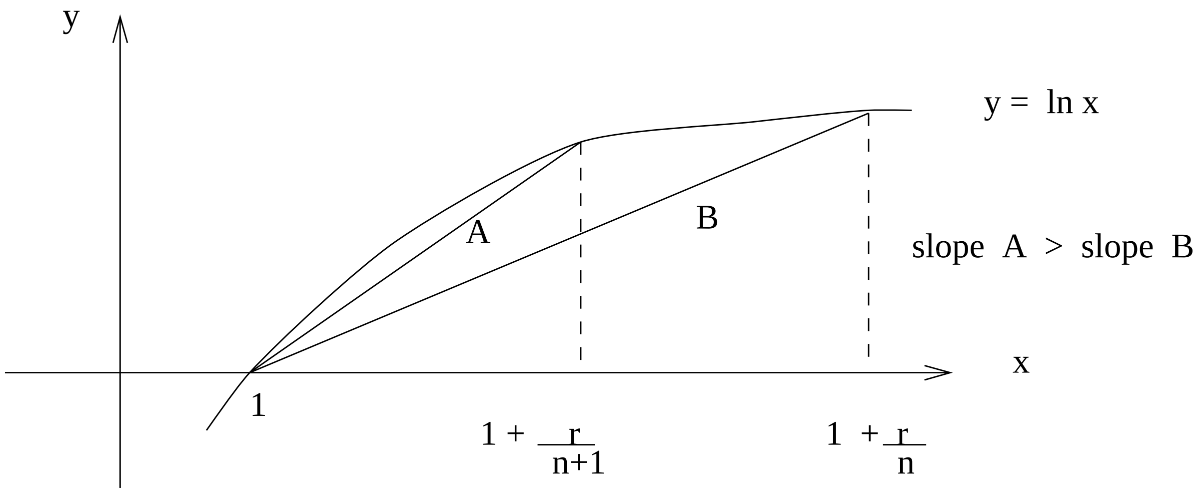
$$\begin{aligned}\lim_{n\to\infty} \frac{n}{r}\ln(1+\frac{r}{n}) &= \lim_{h\to 0} \frac{\ln(1+h)}{h} \\ &= \lim_{h\to 0} \frac{\ln(1+h)-\ln(1)}{h} \\ &= g'(1) \qquad \text{(where } g = \ln x\text{)}\end{aligned}$$

(c)

$$\begin{aligned}\lim_{n\to\infty}(1+\frac{r}{n})^n &= \lim_{n\to\infty} e^{n\ln(1+\frac{r}{n})}\\ &= e^{rg'(1)} \qquad \text{since the function } e^x \text{ is continuous and we}\\ &\qquad\qquad \text{know } \lim_{n\to\infty} n\ln(1+\frac{r}{n}) = rg'(1)\\ &= e^r\end{aligned}$$

7. **(a)** For a principal amount of money, P, $P(1+\frac{r}{n})^n$ is the sum of money resulting from compound interest, compounded n times at the interest rate r, so we should have $P(1+\frac{r}{n})^n < P(1+\frac{r}{n+1})^{n+1}$ which implies $(1+\frac{r}{n})^n < (1+\frac{r}{n+1})^{n+1}$.

(b) In the graph of $y = \ln x$ we can see that $\frac{n+1}{r}\ln(1+\frac{r}{n+1}) > \frac{n}{r}\ln(1+\frac{r}{n})$, the slopes of the lines A and B respectively.



Therefore $(n+1)\ln(1+\frac{r}{n+1}) > n\ln(1+\frac{r}{n})$ and $e^{(n+1)\ln(1+\frac{r}{n+1})} > e^{n\ln(1+\frac{r}{n})}$ so $(1+\frac{r}{n+1})^{n+1} > (1+\frac{r}{n})^n$.

8. **(a)** $f(x) = \begin{cases} 0; & 0 \le x < 1 \\ 1; & x = 1 \end{cases}$, so $f(x)$ is discontinuous at $x = 1$.

(b) $f(x) = \begin{cases} 0; & 0 \le x < 1 \\ 1; & x > 1 \end{cases}$, $f(x)$ is undefined at $x = 1$. Thus $f(x)$ is discontinuous at $x = 1$.

9. For $m < 0$, the function is undefined at $x = 0$ and is therefore not continuous there. For $m = 0$, $f(x) = 1$ and is therefore continuous everywhere. For $m > 0$, $\lim_{x\to 0} x^m = 0 = f(0)$, so $f(x)$ is continuous at $x = 0$. Therefore, it is required that $m \ge 0$ for $f(x)$ to be continuous at $x = 0$.

10. (a) $f(x) = (x-a)^m f_1(x)$, where $f_1(a) \neq 0$ $g(x) = (x-a)^n g_1(x)$, where $g_1(a) \neq 0$ and $m \geq n \geq 1$.

(b) $h(x) = \frac{(x-a)^{m-n} f_1(x)}{g_1(x)}$

11. (a) When $x^2 - 1 = 0$, then $f(x)$ is not defined. So we need only consider

i) when $x^2 - 1 > 0$. Here $f(x) = \frac{x^2-1}{x^2-1} = 1$ is continuous everywhere where $x^2 - 1 > 0$.

ii) when $x^2 - 1 < 0$. Similarly, $f(x) = \frac{-(x^2-1)}{x^2-1} = -1$ is continuous everywhere where $x^2 - 1 < 0$.

(b) $f(x)$ is not defined when $x^2 - 1 = 0$, i.e., when $x \pm 1$

i) If $x = 1$, then since

$A)$ $$\lim_{x \to 1^-} \frac{|x^2-1|}{x^2-1} = \lim_{x \to 1^-} -\frac{(x^2-1)}{x^2-1} = -1, \qquad \text{and}$$

$B)$ $$\lim_{x \to 1^+} \frac{|x^2-1|}{x^2-1} = \lim_{x \to 1^+} \frac{(x^2-1)}{x^2-1} = 1, \qquad \text{then}$$

no matter what our choice for $f(1)$, f cannot be continuous at $x = 1$ since the left-hand and right-hand limits differ.

ii) Similarly, if $x = -1$, then

$A)$ $$\lim_{x \to -1^-} \frac{|x^2-1|}{x^2-1} = \frac{(x^2-1)}{x^2-1} = 1, \qquad \text{and}$$

$B)$ $$\lim_{x \to -1^+} \frac{|x^2-1|}{x^2-1} = -\frac{(x^2-1)}{x^2-1} = -1,$$

hence, no value of f can be found at -1 to make f continuous at -1.