RESEARCH STATEMENT

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My research is mainly in the areas of Conformal/Quasiconformal analysis and geometric measure theory. I am also interested in the applications of these to the geometry of random and deterministic fractals (SLE, Brownian motion, fractal percolation, self-similar/self-affine iterated function systems), holomorphic dynamics, hyperbolic geometry, Teichmüller theory and Gromov hyperbolic spaces.

In Section 1 I list some results from my past and current research. In Section 2 I recall several standard definitions in the field. A more detailed account of my results and of the relevant open problems follows in Sections 3 through 8. Section 9 is devoted to some further projects on which I plan to work on.

1. Summary of past and current research

1.1. **Comformal dimension =1: Details in §3.** In [17] and [18] I investigate quasiconformal geometry of a class of Cantor sets. In particular I answer a question of Bishop and Tyson from [5] by proving the following: *There are sets of zero length and conformal dimension 1.*

1.2. **Conformal dimension > 1 & Fuglede modulus: Details in §4.** In [18] I generalize a result of Tyson from [38] by showing that: *If X contains a "nontrivial" family of sets of conformal dimension 1 then X is minimal.* Here "nontriviality" is a condition involving a notion of the modulus of a family of measures in the sense of Fuglede. One corollary of the results in [18] is a partial confirmation for the following conjecture from [18]: *$Z \subset \mathbb{R}$ is minimal if and only if $Y \times Z$ is minimal for every Y.*

1.3. **Self-affine and random Bedford-McMullen fractals: Details in §5.** In a joint work with Ilia Binder [2], we obtain new *lower bounds for conformal dimension of self-affine Bedford-McMullen sets*. We also define a class of random fractals, a self-affine version of Mandelbrot's fractal percolation, and show that they are *almost surely minimal for conformal dimension*. We conjecture that conformal dimension of the graph of 1-dimensional Brownian motion is almost surely equal to 3/2 (i.e. the graph is a.s. minimal).

1.4. **Dimension of Path Families and Rigidity: Details in §6.** In [19] I define a *quasiconformally invariant notion of dimension* (denoted by $\dim_{\mathrm{mod}}$) for families of curves. This allows me to calculate the conformal dimension of certain spaces using classical extremal length estimates. Also, using these invariants I prove the *quasiconformal rigidity for self-affine carpets in the plane*.

1.5. **Central Sets and Medial Axis: Details in §7.** Let D be a domain in the plane. Consider the set of the centers of maximal discs inscribed in D. Fremlin proved in [15] that this set has zero Lesbegue measure for any planar domain and asked if there is a domain with the central set of Hausdorff dimension strictly larger than 1. In [4], we construct examples of *domains with central sets of Hausdorff dimension* 2.

1.6. **Euclidean Quasiconvexity: Details in §8.** In [21] together with David Herron we investigate the geometry of quasiconvex domains in Euclidean spaces. We construct *Cantor sets in $\mathbb{R}^n$ of Hausdorff dimension $n-1$ s.t. the complement is not quasiconvex.* This is sharp since the complement of every set of dimension less than $n-1$ is quasiconvex. Also we *characterize finitely connected quasiconvex plane domains*.

2. Definitions

The following definitions are quite standard and we refer to [22] for more details.

2.1. **Quasisymmetry (QS).** A homeomorphism f between metric spaces (X, d_X) and (Y, d_Y) is called *quasisymmetric* (which I will abbreviate as QS) if there is a self-homeomorphism $\eta : [0, 1) \to [0, 1)$ such that for all $x, y, z \in X$ and $t > 0$

$$d_X(x, y) \le t d_X(y, z) \quad \Rightarrow \quad d_Y(f(x), f(y)) \le \eta(t) d_Y(f(y), f(z)). \tag{2.1}$$

This definition was given by Tukia and Vaisala [37] as a generalization of the notion of quasiconformal/quasisymmetric maps of Euclidean spaces.

2.2. **Conformal dimension.** One of the most important quasisymmetric invariants, *Conformal dimension* (denoted by $\mathcal{C}\dim$ below), was introduced by Pansu [30, 31] in the following way:

$$\mathcal{C}\dim X = \inf_f \dim_H f(X), \tag{2.2}$$

here the infimum is taken over all *quasisymmetric* f defined on X. Clearly for a general metric space X one has $\mathcal{C}\dim X \le \dim_H X$. We say X is *minimal* (for conformal dimension) if

$$\mathcal{C}\dim X = \dim_H X.$$

2.3. **Modulus of a family of measures/curves: see** [1, 16, 22]**.** Let (X, d_X, μ) be a metric measure space, $\mathcal{E} = \{E\}$ a family of subsets of X and $\mathcal{L} = \{\lambda_E\}_{E \in \mathcal{E}}$ a family of measures. We say a Borel measurable function $\rho : X \to [0, \infty]$ is *$\mathcal{L}$-admissible*, and write $\rho \wedge \mathcal{L}$,

$$\int \rho d\lambda \ge 1, \quad \forall \lambda_E \in \mathcal{L}.$$

The *p-modulus of $\mathcal{L}$ with respect to μ* is

$$\mathrm{mod}_p(\mathcal{L}, \mu) = \inf_{\rho \wedge \mathcal{L}} \int_X \rho^p d\mu.$$

A particular case is the modulus of a family of curves: $\mathrm{mod}_p \Gamma$. In that case the admissibility condition $\rho \wedge \Gamma$ is understood as $\int_\gamma \rho ds \ge 1$, where ds denotes the arc length element.

2.4. **Geometric Quasiconformality.** A metric measure space (X, μ, d_X) is *Ahlfors Q-regular* if there is a constant $C \ge 1$ such that

$$\frac{1}{C} r^Q \le \mu(B(x, r)) \le C r^Q$$

for every ball $B(x, r) \in X$. For instance Euclidean n-space is n regular and the 3 dimensional Heisenberg group is 4-regular with respect to its Carnot-Caratheodory metric. In an Ahlfors Q-regular space Q-modulus with respect to μ is called *the conformal modulus* and denoted by just $\mathrm{mod}\Gamma$.

A homemorphism $F : X \to Y$ between Ahlfors Q-regular spaces is *K-quasiconfomal* (see e.g. [1]) if for every curve family $\Gamma \subset X$ the following inequality holds

$$\mathrm{mod} f(\Gamma) \le K \mathrm{mod}\Gamma.$$

3. Sets of Conformal dimension 1: [17, 18, 20]

Beurling and Ahlfors have shown that QS selfmaps of the line can be singular. Tukia [36] gave examples of QS maps which map a set of full measure onto a set of arbitrary small positive Hausdorff dimension. On the other hand Ward and Staples [35] gave first examples of so called *quasisymmetrically thick sets*, i.e. subsets of a line with QS images of positive measure. In [5] Bishop and Tyson constructed examples of Cantor sets of conformal dimension d for every $d > 1$ and asked if there are sets minimal for conformal dimension which are not QS thick. We show that such sets do exist.

Theorem 3.1 (Hak. [18])**.** *Let $E \subset \mathbb{R}$ be a regular Cantor set and f an η-quasisymmetric map with $\eta(t) \lesssim \max(t^p, t^{1/p})$ for some $p \geq 1$. Then*

$$\dim_H E = 1 \quad \Longrightarrow \quad \dim_H f(E) \geq 1.$$

We refer to [18] for the precise definition of regular Cantor sets, but examples include the *middle interval Cantor sets*, which are constructed by removing at each step an interval from the middle of every interval of the previous generation. Theorem 3.1 is proved by carefully constructing an appropriate 1 dimensional measure on $f(E)$.

Corollary 3.2 (Hak [17, 18])**.** *There are subsets of a line of conformal dimension* 1 *and length* 0 *(and therefore they are not quasisymmetrically thick.)*

Corollary 3.3 (Hak. [17, 18])**.** *There are rigid Cantor sets, i.e sets with all quasisymmetric images of* 0 *length and Hausdorff dimension* 1*.*

Corollary 3.2 follows from the fact that qs maps on uniformly perfect spaces satisfy the condition of Theorem 2.1, see [22]. To obtain Corollary 3.3 we combine Theorem 3.1 with a theorem of Jang-Mei Wu from [39] where a sufficient condition for a Cantor set to be quasisymmetrically null (every image has zero length) is given. In [5] Bishop and Tyson also asked if all quasisymmetrically thick sets are minimal for conformal dimension. Theorem 3.1 implies that this is the case for all the currently known quasisymmetrically thick sets (from [9] and [35]).

Results of [18] indicate that it is important to know which sets have conformal dimension 1 and in particular which subsets of the line have this property. Here are some related open problems.

Problem 3.4. *Below $E \subset \mathbb{R}$*

- *(folklore) Characterize sets $E \subset \mathbb{R}$ which are quasisymmetrically thick/null/minimal.*
- *(Bishop, Tyson* [5]*) If E is quasisymmetrically thick then $\mathcal{C}\dim E = 1$.*

The following problem is due to Jang-Mei Wu. Let $N(K)$ be the class of subsets of $\mathbb{R}$ such that $f(E)$ has zero Lebesgue measure for all K-quasisymmetric maps of the line.

Problem 3.5. *Show that if $K_1 \neq K_2$ then $N(K_1) \neq N(K_2)$.*

4. Fuglede modulus and sets of Conformal Dimension > 1: [18, 2]

Existence of families of rectifiable curves in a space X is fundamentally important for the quasiconformal geometry of X (see e.g. [23, 24, 30, 38]). In [38] Tyson showed that if (X, μ) is a doubling, upper Q-regular metric measure space which contains a curve family $\Gamma \subset X$ of positive Q modulus then $\mathcal{C}\dim X \geq Q$. We refer the reader to Section 2 for the definition of a modulus of a family of curves and measures.

We recall that a metric measure space (X, μ) is *doubling* if $\mu(B(x, 2r) \leq C\mu(B(x, r))$ for some constant C and all x and $r > 0$. (X, μ) is *upper Q-regular* if $\mu(B(x, r)) \leq Cr^Q$ for fixed $C < \infty$ and every x and r. Tyson's theorem implies that for every compact $Y \subset \mathbb{R}^n$

$$\mathcal{C}\dim(Y \times (0, 1)) = \dim_H(Y \times (0, 1)).$$

In [18] we generalize Tyson's theorem by considering Fuglede modulus of a family of measures supported on minimal sets of dimension 1.

Theorem 4.1 (Hak. [18])**.** *Let $Q > 1$, (X, μ) be a doubling, upper Q-regular metric measure space. Suppose there is a family of measures $\mathcal{L} = \{\lambda\}$ in X satisfying the following conditions*

- *$\forall \varepsilon > 0$ there is a constant $C_\varepsilon > 0$ s.t. $\lambda(B(x,r)) \geq C_\varepsilon r^{1+\varepsilon}, \forall \lambda \in \mathcal{L}$,*
- *$\forall \lambda \in \mathcal{L}$ there is a set $E \subset X$ s.t. $\lambda(E) > 0$ and $\mathcal{C}\dim E = 1$.*

If $\mathrm{mod}_p(\mathcal{L}, \mu) > 0$ for some $p \in [1, Q)$ then $\mathcal{C}\dim X \geq Q$.

Corollary 4.2. *Suppose $E \subset \mathbb{R}$ and $\mathcal{C}\dim E = 1$. Then for every compact $Y \subset \mathbb{R}^n$*

$$\mathcal{C}\dim(Y \times E) = \dim_H(Y \times E),$$

provided there is a measure λ on E such that $r^{1+\varepsilon} \lesssim \mu(B(x,r)) \lesssim r^{1-\varepsilon}$ for every $x \in E$ and small $r > 0$.

Combining with Theorem 3.1 it follows that if E is a middle interval Cantor set of dimension one then $Y \times E$ is minimal for every $Y \subset \mathbb{R}^n$. In particular E can have 0 length. Corollary 4.2 gives an affirmative answer to the following question from [5] in the particular case when E is regular enough.

Problem 4.3 (Bishop, Tyson [5])**.** *Suppose $\mathcal{C}\dim E = 1$. Is $Y \times E$ minimal for every compact $Y \subset \mathbb{R}^k$?*

One of the main tools used in proving Theorem 4.1 is a version of discretized modulus used by Cannon in [11] and Heinonen and Koskela in [23].

5. Self-affine and random Bedford-McMullen sets: [2]

5.1. **Self-affine sets of Bedford and McMullen.** These sets can be constructed as follows [29]. Let $m < n$. Given a subset $D \subset \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$, we let

$$K(D) = \left\{ \sum_{k=1}^{\infty} \left(\frac{a_k}{m^k}, \frac{b_k}{n^k} \right) : (a_k, b_k) \in D \text{ for all } k \right\}.$$

Let $r(j)$ be the number of rectangles in the j-th column of D. It is easy to see that Minkowski dimension of $K(D)$ is

$$\dim_M K(D) = 1 + \log_n \frac{1}{m} \sum_{j=1}^{m} r_j \quad (> \mathcal{C}\dim K(D)). \tag{5.1}$$

Theorem 5.1 (Binder, Hak. [2])**.** *For a self-affine Bedford-McMullen set $K(D)$ we have*

$$\mathcal{C}\dim K(D) \geq 1 + \log_n \left(\prod_{i=1}^{m} r_j \right)^{1/m}. \tag{5.2}$$

Theorem 5.1 is proved by constructing a background measure μ and a family of measures $\mathcal{L}$ on K such that $\mathrm{mod}_1(\mathcal{L}, \mu) > 0$ (and therefore $\mathrm{mod}_Q(\mathcal{L}, \mu) > 0, \forall Q > 1$) and applying Theorem 4.1. The difficulty is to show that μ and $\mathcal{L}$ can be chosen in such a way that conditions of Theorem 4.1 are satisfied. Theorem 5.1 raises the following questions.

Problem 5.2. *Is inequality (5.2) strict? Is $K(D)$ minimal? Is $\mathcal{C}\dim K(D)$ achieved?*

5.2. **Random Bedford-McMullen sets.** We say a pattern of rectangles is a *rearrangement* of D if it is obtained from D by rearranging its columns. Let $\mathcal{R}(D) = \{D_1, \dots, D_{m!}\}$ denote the collection of all rearrangements of D. Construct a set similarly to a Bedford-McMullen set as follows. Start from the unit square and in step k divide every remaining rectangle into $m \times n$ rectangles with sides m^{-k} and n^{-k}. Choose a subset of these rectangles corresponding a rearrangement of D uniformly with probability $1/(m!)$. Continuing this process we obtain a random set K, which we call a *random Bedford-McMullen set*.

Theorem 5.3 (Binder, Hak. [2]). *A random Bedford-McMullen set K is almost surely minimal for conformal dimension and*

$$\mathcal{C}\dim K = \dim_M K(D) = 1 + \log_n \frac{1}{m} \sum_{j=1}^{m} r(j).$$

5.3. **Problems.** Theorems 4.1, 5.1 and 5.3 motivates the next three questions. Let $T_1, \dots, T_k$ be contracting linear maps of $\mathbb{R}^n$. By a theorem of Falconer [14] and Solomyak [34] for almost every $(a_1, \dots, a_k) \in \mathbb{R}^{kn}$ the unique compact set F satisfying $F = \bigcup_{i=1}^{k}(T_i(F) + a_i)$ has the property that $\dim_H F = \dim_M F$, provided $\|T_i\| < 1/2, \forall i$.

Problem 5.4 (Binder-Hak. [2]). *Is it true that $\mathcal{C}\dim F(\boldsymbol{a}) = \dim_M F(\boldsymbol{a})$ for almost every $\boldsymbol{a} \in \mathbb{R}^{kn}$?*

By previous results very often conformal dimension of X is equal to its Hausdorff dimension if X has cross-sections of dimension $\dim_H X - 1$. One such example is the graph of the 1-dimensional Brownian motion.

Problem 5.5 (Binder-Hak. [2]). *Is it true that the conformal dimension of the graph of the 1-d Brownian motion is almost surely equal to its Hausdorff dimension (and hence is equal to $3/2$)?*

Another class of random fractals are Schramm's SLE curves.

Problem 5.6 (Binder). *Is SLE_κ almost surely minimal for $4 \le \kappa < 8$?*

6. Dimension of Path Families and Rigidity

Motivated by the theory of Gromov hyperbolic spaces/groups much work recently has been devoted to understanding the quasiconformal geometry of fractals in the plane, see e.g. [7, 8, 27]. Often such fractals have 0 area and necessarily any family of curves inside will be of 0 conformal modulus. So it is natural to search for quasiconformally invariant quantities for such families of 0 modulus. In [19, 20] I define such invariants and apply these ideas to show quasiconformal non-equivalence and rigidity of self-affine carpet, products and other spaces.

6.1. **Dimension of Path Families and Rigidity:** [19, 20]. A *quadrilateral* $Q(E, F) \subset \mathbb{C}$ is a Jordan domain $Q \subset \mathbb{C}$ with two distinguished disjoint arcs E and F on the boundary. The collection of all curves in Q connecting E and F will be denoted $(Q; E, F)$. We will denote the conformal modulus of $(Q; E, F)$ by $mod(Q)$. We say Γ is a *disjoint family of curves* if all the curves in Γ are pairwise disjoint.

Definition 6.1 (Hak. [19]). *Let $\Gamma \subset (Q; E, F)$ be a family of curves.*

- *For a disjoint family of curves Γ define*

$$m(\Gamma, t) = \liminf_{\varepsilon \to 0} \left\{ \sum_{i=1}^{\infty} \operatorname{mod}^t(Q_i) : \Gamma \subset \bigcup_{i=1}^{\infty} (Q_i, E_i, F_i), \operatorname{mod}(Q_i) < \varepsilon \right\}, \tag{6.1}$$

where $(Q_i; E_i, F_i)$ is a subfamily of $(Q; E, F)$ for every $i \in \mathbb{N}$.

- *Next, for an arbitrary family of curves* $\Gamma \subset (Q; E, F)$ *define*

$$m(\Gamma, t) = \sup_{\Gamma' \subset \Gamma} m(\Gamma', t), \tag{6.2}$$

where the supremum over all disjoint subfamilies $\Gamma' \subset \Gamma$.

The following result implies that $m(\Gamma, t)$ should be thought of as a conformally invariant transverse measure on families of curves.

Theorem 6.2 (Hak. [19])**.** $m(\cdot, t)$ *has the following properties*

- *For every* $t \in [0, 1]$, $m(\cdot, t)$ *is an outer measure on the space of curve families of* C.
- *If* f *is a* K*-quasiconformal map of the plane then for every* $t \in [0, 1]$

$$\left(\frac{1}{K}\right)^t m(\Gamma, t) \leq m(f(\Gamma), t) \leq K^t m(\Gamma, t). \tag{6.3}$$

In particular $m(\Gamma, t)$ *is a conformal invariant for every* $t \in [0, 1]$.

It is natural to introduce the following definitions.

Definition 6.3 (**Mod-dimension of a path family**, Hak. [19, 20])**.** *For every curve family* $\Gamma \subset \mathbb{C}$ *define*

$$\dim_{\text{mod}}(\Gamma) = \inf\{t | m(\Gamma, t) = 0\} = \sup\{t | m(\Gamma, t) = \infty\} \tag{6.4}$$

From 6.3 we obtain the following.

Theorem 6.4 (**QC Invariance of dimension**, Hak. [19])**.** *If* $f : \mathbb{C} \to \mathbb{C}$ *is a QC map and* $\Gamma \subset \mathbb{C}$ *is any curve family then*

$$\dim_{\text{mod}}(f(\Gamma)) = \dim_{\text{mod}}(\Gamma). \tag{6.5}$$

We say that $X \subset \mathbb{R}^2$ is *quasiconformally rigid* if every every QC map f which fixes X as a set is the restriction of a Euclidean isometry when restricted to X. In [19] I prove

Theorem 6.5. *if* X *is a self-affine (but not self-similar) set homeomorphic to the Sierpinski carpet then* X *is QC-rigid.*

This result is inspired by a theorem of Bonk and Merenkov from [8] which shows that many self similar carpets are QC-rigid. The main tool in [8] is the beautiful notion of "carpet modulus" which has some similarities to Oded Schramm's transboundary extremal length [32] and can be explicitly computed for square carpets. Carpet modulus does not help in the self-affine situation, so instead we use $\dim_{\text{mod}}$ and the theory developed in [19] and [20].

Let S below be the standard Sierpinski carpet. The following problem is from [19].

Problem 6.6. *There is no QS embedding of* $E \times (0, 1)$ *into* S *whenever* $\dim_H E > \log_3 2$.

If true this "non-squeezing" phenomenon will give an alternative proof of a theorem from [8], which states that $f(S_p) \neq S_q$ whenever $f \in QC$ and $p \neq q$. Here S_p denotes a Sierpinski carpet corresponding to an odd $p \in \mathbb{N}$.

The problems below are quite well-known and can be found for instance in [27] and [22].

Problem 6.7. *Is* $\mathcal{C}\dim S$ *attained? Find* $\mathcal{C}\dim S$. *Is* S *quasisymmetric to a Loewner space?*

7. Central sets of dimension 2: [4]

Let D be a domain in $\mathbb{R}^2$. Its *skeleton or medial axis* is

$$S(D) = \{x \in D : \exists \text{ distinct } y, y' \in \partial D \text{ s.t. } d(x,y) = d(x,y') = d(x, \partial D)\},$$

where d denotes the Euclidean metric in $\mathbb{R}^2$. A ball $B = B(a,r) \subset D$ is called maximal if it is not contained in any larger ball $B' \subset D$. The set of centers of maximal balls of D is the central set of D and is denoted by $C(D)$. Clearly $S(D) \subset C(D)$. On the other hand $C(D) \setminus S(D)$ is not always empty. For example $C(D) \setminus S(D)$ for the domain bounded by an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; (a > b)$, consists of the two centers of curvature of the points $(-a, 0)$ and $(a, 0)$. Fremlin showed in [15] that for every planar domain $Q \subset \mathbb{R}^2$:

$$a) \dim_H S(D) \leq 1, \quad b) \mathcal{H}_2 C(D) = 0.$$

In the same paper Fremlin asked if a central set can have Hausdorff dimension strictly bigger than 1. With Christopher Bishop we show that surprisingly it is indeed possible.

Theorem 7.1 (Bishop, Hak. [4])**.** *Given any Hausdorff measure function $\varphi(t)$ such that $t^2/\varphi(t) \to 0$, there is a domain D such that $\mathcal{H}_\varphi(C(D)) > 0$.*

In particular Hausdorff dimension of the central set can be 2. The proof of the theorem is constructive and in fact we are able to show the following.

Lemma 7.2. *For any $\varepsilon > 0$ there is a domain D^ε with $C^{1+\varepsilon}$ boundary such that $dim_H(C(D^\varepsilon)) = 2$ and $\partial D^\varepsilon \subset \{1 < |z| < 1 + \varepsilon\}$.*

We suspect that ∂D can in fact be Lipschitz.

8. Quasiconvexity [21]

A metric space (X, d_X) is C-quasiconvex if there is a constant $C < \infty$ s.t. for every $x, y \in X$ there is a rectifiable curve $\gamma \subset X$ connecting x and y such that

$$l(\gamma) \leq C d_X(x,y),$$

where $l(\gamma)$ is the length of γ.

It is known that if $E \subset \mathbb{R}^n, n \geq 2$, is such that $\dim_H E < n - 1$ then the complement E is quasiconvex. In [21] we show that this is sharp.

Theorem 8.1 (Herron, Hak. [21])**.** *For every $\alpha \in [n-1, n]$ there are compact totally disconnected subsets of $\mathbb{R}^n$ of Hausdorff dimension α and non-quasiconvex complements.*

It would be interesting to characterize metrically Cantor sets in $\mathbb{R}^n$ with quasiconvex complements. We think that any quasiconformal image of a non-quasiconvex domain in $\mathbb{R}^n$ is non-quasiconvex as well. If this is true then every quasiconformal image of a Cantor set with a non-quasiconvex complement would actually have Hausdorff dimension at least $n - 1$.

In [21] (see Corollary **F**) we also characterize finitely connected, quasiconvex plane domains.

Theorem 8.2. *Let $D \subset \mathbb{R}^2$ be a finitely connected domain. Then D is c-quasiconvex if and only if*

- *Every boundary component of D is a point or a Jordan curve, and*
- *Each pair of points $\xi, \eta \in \partial D$ can be joined by a b-quasiconvex path in $D \cup \{\xi, \eta\}$ for every $b > c$.*

9. Future Projects

9.1. **Julia sets/Limit sets.** Conformal/holomorphic dynamics is a source of many beautiful fractals such as Julia sets of rational maps and limit sets of Kleinian groups. Understanding the quasiconformal geometry of these sets is one of my motivations for the study of the quasiconformal geometry in general metric spaces. Problems below are very much related to the questions about conformal dimension and I hope that techniques developed in my papers may be helpful in solving some of them.

Problem Set 9.1 (Limit sets)**.** *Let G be a geometrically finite Kleinian groups with the limit set Λ and the set of discontinuity $\Omega = \mathbb{S}^2 \setminus \Lambda$. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a QC deformation of G to another Kleinian group $G_f = fGf^{-1}$, with the limit set $\Lambda_f = f(\Lambda)$.*

- *(Canary, Minsky, Taylor* [10]*) If Λ is homeomorphic to the Sirepinski carpet and every component of Ω is a round circle then* $\dim_H \Lambda_f \geq \dim_H \lambda$.
- *(Bishop) Suppose G is a Koebe group, i.e. Ω has an invariant component and all the factor groups of G are Fuchsian groups. Then* $\dim_H \Lambda_f \geq \dim_H \Lambda$.

For definition and properties of Koebe groups we refer to the article of Maskit [28]*.*

Problem Set 9.2 (Julia sets)**.** *Let f be a complex polynomial and let $J(f)$ be its Julia set.*

- *Is* $\mathcal{C}\dim J(f) = 1$ *if f is hyperbolic?*
- *Is there an f s.t.: 1)* $\mathcal{C}\dim_H J(f) \in (1,2)$ *(and is achieved)? 2) J_f is minimal with* $\dim_H J_f \in (1,2)$*?*
- *Is there a QC-nonremovable Julia set of zero area?*

9.2. **QC Jacobians & Lipschitz planes.** Given a *QC* map f let J_f be the Jacobian of f. J_f is defined at Lebesgue a.e. point $x \in \mathbb{C}$. The well-known *Quasiconformal Jacobian Problem* of Guy David and Stephen Semmes asks for the characterization of weights $w : \mathbb{C} \to [0,\infty)$ for which there is a constant $C > 0$ s.t.

$$\frac{1}{C}w(x) \leq J_f(x) \leq Cw(x) \text{ for a.e. } x \in \mathbb{C}.$$

In [6] it is shown that this is equivalent to the problem of bi-Lipschitz parametrization of the plane, The following problems are particular cases of the Jacobian problem, but are still open.

Problem Set 9.3. *Is there a set $E \subset \mathbb{R}^2$ of zero area s.t.*

- *(Bishop* [3]*) Every quasiconformal image of E contains a rectifiable curve?*
- *(Bonk, Heinonen, Saksman* [6]*) No QC Jacobian blows up at every point of E?*

We refer to [6] for more precise formulation of the last problem as well as for further problems.

9.3. **Double suspension of homology spheres & conformal dimension.** Let M be a homology 3-sphere, not homeomorphic to $\mathbb{S}^3$. By the celebrated results of Edwards and Cannon (see [13], [12]) double suspension $\Sigma^2 M$ of M is homeomorphic to $\mathbb{S}^5$. Such a double suspension is known as *Edwards sphere*. For every Edwards sphere there is a "wild" embedding $\mathcal{S} = i(\mathbb{S}^1)$ of $\mathbb{S}^1$ such that $\pi_1(\mathbb{S}^5 \setminus i(\mathcal{S})) = \pi_1(M) \neq 0$. It follows that $\dim_H \mathcal{S} > 1$. Sullivan and Siebenmann [33] used this to show that $\Sigma^2 M$ is not bi-Lipschitz to $\mathbb{S}^5$ and asked the following (see also [25]).

Problem 9.4. *Is there a quasisymmetric map from the polyhedral Edwards sphere $\Sigma^2 M$ onto S^5?*

This wild circle has certain cross-sections of Hausdorff dimension bounded away from 0. Therefore by the results of [2] and [18] it is reasonable to expect that $\mathcal{C}\dim \mathcal{S} > 1$. If true, this would give a negative answer to the question above.

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