UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2008 EXAMINATIONS

MAT335H1

Duration - 3 hours

No Aids Allowed

1. (15 points) True or False questions. ***Give a short explanation for your answer and state the relevant theorem.***

(a) If a continuous function of the real line has a periodic point of period $10^{10}+145$, then it must have a periodic point of period 101.

`No.` In Sarkovski's ordering $101 \triangleleft 10^{10}+145$ and therefore there is a continuous function on the line which has a periodic cycle of order 10^{10} and no cycle of 101.

(b) There is a parameter a such that $x^3 + a$ has 4 distinct attracting cycles.

`FALSE. Use Sarkovski.`

(c) Suppose $F : \mathbb{R} \to \mathbb{R}$ is chaotic in the interval (a, b). Can F have an attracting cycle?

NO. If F has an attracting cycle $x_0 \mapsto x_1 \mapsto \ldots \mapsto x_{n-1} \mapsto x_n = x_0$ then there is and $\varepsilon_0 > 0$ s.t. if $|y_0 - x_0| < \varepsilon_0$ then for $y_i = F^{\circ i}(y_0)$ with $i = 0, 1 \ldots, n-1$ we have $F^{\circ nk}(y_i) \to x_i$ as $k \to \infty$. This means that F is neither transitive, nor has sensitive dependence on initial conditions and periodic orbits are not dense. In the test pick one of the properties and explain why it does not hold.

How about a repelling cycle?

YES. x^2-2 is chaotic and has infinitely many repelling cycles, which are dense in $[-2, 2]$. Note that in the period 3-window there are also infinitely many repelling cycles, but they are not dense because there is one attracting cycle.

(d) Can an end point of the middle-thirds Cantor set infinitely many $1's$ and infinitely many $0's$ in its diadic decomposition?

NO

(e) Let $\Sigma' \subset \Sigma$ be the space of sequences $(s_0, s_1, \ldots)$ with $s_j = 0$ or 1 and such that if $s_j = 0$ then $s_{j+1} = 1$. How many periodic points of period 10 does the subshift map σ have on Σ'?

`103 (Hint: Use the recursive formula from the text.)`

2. (15 points) Give the precise definitions of:

(a) Transitivity, sensitive dependence on initial conditions

(b) Chaotic dynamical system

(c) Hausdorff dimension (check the *updated* notes on the course web page).

(d) Julia set. Mandelbrot set.

`See the book`

3. (10 points) State the chain rule for the Schwarzian derivative.
Calculate SM, SF and SG, where $G = M \circ F$ with

$$M(x) = \frac{-2x+7}{\sqrt{2}x - 11} \quad \text{and} \quad F(x) = 5x^2 - 39.$$

4. (20 points) Let F be a continuous function of the line

(a) Show that if F has a cycle of period 3 then it has a cycle of period n for every $n \in \mathbb{N}$.

(b) <u>Assume</u> that if F has a cycle of period 4 then it has a cycle of period 2. <u>Show</u> that period 2^k implies period 2^n for $n < k$.

`See the book.`

5. (20 points) Show that $L_4(x) = 4x(1-x)$ is chaotic on $[0,1]$ by following these steps:

(a) Show that $V(x) = 1 - |x - \frac{1}{2}|$ is chaotic on $[0,1]$.

`Prove each of the 3 properties.`

(b) Give the definition of conjugacy. Show that L_4 is semi-conjugate to V on $[0,1]$.

(c) Explain why the previous two steps imply that L_4 is chaotic.

6. (25 points) Suppose $F : \mathbb{R} \to \mathbb{R}$ is continuous and there are intervals $I_0, I_1 \subset \mathbb{R}$ such that $F'(x) > C > 1$ for all $x \in I_0 \cup I_1$ and

$$F(I_0) \supset I_0 \cup I_1 \quad \textit{and} \quad F(I_1) \supset I_0 \cup I_1.$$

Let $\Lambda = \{x \in I_0 \cup I_1 | \, F^{\circ n}(x) \in I_0 \cup I_1, \text{ for all } n \in \mathbb{N}\}$. Show that F is <u>chaotic on Λ</u> by following these steps:

(a) Define the space Σ and the itinerary map $S : \Lambda \to \Sigma$

(b) Show that S conjugates F on Λ to the shift σ on Σ

(c) Show that the shift $\sigma : \Sigma \to \Sigma$ is chaotic.

`see the book for a,b,c`

(d) Does this imply that F is <u>chaotic on $I_0 \cup I_1$</u>?

`NO`

Why?

`Similar to 1(c).`

7. (15 points) Define Minkowski dimension of a set. Calculate $\dim_M$ for:

(a) Middle-fifth Cantor set:
$\log_3 2$

(b) von-Koch Snowflake:
$\log_3 4$

(c) Sierpinski Carpet
$\log_3 8$

(d) Give an example of a set $E \subset \mathbb{R}$ such that $\dim_H E = 0$ but $\dim_M E = 1$.
`rational numbers.`

8. (10 points) One dimensional dynamics.

(a) Define stochastic maps.
`see the notes`

(b) Suppose Q_c is hyperbolic. How big is the set of points $x \in I_c$ whose orbit does not converge to an attracting cycle?
`This set has zero measure.`

(c) How big is the set of Stochastic/neither Hyperbolic nor Stochastic parameters $c \in [-2, 1/4]$?
There is a set of parameters c of positive length such that $x^2 + c$ is Stochastic by the theorem of Jackobson.
According to the theorem of Lyubich the set of parameters which are neither hyperbolic nor stochastic has measure zero.