Cusp forms on the exceptional group of type $E_7$

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Abstract
Let $G$ be the connected reductive group of type $E_{7,3}$ over $\mathbb{Q}$ and $\mathfrak{T}$ be the corresponding symmetric domain in $\mathbb{C}^{27}$. Let $\Gamma = G(\mathbb{Z})$ be the arithmetic subgroup defined by Baily. In this paper, for any positive integer $k \geq 10$, we will construct a (non-zero) holomorphic cusp form on $\mathfrak{T}$ of weight $2k$ with respect to $\Gamma$ from a Hecke cusp form in $S_{2k-8}(\text{SL}_2(\mathbb{Z}))$. We follow Ikeda’s idea of using Siegel’s Eisenstein series, their Fourier–Jacobi expansions, and the compatible family of Eisenstein series.

Contents
1 Introduction 1
2 Cayley numbers and exceptional Jordan algebras 3
3 Exceptional group of type $E_{7,3}$ 5
4 Jacobi group in $E_{7,3}$, Weil representation, and theta functions 8
5 Modular forms on the exceptional domain and Jacobi forms 12
6 Eisenstein series and their Fourier coefficients 15
7 Fourier–Jacobi expansion of Eisenstein series on $E_{7,3}$ 17
8 Compatible family of Eisenstein series 21
9 Construction of cusp forms on the exceptional domain 22
10 Hecke operators 25
11 The degree-56 standard $L$-function 27
Acknowledgements 29
Appendix 29
References 31

1. Introduction
Let $G$ be the exceptional Lie group of type $E_{7,3}$ over $\mathbb{Q}$ and $\mathfrak{T} \subset \mathbb{C}^{27}$ the corresponding bounded symmetric domain. The purpose of this paper is to construct holomorphic cusp forms on $\mathfrak{T}$ from cusp forms for $\text{SL}_2$ over $\mathbb{Q}$. In [Ike01], Ikeda originally gave a (functorial) construction of a Siegel cusp form for $\text{Sp}_{2n}$ (rank $2n$) from a normalized Hecke eigenform on the upper half-plane $\mathbb{H}$ with respect to $\text{SL}_2(\mathbb{Z})$ which has been conjectured by Duke and Imamoglu. (Independently Ibukiyama formulated a conjecture in terms of Koecher–Maass series.) He made use of the uniform property of the Fourier coefficients of Siegel Eisenstein series for $\text{Sp}_{2n}$ over $\mathbb{Q}$ and together with various deep facts established in [Ike01] to prove the Duke–Imamoglu conjecture. After this work, his construction was generalized to unitary groups $U(n,n)$.
[Ike08], quaternion unitary groups $\text{Sp}(n,n)$ [Yam10], and symplectic groups $\text{Sp}_{2n}$ over totally real fields [Ike, IH13]. Historically, in the case of $\text{Sp}_2$, the resulting cusp form is called Saito–Kurokawa lift, which has been studied thoroughly [Kur78, Pia83, CP88]. Our method follows his construction. The main obstruction is the hugeness of $E_{7,3}$. In the aforementioned works, the theory of Jacobi forms has been understood well, since the Heisenberg group inside the group in consideration is easy to handle. On the other hand, much less is known in the case of $E_{7,3}$. Therefore we have to consider a suitable Heisenberg subgroup in $E_{7,3}$ which has not been studied. To do this we analyze it in terms of roots.

We now explain our main theorem. We refer to the next section for the several notations which appear below. Let $\Gamma = G(\mathbb{Z})$ be the arithmetic subgroup defined by Baily in [Bai70] which is constructed by using the integral Cayley numbers $\sigma$. For a positive integer $k \geq 10$, let $E_{2k}$ be the Siegel Eisenstein series on $\mathbb{T}$ of weight $2k$ with respect to $\Gamma$. Then it has the Fourier expansion of form

$$E_{2k}(Z) = \sum_{T \in \mathcal{O}(\mathbb{Z})^+} a_{2k}(T) \exp(2\pi i (T, Z)), \quad Z \in \mathbb{T},$$

$$a_{2k}(T) = C_{2k} \det(T)^{(2k-9)/2} \prod_{p \mid \det(T)} \mathcal{J}^p_T(p^{(2k-9)/2}),$$

where $C_{2k} = 2^{15} \prod_{n=0}^{2} (2k - 4n)/(B_{2k-4n})$, and $\mathcal{J}^p_T(X)$ is a Laurent polynomial over $\mathbb{Q}$ in $X$ which is depending only on $T$ and $p$.

Let $S_{2k-8}(\text{SL}_2(\mathbb{Z}))$ be the space of elliptic cusp forms of weight $2k - 8 \geq 12$ with respect to $\text{SL}_2(\mathbb{Z})$. For each normalized Hecke eigenform $f = \sum_{\tau \in \mathbb{H}} c(\tau) q^\tau, \tau \in \mathbb{H}$ in $S_{2k-8}(\text{SL}_2(\mathbb{Z}))$ and each rational prime $p$, we define the Satake $p$-parameter $\alpha_p$ by $c(p) = p^{(2k-9)/2}(\alpha_p + \alpha_p^{-1})$. For such $f$, consider the following formal series on $\mathbb{T}$:

$$F(Z) = \sum_{T \in \mathcal{O}(\mathbb{Z})^+} A(T) \exp(2\pi i (T, Z)), \quad Z \in \mathbb{T}, \quad A(T) = \det(T)^{(2k-9)/2} \prod_{p \mid \det(T)} \mathcal{J}^p_T(\alpha_p).$$

Then we will show the following theorem.

**Theorem 1.1.** The function $F(Z)$ is a non-zero Hecke eigen cusp form on $\mathbb{T}$ of weight $2k$ with respect to $\Gamma$.

If $f$ has integer Fourier coefficients, then $F$ also has integer Fourier coefficients (Remark 9.2). By virtue of Theorem 1.1, $F = F(Z)$ gives rise to a cuspidal automorphic representation $\pi_F = \pi_\infty \otimes \pi_p$ of $G(\mathbb{A})$. Then $\pi_\infty$ is a holomorphic discrete series of the lowest weight $2k$ associated to $-2k\pi \tau_7$ in the notation of [Bou02] (cf. [Kna86, p. 158]). For each prime $p$, $\pi_p$ is unramified. In fact, $\pi_p$ turns out to be a degenerate principal series $\text{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} |\nu(g)|^{2p}$, where $p^{s_p} = \alpha_p$. Then for each local component $\pi_p$, one can associate the local $L$-factor $L(s, \pi_p, St)$ of the standard $L$-function of $\pi_F$ by using the Langlands–Shahidi method. Put $L(s, \pi_F, St) = \prod_p L(s, \pi_p, St)$ and let $L(s, \pi_f) = \prod_p (1 - \alpha_p p^{-s})(1 - \alpha_p^{-1} p^{-s})$ be the automorphic $L$-function of the cuspidal representation $\pi_f$ attached to $f$. Then we have the following theorem.

**Theorem 1.2.** The degree-56 standard $L$-function $L(s, \pi_F, St)$ of $\pi_F$ is given by

$$L(s, \pi_F, St) = L(s, \text{Sym}^3 \pi_f)L(s, \pi_f)^2 \prod_{i=1}^{4} L(s \pm i, \pi_f)^2 \prod_{i=5}^{8} L(s \pm i, \pi_f),$$

where $L(s, \text{Sym}^3 \pi_f)$ is the symmetric cube $L$-function.
This paper is organized as follows. In §2, we fix notations on Cayley numbers and exceptional Jordan algebras and review their properties. In §3, we review the exceptional group of type $E_{7,3}$ and prove many facts which are not available in the literature. In §4, we define the Jacobi group inside the exceptional group using the root subgroups, and recall Weil representations and theta functions. In §5, we review modular forms on the exceptional domain and define Jacobi forms of matrix indices and study the Fourier–Jacobi coefficients of a modular form both in classical setting and in adelic setting. In §6, we review the result of Karel on Fourier coefficients of Eisenstein series and interpret Eisenstein series in terms of degenerate principal series, following [Kud08]. Section 7 is the main technical part, where we prove the analogue of Ikeda’s result [Ike94], namely, the Fourier–Jacobi coefficients of Eisenstein series are a sum of products of theta functions and Eisenstein series. In §9, by following Ikeda [Ike01, Ike08], we construct a holomorphic cusp form on the exceptional group of type $E_{7,3}$. Our situation is similar to unitary group case, in that we do not need to consider half-integral modular forms. In §10, we review the Hecke operators from Karel’s thesis [Kar72] and modify it to fit into representation theory. Then we prove that our cusp form is a Hecke eigenform with respect to this modified action. The degree-56 standard $L$-function helps us to speculate on the Arthur parameter of $\pi$. We make a brief remark on it at the end of §11. In the Appendix, we compute the discriminant of some quadratic forms and prove the orthogonal relation of theta functions we need.

2. Cayley numbers and exceptional Jordan algebras

In this section we will recall the Cayley numbers and the exceptional Jordan algebras. We refer to [Bai70, Cox46, Kim93]. For any field $K$ whose characteristic is different from 2 and 3, the Cayley numbers $\mathbb{C}_K$ over $K$ is an eight-dimensional vector space over $K$ with basis $\{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ satisfying the following rules for multiplication:

1. $xe_0 = e_0x = x$ for all $x \in \mathbb{C}_K$;
2. $e_i^2 = -e_0$ for $i = 1, \ldots, 7$;
3. $e_i(e_i+1) = (e_i+1)e_i = -e_0$ for any $i$ (modulo 7).

For each $x = \sum_{i=0}^7 x_ie_i \in \mathbb{C}_K$, the map $x \mapsto \bar{x} = x_0e_0 - \sum_{i=1}^7 x_ie_i$ defines an anti-involution of $\mathbb{C}_K$. The trace and the norm on $\mathbb{C}_K$ are defined by

$\text{Tr}(x) := x + \bar{x} = 2x_0, \quad N(x) := x\bar{x} = \sum_{i=0}^7 x_i^2$.

The space of Cayley numbers $\mathbb{C}_K$ is neither commutative nor associative. In spite of this, we have

$\text{Tr}(xy) = \text{Tr}(yx), \quad \text{Tr}(x\bar{y}) = \text{Tr}(\bar{y}x), \quad \text{Tr}((xy)z) = \text{Tr}(x(yz))$.

We denote by $\mathfrak{o}$ the space of integral Cayley numbers which is a $\mathbb{Z}$-submodule of $\mathbb{C}_K$ given by the following basis:

$\alpha_0 = e_0, \quad \alpha_1 = e_1, \quad \alpha_2 = e_2, \quad \alpha_3 = -e_4,$
$\alpha_4 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4), \quad \alpha_5 = \frac{1}{2}(-e_0 - e_1 - e_4 + e_5),$  
$\alpha_6 = \frac{1}{2}(-e_0 + e_1 - e_2 + e_6), \quad \alpha_7 = \frac{1}{2}(-e_0 + e_2 + e_4 + e_7)$.

As shown in [Cox46], $\mathfrak{o}$ is stable under the operations of the anti-involution, multiplication, and addition. Further we have $\text{Tr}(x), N(x) \in \mathbb{Z}$ if $x \in \mathfrak{o}$. By using this integral structure, for any $\mathbb{Z}$-algebra $R$, one can consider $\mathbb{C}_R = \mathfrak{o} \otimes_{\mathbb{Z}} R$. 

Page 3 of 32
Let $\mathfrak{J}_K$ be the exceptional Jordan algebra consisting of the element

$$X = (x_{ij})_{1 \leq i, j \leq 3} = \begin{pmatrix} a & x & y \\ \bar{x} & b & z \\ \bar{y} & \bar{z} & c \end{pmatrix},$$  \hspace{1cm} (2.1)$$

where $a, b, c \in K e_0 = K$ and $x, y, z \in \mathcal{C}_K$. In general, the matrix multiplication $X \cdot Y$ for two elements $X, Y \in \mathfrak{J}_K$ does not belong to $\mathfrak{J}_K$, but the square $X^2 = X \cdot X$ always does. The composition of $\mathfrak{J}_K$ is given by

$$X \circ Y = \frac{1}{2}(X \cdot Y + Y \cdot X).$$

For the above $X$, we define the trace by $\text{Tr}(X) := a + b + c$, and define an inner product on $\mathfrak{J}_K \times \mathfrak{J}_K$ by $(X, Y) := \text{Tr}(X \circ Y)$. Moreover we define

$$\det(X) := abc - aN(z) - bN(y) - cN(x) + \text{Tr}((xz)y)$$

and a symmetric tri-linear form $(\ast, \ast, \ast)$ on $\mathfrak{J}_K \times \mathfrak{J}_K \times \mathfrak{J}_K$ by

$$(X, Y, Z) := \frac{1}{2}\{\det(X + Y + Z) - \det(X + Y) - \det(Y + Z) - \det(Z + X)$$

$$+ \det(X) + \det(Y) + \det(Z)\}.$$

Then we define a bilinear pairing $\mathfrak{J}_K \times \mathfrak{J}_K \rightarrow \mathfrak{J}_K$, $(X, Y) \mapsto X \times Y$ by requiring the identity

$$\mathfrak{J}(X, Y, Z) = (X \times Y, Z) = \text{Tr}((X \times Y) \circ Z)$$

for any $Z \in \mathfrak{J}_K$.

In particular, for $X_i, i = 1, 2$ with entries as in (2.1), we have

$$X_1 \times X_2 = \begin{pmatrix} b_1 c_2 + c_1 b_2 - \bar{z}_1 z_2 + z_2 \bar{z}_1 & A & B \\ \bar{A} & a_1 c_2 + c_1 a_2 - \bar{z}_1 y_2 + y_2 \bar{z}_1 & C \\ \bar{B} & \bar{C} & a_1 b_2 + b_1 a_2 - \bar{z}_1 x_2 + x_2 \bar{z}_1 \end{pmatrix},$$  \hspace{1cm} (2.2)$$

where $A = (-c_1 x_2 - c_2 x_1)/2 + (y_1 \bar{z}_2 + y_2 \bar{z}_1)/2$, $B = (-a_1 y_2 - b_2 y_1)/2 + (x_1 z_2 + x_2 z_1)/2$, and $C = (-a_1 z_2 - a_2 z_1)/2 + (\bar{x}_1 y_2 + \bar{y}_2 z_1)/2$.

By using integral Cayley numbers, we define a lattice

$$\mathfrak{J}(Z) := \{X = (x_{ij}) \in \mathfrak{J}_Q \mid x_{ii} \in \mathbb{Z}, \text{ and } x_{ij} \in \mathfrak{o} \text{ for } i \neq j\},$$

and put $\mathfrak{J}(R) = \mathfrak{J}(Z) \otimes_Z R$ for any $Z$-algebra $R$. Although the composition $\circ$ does not preserve the integral structure, the inner product $(\ast, \ast)$ does. Hence $(\mathfrak{J}(R), \mathfrak{J}(R)) \in R$. Then one can show that the lattice $\mathfrak{J}(Z)$ in $\mathfrak{J}_Q$ is the self-dual with respect to $(\ast, \ast)$, namely

$$\mathfrak{J}\overline{(Z)} := \{X \in \mathfrak{J}_Q \mid (X, Y) \in \mathbb{Z} \forall Y \in \mathfrak{J}(Z)\} = \mathfrak{J}(Z).$$

We also define $\mathfrak{J}_2(R)$ as the set of all matrices of forms

$$X = \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix}, \hspace{0.5cm} a, b \in R, \hspace{0.5cm} x \in \mathcal{C}_R.$$  

Similarly we define the inner product on $\mathfrak{J}_2(R) \times \mathfrak{J}_2(R)$ by $(X, Y) := \frac{1}{2} \text{Tr}(XY + YX)$. For any such $X$, we define $\det(X) := ab - N(x)$. For $X$ as above, $r \in R$, and $\xi = (\xi_i \xi_j), \xi_i \in \mathcal{C}_R (i = 1, 2)$, it is easy to see that

$$\det \begin{pmatrix} X & X \xi \\ \xi X & r \end{pmatrix} = \det(X)(r - \xi X \xi) = \det(X)(r - (X, \xi))$$  \hspace{1cm} (2.3)$$

$$= \det(X)(r - (X, \xi)).$$
Cusp forms on the exceptional group of type $E_7$

which will be used later (§9). Henceforth we identify $\mathcal{H}(R)$ with a subspace of $\mathcal{H}(R)$ by $(\frac{a}{x}, \bar{z}) \mapsto (\frac{a}{x} x, 0)$. We define
\[ R_3(K) = \{ X \in \mathcal{H}_K \mid \det(X) \neq 0 \} \]
and define the set $R_3^+(K)$ consisting of squares of elements in $R_3(K)$. It is known that $R_3^+(\mathbb{R})$ is an open, convex cone in $\mathcal{H}_\mathbb{R}$. We denote by $\overline{R_3^+(\mathbb{R})}$ the closure of $R_3^+(\mathbb{R})$ in $\mathcal{H}_\mathbb{R} \simeq \mathbb{R}^{27}$ with respect to Euclidean topology. For any subring $A$ of $\mathbb{R}$, set
\[ \mathcal{H}(A)_+ := \mathcal{H}(A) \cap R_3^+(\mathbb{R}), \quad \mathcal{H}(A)_{\geq 0} := \mathcal{H}(A) \cap \overline{R_3^+(\mathbb{R})}. \]
We also define
\[ \mathcal{H}_2(A)_+ = \left\{ \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \in \mathcal{H}_2(A) \mid a, b \in A \cap \mathbb{R}_{>0}, ab - N(x) > 0 \right\}, \]
and
\[ \mathcal{H}_2(A)_{\geq 0} = \left\{ \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \in \mathcal{H}_2(A) \mid a, b \in A \cap \mathbb{R}_{>0}, ab - N(x) \geq 0 \right\}. \]
We define the exceptional domain as follows:
\[ \mathcal{T} := \{ Z = X + Y \sqrt{-1} \in \mathcal{H}_\mathbb{C} \mid X, Y \in \mathcal{H}_\mathbb{R}, Y \in \overline{R_3^+(\mathbb{R})} \} \]
which is a complex analytic subspace of $\mathbb{C}^{27}$. We also define
\[ \mathcal{T}_2 := \{ X + Y \sqrt{-1} \in \mathcal{H}_2(\mathbb{C}) \mid X, Y \in \mathcal{H}_2(\mathbb{R}), Y \in \mathcal{H}_2(\mathbb{R})_+ \}. \]

3. Exceptional group of type $E_{7,3}$

In this section we recall the exceptional group of type $E_{7,3}$. Put $\mathfrak{g} = \mathfrak{g}_K$ where $K$ is a field whose characteristic is different from 2 and 3. Define two subgroups of $\text{GL}(\mathfrak{g})$ by
\[ M = \{ g \in \text{GL}(\mathfrak{g}) \mid \det(gX) = \nu(g) \det(X), \text{for } \nu(g) \neq 0 \} \]
\[ M' = \{ g \in M \mid \nu(g) = 1 \}. \]

Then $M$ is an algebraic group over $\mathbb{Q}$ of type $GE_6$, and $M'$ is the derived group of $M$, which is a simple group of type $E_{6,2}$. The center of $M'$ is the group of cube roots of unity.

There is an automorphism $g \mapsto g^*$ of $M$ of order 2 by the identity
\[ (gX, g^*Y) = (g^*X, gY) = (X, Y). \]

(3.1)

Then $g^*$ is the inverse adjoint of $g$. It satisfies $g(X \times Y) = (g^*X) \times (g^*Y)$.

Let $G$ be the algebraic group over $\mathbb{Q}$ as in [Bai70]. Let $X, X'$ be two $K$-vector spaces, each isomorphic to $\mathfrak{g}$, and $\Xi, \Xi'$ be copies of $K$. Let $W = X \oplus \Xi \oplus X' \oplus \Xi'$, and for $w = (X, \xi, X', \xi') \in W$, define a quartic form $Q$ on $W$ by
\[ Q(w) = (X \times X, X' \times X') - \xi \det(X) - \xi' \det(X') - \frac{1}{4}((X, X') - \xi\xi')^2, \]
and a skew-symmetric bilinear form $\{ , \}$ by
\[ \{w_1, w_2\} = (X_1, X'_2) - (X_2, X'_1) + \xi_1 \xi'_2 - \xi_2 \xi'_1. \]
Then \[ G(K) = \{ g \in GL(W_K) \mid Qg = Q, g\{, \} = \{, \} \}. \]

This defines a connected algebraic \( \mathbb{Q} \)-group of type \( E_{7,3} \). The center of \( G(\mathbb{R}) \) is \( \{ \pm \text{id} \} \) and the quotient of \( G(\mathbb{R}) \) by its center is the group of holomorphic automorphisms of \( \mathbb{C} \). The real rank of \( G \) is 3, and it is split over \( \mathbb{Q}_p \) for any prime \( p \).

The group \( M \) can be considered as a subgroup of \( G \) by defining the action \( g(X, \xi, X', \xi') = (gX, \nu(g)\xi, g^*X', \nu(g)^{-1}\xi'). \)

Let \( N \) be the subgroup of all transformations \( p_B \) for \( B \in \mathfrak{F} \) as in [Bai70]. Recall the definition.

\[
p_B \begin{pmatrix} X \\ X' \\ \xi' \end{pmatrix} = \begin{pmatrix} X + \xi' B \\ \xi + (B, X') + (B \times B, X) + \xi' \det(B) \\ X' + 2B \times X + \xi' B \times B \end{pmatrix}.
\]

The relative root system of \( G \) over \( \mathbb{Q} \) is of type \( C_3 \), and we denote the positive roots by \( \{ e_1 \pm e_2, e_1 \pm e_3, e_2 \pm e_3 \} \), and let \( \Delta = \{ e_1 - e_2, e_2 - e_3, 2e_1, 2e_2, 2e_3 \} \) be the set of simple roots. We describe their root spaces: for a positive root \( \alpha \), let \( U_\alpha \) be the root subspace. For \( 1 \leq i \leq j \leq 3 \), let \( e_{ij} \) be the \( 3 \times 3 \) matrix with a 1 in the intersection of the \( i \)th row and \( j \)th column and zeros elsewhere, and let \( e_i = e_{ii} \). Then for \( a, b, c \in K, x, y, z \in \mathfrak{C}_K \),

\[
U_{2e_1} = \{ p_{ae_1} \}, \quad U_{2e_2} = \{ p_{ae_2} \}, \quad U_{2e_3} = \{ p_{ae_3} \}
\]

\[
U_{e_1 + e_2} = \{ p_{2e_{12}} \}, \quad U_{e_1 + e_3} = \{ p_{2e_{13}} \}, \quad U_{e_2 + e_3} = \{ p_{2e_{23}} \}
\]

\[
U_{e_1 - e_2} = \{ m_{\pm e_{21}} \in \text{GL}(3) : m_{\pm e_{21}} X = (I + xe_{12})X(I + \bar{e}_{21}), X \in \mathfrak{F} \}
\]

\[
U_{e_1 - e_3} = \{ m_{\pm e_{31}} \in \text{GL}(3) : m_{\pm e_{31}} X = (I + ye_{13})X(I + \bar{y}e_{31}), X \in \mathfrak{F} \}
\]

\[
U_{e_2 - e_3} = \{ m_{\pm e_{32}} \in \text{GL}(3) : m_{\pm e_{32}} X = (I + ze_{23})X(I + \bar{z}e_{32}), X \in \mathfrak{F} \}
\]

Remark 3.1. Note that we are using different ordering of roots from [Bai70]. In [Bai70], \( N \) consists of root spaces of negative non-compact roots. However, it is more convenient to make it correspond to positive roots so that it may correspond to the upper triangular matrices of the form \( \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \) in the \( \text{Sp}_{2n} \) case.

Note the following:

\[
m_{\mp e_{ij}}^* = m_{\pm e_{ji}}.
\]

Let \( H \) be the group generated by \( U_{2e_3} \) and \( \iota_{e_3} \), where \( \iota_{e_3} \) is the Weyl group element of \( 2e_3 \), which is given by \( \iota_{e_3} = p_{e_3}p_{-e_3}p_{e_3} \), where \( p_{e_3} \) generates the root subspace of \(-2e_3\). Then \( H \cong \text{SL}_2 \). Let \( \iota = \iota_{e_1} \iota_{e_2} \iota_{e_3} \). Then \( \iota^{-1} = -\iota \), and \( \iota_B = \iota p_B \iota^{-1} \) will generate the opposite unipotent subgroup \( N \) of \( N \). This \( \iota \) plays the role of \( \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \) is \( \text{Sp}_{2n} \). Its action is given by

\[
\iota(X, \xi, X', \xi') = (-X', -\xi', X, \xi).
\]

We define two maximal parabolic \( \mathbb{Q} \)-subgroups:

\[
P = MN, \quad Q = LV,
\]

where \( V \) is generated by \( U_\alpha \) for \( \alpha = e_1 \pm e_3, e_2 \pm e_3, e_1 + e_2, 2e_1, 2e_2 \). Then \( P \) is the Siegel parabolic subgroup associated to \( \Delta - \{ 2e_3 \} \), and \( Q \) is the parabolic subgroup associated to \( \Delta - \{ e_2 - e_3 \} \).

Then \( V \) is the Heisenberg group, and the derived group of \( L \) is \( L' = H \times \text{Spin}(9, 1) \).
Lemma 3.2. For \( g \in \mathbb{M} \) and \( p_B \in \mathbb{N} \),
\[
g p_B = p_{B_1} g, \quad B_1 = \nu(g) g B.
\]

Proof. By explicit computation, we see that
\[
\begin{aligned}
g p_B \begin{pmatrix} X \\ \xi \\ X' \\ \xi' \end{pmatrix} &= \begin{pmatrix} g X + \xi' g B \\
\nu(g)(\xi + (B, X') + (B \times B, X) + \xi' \det(B)) \\
g^* X' + 2 g^*(B \times X) + \xi' g^*(B \times B) \\
\nu(g)^{-1} \xi' \end{pmatrix}, \\
p_{B_1} g \begin{pmatrix} X \\ \xi \\ X' \\ \xi' \end{pmatrix} &= \begin{pmatrix} g X + \nu(g)^{-1} \xi' B_1 \\
\nu(g)(\xi + (B_1, g^* X') + (B_1 \times B_1, g X) + \nu(g)^{-1} \xi' \det(B_1)) \\
g^* X' + 2 B_1 \times g X + \nu(g)^{-1} \xi' B_1 \times B_1 \\
\nu(g)^{-1} \xi' \end{pmatrix}.
\end{aligned}
\]

By comparing coefficients, we see that \( B_1 = \nu(g) g B \). \(\square\)

Denote the element of \( V = V(K) \) by
\[
v(x, y, z) = m_{x_1 x_3} m_{x_2 x_3} \cdot p_{y_1 y_3} p_{y_2 y_3} \cdot p_z,
\]
where
\[
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad z = \begin{pmatrix} a \\ w \\ b \end{pmatrix},
\]
where \( x_1, x_2, y_1, y_2, w \in \mathbb{C}_K \) and \( a, b \in K \). We identified \( z \) with \( (0 0) \) in the definition of \( p_z \). Then by using the above lemma, we can show that
\[
v(x, y, z) v(x', y', z') = v(x + x', y + y', z + z' - y(t^i x^i) - x'(t^i y^i)). \tag{3.2}
\]

Now let
\[
\begin{aligned}
X &= X(K) = \{ m_{x_1 x_3} m_{x_2 x_3} \in V \mid x_1, x_2 \in \mathbb{C}_K \}, \\
Y &= Y(K) = \{ p_{y_1 y_3} p_{y_2 y_3} \in V \mid y_1, y_2 \in \mathbb{C}_K \}, \\
Z &= Z(K) = \{ p_z \in V \mid z \in \mathbb{J}_2(K) \} \simeq \mathbb{J}_2(K).
\end{aligned}
\]

We identify \( X \) (respectively \( Y \)) with \( \mathbb{C}_K^2 \) by \( m_{x_1 x_3} m_{x_2 x_3} \mapsto x = (x_1 x_2) \) (respectively by \( p_{y_1 y_3} p_{y_2 y_3} \mapsto y = (y_1 y_2) \)). Then we have the decomposition
\[
V = V(K) = X \cdot Y \cdot Z. \tag{3.4}
\]

We hope that it is clear from the context when \( X, Y, Z \) denote the sets and when they denote the elements of \( \mathbb{J} = \mathbb{J}_K \).

For any \( S \in \mathbb{J}_2(K) \), define \( \text{tr}_S : Z = \{ v(0, 0, z) \} \rightarrow K, \text{tr}_S(v(0, 0, z)) = \frac{1}{2} (S, z) \). Since \( Z \) is the center of \( V \), \( \text{Ker} (\text{tr}_S) \) is a normal subgroup of \( V \), and we may consider the quotient \( V_0 = V/\text{Ker}(\text{tr}_S) \).

Define the alternating form on \( X \oplus Y \) by
\[
\langle (x, y), (x', y') \rangle_S = \text{Tr}(S(x^t y^T) + y' (t^ix^i) - x'(t^i y^i) - y'(t^i x^i)).
\]

Consider the map \( g_S : V \rightarrow X \oplus Y \oplus K \) defined by
\[
v(x, y, z) = v(x) v(y) v(z) \mapsto \left( x, y, \text{Tr} \left( \frac{1}{2} S z \right) + \text{Tr} \left( \frac{S}{2} (y^t x + x^t y) \right) \right). \tag{3.5}
\]
From (3.2), we see that \( g_S(v(x, y, z)v(x', y', z')) = (x + x', y + y', z'') \) where
\[
z'' = \text{Tr}
\left(\frac{1}{2}S_z\right) + \text{Tr}
\left(\frac{S}{2}(y^t \bar{x} + x^t \bar{y})\right) + \text{Tr}
\left(\frac{1}{2}S_{z'}\right) + \text{Tr}
\left(\frac{S}{2}(y'^t \bar{x} + x'^t \bar{y})\right) + \frac{1}{2}((x, y), (x', y'))s.
\]

Since \( \text{Ker}(g_S) = \text{Ker}(tr_S) \), if \( \text{det}(S) \neq 0 \) then we obtain the isomorphism
\[
g_S : V_0 = V/\text{Ker}(tr_S) \xrightarrow{\sim} X \oplus Y \oplus K. \quad (3.6)
\]

Next we compute the action of \( H(K) \) on \( V = V(K) \): recall that \( H(K) = \langle p_{bc,3}, t_{e_3} \rangle \cong \text{SL}_2(K) \) for \( b \in K \). We identify \( \gamma = (a \ b \ c \ d) \in \text{SL}_2(K) \) with the corresponding element in \( H(K) \) under the isomorphism. Observe [Fre53] that
\[
e_3 \times X = \frac{1}{2} \begin{pmatrix} b & -x & 0 \\ -\bar{x} & a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 \times (e_3 \times X) = \frac{1}{4} \begin{pmatrix} a & x & 0 \\ \bar{x} & b & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then, for \( i = 1, 2 \),
\[
t^{-1}_e p_{xe_3} t_e = m_{-xe_3}, \quad t^{-1}_e m_{xe_3} t_e = p_{xe_3}.
\]

For \( 1 \leq i < j \leq 2 \), we have \( t^{-1}_e p_{xe_j} t_e = p_{xe_j} \). Hence
\[
p^{-1}_{be_3} v(x, y, z)p_{be_3} = v(x, bx + y, z - bx^t \bar{x}), \quad t^{-1}_e v(x, y, z) t_e = v(-y, x, z + x^t \bar{y} + y^t \bar{x}).
\]

Since \( p^{-1}_{ce_3} t_e p^{-1}_{ce_3} t_e = 1 \), and \( h(a) = (a \ 0 \ 0 \ 1) \) is identified with \( p_{ae_3} p^{-1}_{a_e} p^{-1}_{ae_3} \), we see that
\[
p^{-1}_c v(x, y, z) p^{-1}_c = v(x + cy, y - cy^t \bar{y}), \quad h(a)^{-1} v(x, y, z) h(a) = v(ax, a^{-1} y, z).
\]

Here \( h(a) \in M \), and \( \nu(h(a)) = a \); more explicitly,
\[
h(a)(X, \xi, X', \xi') = (X + (a - 1)(e_3, X)e_3, a \xi, X' - (1 - a^{-1})(e_3, X')e_3, a^{-1} \xi').
\]

Hence we have the following lemma.

**LEMMA 3.3.** Let \( \gamma = (a \ b \ c \ d) \in H(K) \). Then \( \gamma^{-1} v(x, y, z) \gamma = v(ax + cy, bx + dy, z') \) where
\[
z' = z - \frac{1}{4}((ax + cy)^t(bx + dy) + (bx + dy)^t(ax + cy) - x^t \bar{y} - y^t \bar{x}).
\]

### 4. Jacobi group in \( E_{7,3} \), Weil representation, and theta functions

#### 4.1 Jacobi group in \( E_{7,3} \)

Let \( A \) be the ring of adeles of \( \mathbb{Q} \) and \( A_f \) its finite part. Let \( \widehat{\mathbb{Z}} \) be the profinite completion of \( \mathbb{Z} \).

For \( R = A, A_f, \widehat{\mathbb{Z}}, \mathbb{Q}, \mathbb{Q}_p, \mathbb{Z}_p, p \leq 2, \) or any field \( R \), one can consider \( X(R), Y(R), Z(R), V(R) \) (analogues of (3.3) and (3.4)) by using \( C \) and the action of \( H(R) \cong \text{SL}_2(R) \) on \( V(R) \) by using the calculation done in §3. Note that we may not get the identification \( H(R) \cong \text{SL}_2(R) \) for an arbitrary ring \( R \) since this map is described in terms of the root system. (The interested readers should consult the notion of Chevalley basis, cf. [Ste68].)

Now we introduce a new coordinate on \( V \) by modifying the group actions, and define the Jacobi group in \( E_{7,3} \). For any \( x, y \in X(R) \) and \( S \in \mathcal{J}_2(R) \) so that \( \text{det}(S) \neq 0 \), we define
\[
\sigma(x, y) := x^t \bar{y} + y^t \bar{x}, \quad \sigma_S(x, y) := (S, \sigma(x, y)) \quad \text{and} \quad \lambda_S(x, y) := \frac{1}{2} \sigma_S(x, y). \quad (4.1)
\]
Cusp forms on the exceptional group of type $E_7$

Clearly $\sigma(x, y) \in \mathbb{Z}_2(R) \simeq Z(R)$. A new coordinate $v_1(x, y, z)$ on $V$ is defined by

$$v_1(x, y, z) := v(x, y, z - \sigma(x, y)), \quad x \in X(R), y \in Y(R) \text{ and } z \in Z(R).$$

Then by (3.2), one has

$$v_1(x, y, z)v_1(x', y', z') = v_1(x + x', y + y', z + z' + \sigma(x, y') - \sigma(x', y)). \quad (4.2)$$

The alternating pairing on $X(R) \oplus Y(R)$ is modified as

$$\langle (x, y), (x', y') \rangle_S := 2(\lambda_S(x, y') - \lambda_S(x', y)) = \sigma_S(x, y') - \sigma_S(x', y), \quad (4.3)$$

and $X(R) \oplus Y(R) \oplus R$ has the Heisenberg structure defined by

$$(x, y, a) \ast (x', y', b) = (x + x', y + y', a + b + \frac{1}{2}\langle (x, y), (x', y') \rangle_S).$$

For any $S \in \mathbb{Z}_2$, the Heisenberg structure on $V$ is modified by passing to $g_S$ (see (3.6) for this map) as

$$g_{1,S} : V(R) \longrightarrow X(R) \oplus Y(R) \oplus R, \quad v_1(x, y, z) \mapsto (x, y, \frac{1}{2}(S, z)). \quad (4.4)$$

Noting $\frac{1}{2}(\sigma_S(x, y') - \sigma_S(x', y)) = \lambda_S(x, y') - \lambda_S(x', y)$, it is easy to see that

$$g_{1,S}(v_1(x, y, z)v_1(x', y', z')) = g_{1,S}(v_1(x, y, z)) \ast g_{1,S}(v_1(x', y', z')),$$

or, equivalently, $g_{1,S}$ preserves the Heisenberg structures in both sides.

The action of $H(R)$ on $V(R)$ with the new coordinates now turn to be much simpler by Lemma 3.3:

$$\gamma^{-1}v_1(x, y, z)\gamma = v_1(ax + cy, bx + dy, z) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H(R). \quad (4.5)$$

By using this action, we define the Jacobi group in $G(R)$ by

$$J(R) := V(R) \rtimes H(R) \quad (4.6)$$

with the coordinates $(v_1(x, y, z), h(\gamma)), \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ under the identification $H(R) \simeq SL_2(R)$.

4.2 Weil representation and theta functions

In this section we shall recall Weil representation and theta series (cf. [Ike94, §1-3]).

For each place $p \neq \infty$, put

$$e_p(x) = \exp(-2\pi \sqrt{-1} \cdot \text{Frac}(x))$$

for $x \in \mathbb{Q}_p$ where Frac($x$) stands for the fractional part of $x$. For $p = \infty$, put $e(x) = e_\infty(x) := \exp(2\pi \sqrt{-1}x)$ for $x \in \mathbb{R}$. Fix a non-trivial additive character $\psi : \mathbb{A} \rightarrow \mathbb{C}^\times$ and decompose it as the restricted tensor product $\psi = \bigotimes'_{p < \infty} \psi_p$. As a standard example, one can take $\psi_{st} := \bigotimes'_{p < \infty} e_p(\ast)$ which will be used later when we translate the adelic setting into the classical setting and vice versa.

Fix $S \in \mathbb{Z}_2$. We denote by $h(a)$ (respectively $n(b)$) the element of $H(Q_p)$ corresponding to $(a, 0, 1), a \in Q_p^\times$ (respectively $(0, b)$, $b \in Q_p$) under $H(Q_p) \simeq SL_2(Q_p)$. Note that $(0, 1)$ corresponds to $\{e_3 \in H(Q_p)$.
For each place $p \leq 1$, the Schrödinger model $\omega_{S,p}$ on $V(Q_p)$ with the central character $\psi_p : z \mapsto \psi_p(\frac{1}{2}(S, z))$, $z \in Z(Q_p)$ realized on the Schwartz space $S(X(Q_p))$ is given by

$$\omega_{S,p}(v_1(x, y, z)) \varphi(t) = \varphi(t + x)\psi_p(\frac{1}{2}S, z) + 2\lambda_S(t, y) + \lambda_S(x, y)$$

for $\varphi \in S(X(Q_p))$ and $v_1(x, y, z) \in V(Q_p)$. Noting the multiplication law (4.2), it is easy to check

$$\omega_{S,p}(v_1(x, y, z)v_1(x', y', z'))(t) = \omega_{S,p}(v_1(x, y, z))(\omega_{S,p}(v_1(x', y', z'))\varphi(t)),$$

By the Stone–von Neumann theorem, $\omega_{S,p}$ is a unique irreducible unitary representation on which $Z(Q_p)$ acts by $\psi_p$.

Recall the conjugate action $H(Q_p)$ on $V(Q_p)$ (see (4.5)) and the alternating pairing (4.3). This induces a homomorphism

$$H(Q_p) \hookrightarrow SP_{V/Z}(Q_p) := SP(V(Q_p)/Z(Q_p), \langle *, * \rangle_S). \quad (4.7)$$

Let $\tilde{SP}_{V/Z}(Q_p)$ be the metaplectic covering of $SP_{V/Z}(Q_p)$. This covering does not split, but by [Kud94], one has a splitting $H(Q_p) \hookrightarrow \tilde{SP}_{V/Z}(Q_p)$ so that the map (4.7) factors through it via the covering map. The Schrödinger model $\omega_{S,p}$ extends to the Weil representation of $V(Q_p) \rtimes \tilde{SP}_{V/Z}(Q_p)$ acting on $S(X(Q_p))$. Then the pullback to $H(Q_p)$ of this representation is given by

$$\omega_{S,p}(h(a))\varphi(t) = \chi_S(a)|a|_{p}^{8}\varphi(ta), \quad \chi_S(a) := \langle \text{disc}(\lambda_S), a \rangle_{Q_p}$$

$$\omega_{S,p}(n(b))\varphi(t) = \psi_p(\lambda_S(t, t)b)\varphi(t)$$

$$\omega_{S,p}(te_3)\varphi(t) = (F_S \varphi)(-t),$$

where $\langle *, * \rangle_{Q_p}$ stands for the Hilbert symbol on $Q_p \times Q_p$ and

$$(F_S \varphi)(t) = \int_{X(Q_p)} \varphi(x)\psi_p(\lambda_S(t, x)) \, dx,$$

where $dx$ means the Haar measure on $X(Q_p)$ which is self-dual with respect to the Fourier transform $F_S$. Note that the index 8 of $|a|_{p}^{8}$ in the first formula comes from the fact that $\frac{1}{2} \dim_{Q_p} X(Q_p) = 8$ and we also use $\nu(h(a)) = a$. Furthermore, we always have $\chi_S(a) = 1$ since $\text{disc}(\lambda_S)$ is a square by Lemma A.1 in the Appendix.

The global Weil representation of $\omega_S$ of $J(A)$ acting on the Schwartz space $S(X(A))$ is given by the restricted tensor product of $\omega_{S,p}$. In our setting, $\omega_S$ is much simpler than that of the case $Sp_{2n}$ (compare with [Ike01, §1]): for $\varphi \in S(X(A))$,

$$\omega_S(h(a))\varphi(t) = |a|_A^{6}\varphi(ta), \quad \omega_S(n(b))\varphi(t) = \psi(\lambda_S(t, t)b)\varphi(t), \quad \omega_S(te_3)\varphi(t) = (F_S \varphi)(-t),$$

where $(F_S \varphi)(t) = \int_{X(A)} \varphi(x)\psi(\lambda_S(t, x)) \, dx$.

For each $\varphi \in S(X(A))$, the theta function $\Theta^{\psi_S}(v_1(x, y, z)h; \varphi)$ on $V(A)$ is given by

$$\Theta^{\psi_S}(v_1(x, y, z)h; \varphi) := \sum_{\xi \in X(Q)} \omega_S(v_1(x, y, z)h)\varphi(\xi)$$

$$= \sum_{\xi \in X(Q)} \omega_S(h)\varphi(\xi + x)\psi\left(\frac{1}{2}(S, z) + 2\lambda_S(\xi, y) + \lambda_S(x, y)\right).$$
Cusp forms on the exceptional group of type $E_7$

It is easy to see that this function is invariant under the action of $J(\mathbb{Q})$. By the equivalence of the Schrödinger model and the lattice model (cf. [Tak96]), for any $\varphi \in \mathcal{S}(X(\mathbb{A}_f))$ one has the Poisson summation formula which will be used later:

$$\Theta_{\psi_S}(v_1(x,y,z)(\xi_{-\text{co}h}); \varphi) = \Theta_{\psi_S}(v_1(x,y,z)h; \varphi).$$

(4.8)

To end this section, we discuss the relation between the adelic theta function and the classical theta function. For any $\varphi' \in \mathcal{S}(X(\mathbb{A}_f))$, we extend this function to an element $\varphi$ of $\mathcal{S}(X(\mathbb{A}_f))$ by

$$\varphi((x_p)_{p \in \mathbb{P}}) := \varphi_{\infty}(x_{\infty})\varphi'((x_p)_{p \in \mathbb{P}}), \quad \varphi_{\infty}(x_{\infty}) = e^{-2\pi\sigma_S(x_{\infty},x_{\infty})}.$$ 

Put $X := X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{\oplus 16}$ and let us extend the quadratic form $\sigma_S$ linearly to that on $X$. For each $\varphi \in \mathcal{S}(X(\mathbb{A}_f))$, the classical theta function on $\mathcal{D} := \mathbb{H} \times X$ is given by

$$\theta_{\varphi}(\tau, u) := \sum_{\xi \in X(\mathbb{Q})} \varphi(\xi)e(\sigma_S(\xi, \xi)\tau + 2\sigma_S(\xi, u)).$$

The group $J(\mathbb{R})$ acts on $\mathcal{D}$ by

$$\beta(\tau, u) := \left( \gamma, \frac{u}{ct+d} + x(\gamma \tau) + y \right),$$

where $\beta = v_1(x,y,z)h$ with $v_1(x,y,z) \in V(\mathbb{R})$ and $h = h(\gamma) \in H(\mathbb{R})$ corresponds to $(\frac{a}{b} \; \frac{c}{d}) \in \text{SL}_2(\mathbb{R})$. Here $\gamma \tau = (a\tau + b)/(ct+d)$ and put $j(\gamma, \tau) := ct+d$ for simplicity. For each positive even integer $k$, the automorphy factor on $J(\mathbb{R}) \times \mathcal{D}$ is defined by

$$j_{k,S}(\beta, (\tau, u)) := j(\gamma, \tau)^k e\left( -(S, z) + \frac{c}{j(\gamma, \tau)} \sigma_S(u, u) - \frac{2\sigma_S(x, u)}{j(\gamma, u)} - \sigma_S(x, x)(\gamma \tau) - \sigma_S(x, y) \right),$$

for $\beta = v_1(x,y,z)h$, $h = h(\gamma)$ as above. After lengthy and painful calculation, one can check the cocycle relation:

$$j_{k,S}(\beta', (\tau, u)) = j_{k,S}(\beta, \beta'(\tau, u))j_{k,S}(\beta', (\tau, u)).$$

For each function $f : \mathcal{D} \rightarrow \mathbb{C}$ and $\beta \in V(\mathbb{R})$, we define the ‘slash’ operator $f|_{k,S}[\beta] : \mathcal{D} \rightarrow \mathbb{C}$ by

$$f|_{k,S}[\beta](\tau, u) := j_{k,S}(\beta, (\tau, u))^{-1}f(\beta(\tau, u)).$$

Then the following lemma is easy to deduce from the definition.

**Lemma 4.1. **Keep the notation above. For each $\varphi \in \mathcal{S}(X(\mathbb{A}_f))$ and $h(\gamma) \in H(\mathbb{R})$, $\gamma \in \text{SL}_2(\mathbb{R})$:

1. $\Theta_{\psi_S}(v_1(x,y,z)h; \varphi) = \Theta_{\psi_S}[S, \beta](\sqrt{-1}, 0)$ for any $\beta \in J(\mathbb{R})$;
2. $\Theta_{\omega_S(\gamma^{-1})}[S, \beta](\tau, u) = j(\gamma, \tau)^{-8}\Theta_{\psi_S}(\gamma(\tau, u)).$

**Lemma 4.2. **Keep the notation above. Let $\xi$ be an element of $X(\mathbb{Q})$ so that $\sigma_S(\xi, x) \in \mathbb{Z}$ for all $x \in X(\mathfrak{a})$ and $\varphi_\xi$ be the characteristic function of $\xi + X(\mathfrak{a})$. Then

$$\Theta_{\psi_S}(v_1(x,y,z)h; \varphi_\xi) = \Theta_{\psi_S}(v_1(x_\infty, y_\infty, z_\infty)h; \varphi_\xi).$$

**Proof.** One can decompose any element $v_{1,f} \in J(\mathbb{A}_f)$ as $v_{1,f} = j_1 \cdot v'_1$ so that $j_1 \in J(\mathbb{Q})$ and $v'_1 = v_1(x', y', z') \in V(\mathbb{Z})$. Since the defining this theta function runs over $X(\mathbb{Q}) \cap (\xi + X(\mathfrak{a}))$, we see that $\psi_S^{\mathfrak{a}}((S, z') + 2\sigma_S(a, y') + \sigma_S(x', y')) = 1$ for any $a \in X(\mathbb{Q}) \cap (\xi + X(\mathfrak{a}))$. Then the claim follows from the left invariance of the theta function under $J(\mathbb{Q})$. 

For any $(\tau, u) \in \mathcal{D}$, there exist elements $v_1 \in V(\mathbb{R})$ and $g_\infty \in H(\mathbb{R})$ such that $v_1g_\infty(\sqrt{-1}, 0) = (\tau, u)$ since 1 and $\tau$ are independent over $\mathbb{R}$. From this with Lemma 4.1(1), we make a bridge between the adelic theta functions and the classical theta functions which will be focused in the next section.
5. Modular forms on the exceptional domain and Jacobi forms

We review the definition of modular forms on the exceptional domain \( \mathfrak{T} \) in [Bai70], and define Jacobi forms for our Jacobi group and study their basic properties.

5.1 Modular forms on the exceptional domain

Let \( \Gamma = \mathcal{G}(\mathbb{Z}) \) be the arithmetic subgroup of \( \mathcal{G}(\mathbb{Q}) \) as in [Bai70], defined by \( \mathcal{G}(\mathbb{Z}) = \{ g \in \mathcal{G}(\mathbb{R}) : gW_0 = W_0 \} \), where \( W_0 = \mathfrak{J}(\mathbb{Z}) \oplus \mathbb{Z}e \oplus \mathfrak{J}(\mathbb{Z}) \oplus \mathbb{Z}e' \), and \( e = (0,1,0,0) \) and \( e' = (0,0,0,1) \).

**Lemma 5.1.** The arithmetic group \( \Gamma \) is generated by \( \mathcal{N}(\mathbb{Z}) \) and \( \iota \).

**Proof.** By [Bai70, Theorem 5.2], \( \Gamma \) is generated by \( \mathcal{N}(\mathbb{Z}) \) and \( \mathcal{N}(\mathbb{Z}) \), where \( \mathcal{N} \) is the opposite unipotent subgroup of \( \mathcal{N} \). Since \( \mathcal{N} = \iota^{-1}\mathcal{N}_t \), the result follows.

**Lemma 5.2.** The arithmetic group \( \Gamma \) is generated by \( \mathcal{N}(\mathbb{Z}) \), \( \mathcal{M}(\mathbb{Z}) \) and \( \mathcal{H}(\mathbb{Z}) \), hence by \( \mathcal{N}(\mathbb{Z}) \), \( \mathcal{M}'(\mathbb{Z}) \) and \( \iota e_3 \).

**Proof.** This follows from the above lemma and from the identities, \( \iota = \iota e_1 \iota e_2 \iota e_3 \), and \( \iota e_2 = \varphi_{23} \iota e_3 \varphi_{23}^{-1} \), and \( \iota e_1 = \varphi_{13} \iota e_3 \varphi_{13}^{-1} \), where \( \varphi_{ij} = m_{e_i} m_{e_j} \) for \( i \neq j \).

In [Bai70, Kar74, Kim93], for \( Z \in \mathfrak{T} \) and \( g \in \mathcal{G}(\mathbb{R}) \), the action is defined by the right action:

\[
Z : g = Z_1, \quad p_Z g = p_A k p_Z, \quad \text{for } k \in \mathcal{M}(\mathbb{C}) \text{ and } Z_1 \in \mathcal{H}.
\]

However, following the usual convention, it is more convenient to define the left action by

\[
gZ = Z_1, \quad g p_Z = p_{Z_1} k p_A \quad \text{for } k \in \mathcal{M}(\mathbb{C}) \text{ and } Z_1 \in \mathfrak{T}.
\]

Let \( j(g,Z) = \nu(k)^{-1} \) be the canonical factor of automorphy. Then \( j(g,Z) \) has the following properties:

\[
j(p_B Z) = 1 \quad \text{for all } B \in \mathfrak{J}(\mathbb{R}), \quad j(\iota, Z) = \det(-Z), \quad j(g_1 g_2, Z) = j(1, Z) j(g_1, g_2, Z).
\]

If \( J(Z,g) \) is the functional determinant of \( g \) at \( Z \), then \( J(Z,g) = j(g,Z)^{-1} \). By Lemma 3.2, if \( k \in \mathcal{M}(\mathbb{R}) \), \( k Z \) is just the transformation \( \nu(k) \gamma(x) + k Y \sqrt{-1} \), where \( Z = X + Y \sqrt{-1} \), and for \( kX \) and \( kY \), \( k \) is considered as an element of \( \mathrm{GL}(3) \), and \( j(k,Z) = \nu(k)^{-1} \).

**Definition 5.3.** Let \( F \) be a holomorphic function on \( \mathfrak{T} \) which for some integer \( k > 0 \) satisfies

\[
F(\gamma Z) = F(Z) j(\gamma, Z)^k, \quad Z \in \mathfrak{T}, \gamma \in \Gamma.
\]

Then \( F \) is called a modular form on \( \mathfrak{T} \) of weight \( k \). We denote by \( \mathcal{M}_k(\Gamma) \) the space of such forms.

For a holomorphic function \( F: \mathfrak{T} \rightarrow \mathbb{C} \), the boundary map \( \Phi \) is defined by

\[
\Phi F(Z') = \lim_{\tau \rightarrow \sqrt{-1} \infty} F\left(\begin{array}{cc} Z' & \ast \\ \ast & \tau \end{array}\right),
\]

where \( Z' \in \mathfrak{T} \). We call \( \mathcal{S}_k(\Gamma) = \mathrm{Ker}(\Phi) \cap \mathcal{M}_k(\Gamma) \) the space of cusp forms of weight \( k \) with respect to \( \Gamma \). We should remark that there is only one equivalent class of cusps since \( \mathcal{G}(\mathbb{Q}) = \mathcal{P}(\mathbb{Q}) \mathcal{G}(\mathbb{Z}) \) [Bai70, Theorem 5.2].

Since \( F(Z + B) = F(Z) \) for \( B \in \mathfrak{J}(\mathbb{Z}) \) and \( \mathfrak{J}(\mathbb{Z}) \) is self-dual, \( F \) has a Fourier expansion of the form

\[
F(Z) = \sum_{T \in \mathfrak{J}(\mathbb{Z})_{>0}} a(T)e((T, Z)).
\]

By the Koecher principle, we do not need the holomorphy at the cusps.

If \( F \) is a cusp form, then \( a(T) = 0 \) for \( T \notin \mathfrak{J}(\mathbb{Z})_+ \).
5.2 Jacobi forms of matrix index

We define and study Jacobi forms of matrix index on $\mathcal{D} = \mathbb{H} \times X$ in the classical setting. Set

$$\Gamma_J := J(\mathbb{Q}) \cap G(\mathbb{Z}).$$

**Definition 5.4.** Let $k$ be a positive (even) integer and $S$ be an element of $\mathcal{J}_2(\mathbb{Z})_+$. We say a holomorphic function $\phi : \mathcal{D} \to \mathbb{C}$ is a Jacobi form (respectively Jacobi cusp form) of weight $k$ and index $S$ if $\phi$ satisfies the following conditions:

1. $\phi|_{k,S}[\beta] = \phi$ for any $\beta \in \Gamma_J$;
2. $\phi$ has a Fourier expansion of the form

$$\phi(\tau, u) = \sum_{\xi \in X(\mathbb{Q}), N \in \mathbb{Z}} c(N, \xi)e(N\tau + \sigma_S(\xi, u)),$$

where $c(N, \xi) = 0$ unless $S_{\xi, N} := \begin{pmatrix} S & S \xi \\ N \end{pmatrix}$ belongs to $\mathcal{J}(\mathbb{Z})_{\geq 0}$ (respectively $\mathcal{J}(\mathbb{Z})_+$).

We denote by $J_{k,S}(\Gamma_J)$ (respectively $J^\text{cusp}_{k,S}(\Gamma_J)$) the space of Jacobi forms (respectively Jacobi cusp forms) of weight $k$ and index $S$.

Define the dual of the lattice $\Lambda := X(\mathbb{Z}) = \sigma^2$ with respect to the quadratic form $\sigma_S$ by

$$\tilde{\Lambda}(S) = \{ x \in X(\mathbb{Q}) | \sigma_S(x, y) \in \mathbb{Z} \ \forall y \in \Lambda \}.$$

If $S \in \mathcal{J}_2(\mathbb{Z})_+$, then the quotient $\tilde{\Lambda}(S)/\Lambda$ is a finite group. Fix a complete representative $\Xi(S)$ of $\tilde{\Lambda}(S)/\Lambda$ and denote by $\varphi_\xi$ the characteristic function $\xi + \prod_{p < \infty} X(\alpha_p) \in S(X(\mathbb{A}_f))$. Any Jacobi form turns to be the sum of products of elliptic modular forms and theta functions by following lemma.

**Lemma 5.5.** Assume $S \in \mathcal{J}_2(\mathbb{Z})_+$. Let $\Xi(S)$ be a complete representative of $\tilde{\Lambda}(S)/\Lambda$. Then any $\phi \in J_{k,S}(\Gamma_J)$ has an expression of the form

$$\phi(\tau, u) = \sum_{\xi \in \Xi(S)} \phi_{S,\xi}(\tau)\varphi^S_{\xi}(\tau, u), \quad \phi_{S,\xi}(\tau) = \sum_{N \in \mathbb{Z}} c(N, \xi)e((N - \sigma_S(\xi, \xi))\tau).$$

Furthermore, for each $\xi \in \Xi(S)$, $\phi_{S,\xi}(\tau)$ is an elliptic modular form of weight $k - 8$.

**Proof.** See [Kri96, §2, example (iv)] and also the argument in [Ike01, p. 656].

Let $k$ be a positive even integer and $F$ be a modular form of weight $k$ on $\mathfrak{F}$. Then we have the Fourier–Jacobi expansion

$$F \left( \begin{pmatrix} W \\ t_u \end{pmatrix} \tau \right) = \sum_{S \in \mathcal{J}_2(\mathbb{Z})_{\geq 0}} F_S(\tau, u)e((S, W)), \quad W \in \mathfrak{F}_2, \tau \in \mathbb{H} \text{ and } u \in X(\mathbb{R}) \otimes \mathbb{R} \mathbb{C}.$$  \hfill (5.2)

**Lemma 5.6.** Keep the notation above. Assume $S \in \mathcal{J}(\mathbb{Z})_+$. Then $F_S(\tau, u) \in J_{k,S}(\Gamma_J)$.

**Proof.** It is easy to see that $F_S(\tau, u) = \sum_{T \in \mathcal{J}_2^+} a(T)e(c \tau)e((T, t^u))$, where $\mathcal{J}(\mathbb{Z})_+$ is the set of $T = (S, v) \in \mathcal{J}$. Then the claim follows from the argument at [Ike01, p. 656].
Remark 5.7. Consider any holomorphic function $F(Z), Z = \left( \frac{w}{u} \right), W \in \mathbb{H}, \tau \in \mathbb{H},$ and $u \in X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ on \( \mathbb{H} \) which is invariant under \( \Gamma \cap P(\mathbb{Q}) \). Then one has the Fourier and Fourier-Jacobi expansion

$$F(Z) = \sum_{T \in \mathbb{H}(\mathbb{Z})_{\geq 0}} A_F(T)e((T, Z)) = \sum_{S \in \mathbb{H}(\mathbb{Z})_{\geq 0}} F_S(\tau, u)e((S, W)),$$

as in (5.2). By the proof of Lemma 5.5,

$$F_S(\tau, u) = \sum_{\xi \in \Xi(S)} F_{S, \xi}(\tau)\theta^S_{\varphi, \xi}(\tau, u), \quad F_{S, \xi}(\tau) = \sum_{N \in \mathbb{Z}} A_F(S, \xi, N)e((N - \sigma_S(\xi, \xi))\tau),$$

where \( S, \xi = \left( \frac{S}{S, \xi} \right) \). In this paper, the function \( F_{S, \xi} \) will be called by \( (S, \xi) \)-component of \( F \).

We now discuss the relationship between the adelic setting and classical setting. (See [BJ77] for automorphic forms in the adelic setting.) Let \( \psi \) be a non-trivial additive character of \( \mathbb{Q} \backslash \mathbb{A} \) and, for \( S \in \mathbb{H}(\mathbb{Z})_{>0} \), put \( \psi_S = \psi \circ \text{tr}_S : Z \longrightarrow \mathbb{C}, z \mapsto \psi^1(S, z) \).

**Definition 5.8.** Let \( \tilde{F} \) be an automorphic function on \( G(\mathbb{A}) \). For each \( S \in \mathbb{H}(\mathbb{Z})_{>0} \), the \( S \)-th Fourier-Jacobi coefficient \( F_{\psi, S} \) of \( \tilde{F} \), with respect to \( \psi \), is a function on \( J(\mathbb{Q}) \backslash J(\mathbb{A}) \) given by

$$F_{\psi, S}(v_1 h) = \int_{Z(\mathbb{Q}) \backslash Z(\mathbb{A})} \tilde{F}(zv_1 h)\psi_S^{-1}(z) \, dz, \quad v \in \mathbb{V}(\mathbb{A}), h \in H(\mathbb{A}).$$

Let \( F \) be a modular form in \( \mathcal{M}_k(\Gamma) \), and consider its Fourier-Jacobi expansion

$$F = \sum_{S \in \mathbb{H}(\mathbb{Z})_{>0}} F_S(\tau, u)e((S, W))$$

as above (see (5.2)). We are using \( F_{\psi, S} \) for the Fourier-Jacobi coefficient of \( \tilde{F} \). We hope that this does not cause confusion with \( F_S \), which is the Fourier-Jacobi coefficient of \( F \). Let \( \tilde{F} \) denote the automorphic form on \( G(\mathbb{A}) \) corresponding to \( F \) by the strong approximation theorem, namely,

$$\tilde{F}(g) = j(g, E\sqrt{-1})^{-k}F(g, E\sqrt{-1}) \quad \text{for} \quad g = \gamma \cdot g_1 \cdot k', \quad \xi \in \Xi(G)G(\mathbb{R}) K.$$  

(5.3)

Similarly if we write any element of \( J(\mathbb{A}) \) as \( v_1 h = a \cdot v_{1, \infty} h_\infty \cdot k_j \in J(\mathbb{Q})J(\mathbb{R})K_J \) where \( K_J = K \cap J(\mathbb{A}) \), one has

$$F_{\psi, S}^2(v_1 h) = F_S|_{k, S}[v_{1, \infty} h_\infty](\sqrt{-1}, 0),$$

by Lemma 5.6. It follows from this that \( F_{\psi, S}^2(v_1 h) \) is left invariant under the action of the lattice \( \Lambda = X(\mathbb{O}) \). We also identify \( \Lambda \) with a lattice of \( Y(\mathbb{R}) \) in an obvious way.

Fix \( S \in \mathbb{H}(\mathbb{Z})_{>0} \). For each \( \xi \in \Xi(S) \), we put

$$J_S^{\psi, \xi}(h; F_{\psi, S}^2) := \int_{\mathbb{V}(\mathbb{Q}) \backslash \mathbb{V}(\mathbb{A})} F_{\psi, S}^2(v_1 h)\Theta_{\psi, S}^2(v_1 h; \varphi, \xi) \, dv_1.$$

Since

$$J_S^{\psi, \xi}(h; F_{\psi, S}^2) = \left( \psi_S^2 \right)^2(z)F_{\psi, S}^2(v_1 h), \quad \text{one has}
$$

$$J_S^{\psi, \xi}(h; F_{\psi, S}^2) = \int_{(X \otimes Y)(\mathbb{Q}) \backslash (X \otimes Y)(\mathbb{A})} F_{\psi, S}^2(v_1(x, y, 0) h)\Theta_{\psi, S}^2(v_1(x, y, 0) h; \varphi, \xi) \, dv_1(x, y, 0).$$

By Lemma 4.2,

$$\Theta_{\psi, S}^2(v_1(x, y, 0) h; \varphi, \xi) = \Theta_{\psi, S}^2(v_1(x, y, 0) h; \varphi, \xi).$$

Then one has

$$J_S^{\psi, \xi}(h; F_{\psi, S}^2) = \int_{\Lambda \backslash \Lambda(\mathbb{R})} F_S|_{k, S}[v_{1, \infty} h_\infty](\sqrt{-1}, 0)\Theta_{\psi, S}^2(v_1 h; \varphi, \xi) \, dv_1(x, y, 0).$$
where \( v_{1,\infty}(x_\infty, y_\infty) = v_1(x_\infty, y_\infty, 0) \). Take \( h_\infty = \left( \begin{smallmatrix} y_\infty \nolimits^{-1/2} & x_\infty \nolimits^{-1/2} \\ 0 & y_\infty \end{smallmatrix} \right) \in \mathbb{H}(\mathbb{R}) \) so that \( h_\infty \sqrt{-1} = x_\infty + \sqrt{-1} y_\infty \). Set \( \tau = h_\infty \sqrt{-1} \) and \( v_{1,\infty}(h_\infty \sqrt{-1}, 0) = (\tau, u) \). Put \( L_\tau := \{ \lambda_1 \tau + \lambda_2 \in X(\mathbb{R}) \otimes \mathbb{R} C \mid \lambda_i \in \mathbb{A}, i = 1, 2 \} \). Then by Lemma 4.1, one has

\[
J_\phi^S(h; F_{(\psi^3)^2}) = \int_{\Lambda \setminus X(\mathbb{R}) \otimes \mathbb{A} \setminus X(\mathbb{R})} F_{\phi|_k, k}[v_{1,\infty}(h_\infty)](\sqrt{-1} I, 0) \theta_{\phi/\xi}^S |_{\Lambda \setminus X(\mathbb{R}) \otimes \mathbb{R} C} (\sqrt{-1} I, 0) \, dv_{1,\infty}
\]

(put \( u = x_\infty + \tau y_\infty \))

\[
= \frac{1}{j(g_\infty, i)^{k-8}} \int_{L \setminus X(\mathbb{R}) \otimes \mathbb{C}} F_S(\tau, u) \theta_{\phi/\xi}^S(\tau, u) e^{-4\pi (Im(\tau))^{-1} \sigma_S(Im(u), Im(u))} \left. \frac{\partial (x_\infty, y_\infty)}{\partial u} \right| du
\]

\[
= \frac{2^{-8} y_\infty^{-8}}{j(g_\infty, i)^{k-8}} \int_{L \setminus X(\mathbb{R}) \otimes \mathbb{C}} \sum_{\xi' \in \Xi(S)} F_{S, \xi'}(\tau) \theta_{\phi/\xi'}^S(\tau, u) \theta_{\phi/\xi}^S(\tau, u) e^{-4\pi (Im(\tau))^{-1} \sigma_S(Im(u), Im(u))} \, du
\]

\[
= \frac{2^{-8} y_\infty^{-8}}{j(g_\infty, i)^{k-8}} F_{S, \xi}(\tau) 2^{-24} \det(S)^{-4} y_\infty^8
\]

\[
= 2^{-32} \det(S)^{-4} j(h_\infty, i)^{-8} F_{S, \xi}(\tau).
\]

Here we used the following formula to get the last equality: for each \( \xi \),

\[
\int_{L \setminus X(\mathbb{R}) \otimes \mathbb{C}} \theta_{\phi/\xi}^S(\tau, u) \theta_{\phi/\xi}^S(\tau, u) e^{-4\pi (Im(\tau))^{-1} \sigma_S(Im(u), Im(u))} \, du = \begin{cases} 2^{-24} \det(S)^{-4} y_\infty^8 & \text{if } \xi' = \xi, \\ 0 & \text{otherwise.} \end{cases}
\]

(Apply Lemma A.2 for \( n = 16 \) and combine this with \( \text{disc}(\sigma_S) = 2^4 \text{disc}(\lambda_S) = 2^4 \det(S)^8 \) by Lemma A.1.)

Summing up, we have proved the following.

**Lemma 5.9.** For \( \tau = h_\infty i, j(h_\infty, i)^{(k-8)} J_\phi^S(h_\infty; F_{(\psi^3)^2}) = C_S F_{S, \xi}(\tau) \), where \( C_S = 2^{-32} \det(S)^{-4} \).

In the next section, we will prove \( J_\phi^S(h; F_{(\psi^2)^2}) \) is a section of a degenerate principal series representation of \( \text{SL}_2(\mathbb{A}) \) if \( \hat{F} \) is an adelic Eisenstein series on \( G(\mathbb{A}) \). By Lemma 5.9 above, we will conclude that \( F_{S, \xi}(\tau) \) is an Eisenstein series in the classical sense.

### 6. Eisenstein series and their Fourier coefficients

Recall from [Kim93] an Eisenstein series. Let \( \Gamma_\infty = \Gamma \cap \mathbb{N}(\mathbb{Q}) \). For \( l \) a positive integer and \( s \in \mathbb{C} \),

\[
E_{2l, s}(Z) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} j(\gamma, Z)^{-2l} |j(\gamma, Z)|^{-s}.
\]

When \( s = 0 \) and \( 2l > 18 \), Karel [Kar74] computed the Fourier coefficients and showed that they have bounded denominators. Let

\[
E_{2l}(Z) = E_{2l, 0}(Z) = \sum_{T \in \mathfrak{J}(Z)_+} a_{2l}(T)e((T, Z)).
\]

**Theorem 6.1** [Kar74]. For \( T \in \mathfrak{J}(Z)_+ \), we have

\[
a_{2l}(T) = C_{2l} \det(T)^{2l-9} \prod_{p|\det(T)} f_p(T, p^{1-2l}).
\]
where \( C_{2l} = 2^{15} \prod_{n=0}^{2l} (2l - 4n)/(B_{2l-4n}) \), and \( f^p_\nu \) is a monic polynomial with rational integer coefficients of degree \( d = \text{ord}_p(\det(T)) \). It satisfies the functional equation

\[
X^d f^p_\nu(X^{-1}) = f^p_\nu(X).
\]

Here \( B_{2k} \) is the Bernoulli number; \( \zeta(2k) = (2^{2k-1} \pi^{2k} B_{2k})/(2k)! \). If \( n_{2l} \) is the numerator of \( \prod_{n=0}^{2l} B_{2l-4n} \), then \( n_{2l}E_{2l}(Z) \) has rational integer Fourier coefficients. The functional equation of \( f^p_\nu \) is implicit in [Kar74], and it is stated explicitly in [Kim93, p. 185].

**Corollary 6.2.** Keep the notation in the theorem above. Set \( \tilde{f}^p_\nu(X) := X^d f^p_\nu(X^{-2}) \), where \( d = \text{ord}_p(\det(T)) \). Then

\[
a_{2l}(T) = C_{2l} \det(T)^{(2l-9)/2} \prod_{p|\det(T) \atop p \neq \nu} \tilde{f}^p_\nu(p^{(2l-9)/2}),
\]

and \( \tilde{f}^p_\nu(X) = \tilde{f}^p_\nu(X^{-1}) \).

We can interpret this from the degenerate principal series as in the Siegel case [Kud08]. Let \( K_\infty \) be the stabilizer of \( E \sqrt{-1} \) in \( G(\mathbb{R}) \), where \( E = \text{diag}(1, 1, 1) \in \mathfrak{F}(\mathbb{Q}) \). It is a maximal compact subgroup of \( G(\mathbb{R}) \), and its complexification \( K_\infty \mathbb{C} \) is conjugate in \( G(\mathbb{C}) \) to \( M(\mathbb{C}) \) by the Cayley transform. Let \( K = K_\infty \prod_p K_p \), where \( K_p = G(\mathbb{Z}_p) \). By the strong approximation theorem, \( G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})K \).

For \( s \in \mathbb{C} \), let \( I(s) \) be the degenerate principal series representation of \( G(\mathbb{A}) \) consisting of any smooth, \( K \)-finite function \( f : G(\mathbb{A}) \rightarrow \mathbb{C} \) such that

\[
f(pg) = \delta_\mathbb{P}^{1/2}(p)|\nu(p)|_{\mathbb{A}}^s(g)
\]

for any \( p \in \mathbb{P}(\mathbb{A}) \) and any \( g \in G(\mathbb{A}) \) where \( \mathbb{P} = MN \) is the Siegel parabolic subgroup. Note that the modulus character \( \delta_\mathbb{P} \) is given by \( \delta_\mathbb{P}(mn) = |\nu(m)|_{\mathbb{A}}^s \). We denote it also by \( I(s) = \text{Ind}_{\mathbb{P}(\mathbb{Q}p)}^{G(\mathbb{Q}p)} |\nu(g)|^s \).

Let \( \Phi(g, s) = \Phi_\infty(g, s) \otimes \bigotimes_p \Phi_p(g, s) \) be a standard section in \( I(s) \). Then one can define the Siegel Eisenstein series

\[
E(g, s, \Phi) = \sum_{\gamma \in \mathbb{P}(\mathbb{Q}) \setminus G(\mathbb{Q})} \Phi(\gamma g, s).
\]

It satisfies the functional equation

\[
E(g, s, \Phi) = E(g, -s, M(s)\Phi), \quad M(s) : I(s) \rightarrow I(-s), \quad M(s)\Phi(g) = \int_{N(\mathbb{A})} \Phi(ng, s) \, dn.
\]

Now \( G(\mathbb{R}) = \mathbb{P}(\mathbb{R})K_\infty \), and hence \( \Phi_\infty \) is determined by its restriction to \( K_\infty \). We choose

\[
\Phi_\infty(k, s) = \nu(k)^{2l},
\]

where \( k \in M(\mathbb{C}) \) corresponds to \( k \in K_\infty \) by the Cayley transform. Hence \( \Phi(mnk, s) = |\nu(m)|_{\mathbb{A}}^{s+g} \nu(k)^{2l} \).

By [Bai70, p. 527], given \( Y \sqrt{-1} \in \mathfrak{F} \), there exists \( m \in M(\mathbb{R}) \) such that \( m(E \sqrt{-1}) = Y \sqrt{-1} \). Hence \( pX^m(E \sqrt{-1}) = X + Y \sqrt{-1} \). Let \( g = pxm \).

Now for \( \gamma \in \Gamma \), by Iwasawa decomposition, \( \gamma g = nm'k \) with \( n \in N(\mathbb{R}) \), \( m' \in M(\mathbb{R}) \), and \( k \in K_\infty \). Then

\[
\gamma g(E \sqrt{-1}) = \gamma Z = nm'(E \sqrt{-1}) = X_1 + Y_1 \sqrt{-1}.
\]
Hence \( m'(E\sqrt{-1}) = Y_1\sqrt{-1} \) and \( n = px_1 \). On the other hand,

\[
j(g, E\sqrt{-1}) = j(\gamma, Z)j(g, E\sqrt{-1}) = j(m', E\sqrt{-1})j(k, E\sqrt{-1}).
\]

Here \( j(g, E\sqrt{-1}) = j(m, E\sqrt{-1}) = \det(Y)^{-1}, j(m', E\sqrt{-1}) = \det(Y_1)^{-1} \). By [BB66, p. 500],

**Lemma 6.3.** For \( k \in K, j(k, E\sqrt{-1}) = \nu(k)^{-1}, \) and hence \( |j(k, E\sqrt{-1})| = 1 \).

So

\[
\det(Y_1) = \frac{\det(Y)}{|j(\gamma, Z)|}, j(k, E\sqrt{-1}) = \frac{j(\gamma, Z)}{|j(\gamma, Z)|}.
\]

Therefore,

\[
\Phi_\infty(\gamma, g, s) = \nu(m')^{-s-9}\nu(k)^{2l} = \det(Y)^{s+9}j(\gamma, Z)^{-2l}j(\gamma, Z)^{-s-9+2l}.
\]

Hence as in [Kud08], for \( \Phi(g, s) = \Phi_\infty(g, s) \otimes \bigotimes_p \Phi_p(g, s), \Phi_\infty(g, s) = \nu(k)^{2l}, \) and \( \Phi_p(g, s) = \Phi_p^0(g, s) \), the normalized spherical section for all \( p \),

\[
E(g, s, \Phi) = \det(Y)^{s+9}\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, Z)^{-2l}|j(\gamma, Z)|^{-s-9+2l}.
\]

Hence

\[
E(g, s, \Phi) = \det(Y)^{s+9}E_{2l, s+9-2l}(Z) = j(g, E\sqrt{-1})^{-(s+9)}E_{2l, s+9-2l}(Z).
\]

Summing up, we have proved the following.

**Proposition 6.4.** The adelic Eisenstein series \( E(g, 2l - 9, \Phi) \) on \( G(\mathbb{A}) \) which is associated to a standard section of \( I(2l - 9) \) corresponds to \( E_{2l, 0}(Z) \) via (5.3).

Let \( I(s) = \bigotimes I_p(s) \) and \( I_p(s) \) be the \( p \)-adic degenerate principal series. Then we have the following proposition.

**Proposition 6.5 [Wei03].** \( I_p(s) \) is irreducible except at \( s = \pm 1, \pm 5, \pm 9 \).

**Remark 6.6.** In terms of representation theory, the singular modular forms of weight 4 and 8 constructed in [Kim93] are subrepresentations of \( I(s) \) when \( s = -5, -1 \), respectively.

### 7. Fourier–Jacobi expansion of Eisenstein series on \( E_{7,3} \)

As seen in § 4.2 (see Lemma 5.5 and Lemma 5.6), for each \( S \in \mathcal{I}(\mathbb{Z})_+ \), the \( S \)-th Fourier–Jacobi coefficient of a modular form \( F \) on \( \mathbb{T} \) is represented by the sum of the products of theta series and elliptic modular forms. In this section we shall prove these elliptic modular forms turn to be Eisenstein series on \( \mathbb{H} \) if \( F \) is an Eisenstein series. To do this we generalize the argument in [Ike94, § 3] in our setting and by virtue of Lemma 5.9 this enable us to work on the adelic setting which is much simpler than the classical setting.

Let \( \omega \) be a unitary character of \( \mathbb{Q}^\times \backslash \mathbb{A}^\times \) and \( s \in \mathbb{C} \). Let \( \mathbb{K} = \text{SL}_2(\widehat{\mathbb{Z}}) \times \text{SO}(2) \) be the standard maximal compact subgroup of \( \text{SL}_2(\mathbb{A}) \). We denote by \( I(\omega, s) \), the degenerate principal series representation of \( G(\mathbb{A}) \) consisting of any function \( f : G(\mathbb{A}) \longrightarrow \mathbb{C} \) such that

\[
f(pg) = \delta_{P}^{1/2}(p)|\nu(p)|_{\mathbb{K}}^{s,\omega(\nu(p))}f(g)
\]

for any \( p \in P(\mathbb{A}) \) and any \( g \in G(\mathbb{A}) \). Recall that \( \delta_{P}^{1/2}(mn) = |\nu(m)|_{\mathbb{K}}^{s}. \) Similarly we also define the space \( I_1(\omega, s) \) consisting of any smooth, \( \mathbb{K} \)-finite function \( f : SL_2(\mathbb{A}) \longrightarrow \mathbb{C} \) such that

\[
f(pg) = \delta_{B}^{1/2}(p)|a_1|_{\mathbb{K}}^{s,\omega(\nu(p))}f(g)
\]
for any $p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in B(\mathbb{A})$ and any $g \in \text{SL}_2(\mathbb{A})$. Here $B$ is the Borel subgroup of $\text{SL}_2$ which consists of upper-triangular matrices and $\delta_B^{1/2}(p) = |a|_\mathbb{A}$ for $p = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in B(\mathbb{A})$. For any section $f \in I(\omega, s)$, we define the Eisenstein series on $G(\mathbb{A})$ of type $(\omega, s)$ by

$$E(g; f) := \sum_{p(\mathbb{Q}) \backslash G(\mathbb{Q})} f(gg), \quad g \in G(\mathbb{A}).$$

Let $\psi$ be a non-trivial additive character of $\mathbb{Q} \backslash \mathbb{A}$ and, for $S \in \mathfrak{S}_3(\mathbb{Z})_+$, put $\psi_S = \psi \circ \text{tr}_S : Z(\mathbb{A}) \to \mathbb{C}$. Consider the $S$th Fourier–Jacobi coefficient $E_S(v_1 h; f)$ of $E_S(g; f)$ with respect to $\psi_S$ (see Definition 5.8). For each $\varphi \in \mathcal{S}(X(\mathbb{A}))$, put

$$E_{\psi_S, \varphi}(h) := \int_{V(\mathbb{Q}) \backslash V(\mathbb{A})} E_S(v_1 h, f) \overline{\Theta^{\psi_S}(v_1 h; \varphi)} dv_1, \quad h \in \text{SL}_2(\mathbb{A}).$$

The main purpose in this section is to prove the following key theorem.

**Theorem 7.1.** Keep the notation above. Assume that $\varphi$ is $\mathbb{K}$-finite, and hence the $\mathbb{C}$-span $\langle \omega_S(k) \varphi \mid k \in \mathbb{K} \rangle_\mathbb{C}$ is finite-dimensional. For $\text{Re}(s) \geq 0$ we have the following:

1. $R(h; f, \varphi) := \int_{V(\mathbb{A})} f(hv_1 \cdot v_1 \cdot v_{e_3} \cdot h) \overline{\omega_S(v_1 (v_{e_3} \cdot h) \varphi(0))} dv_1$ is a section of $I(\omega, s)$;

2. $E_{\psi_S, \varphi}$ is an Eisenstein series on $\text{SL}_2(\mathbb{A})$ associated to $R(h; f, \varphi)$.

To prove this, we need some lemmas. Let $P = P(\mathbb{Q}), G = G(\mathbb{Q}), Q = Q(\mathbb{Q})$. Note that $Q$ is the normalizer of $V(\mathbb{Q})$ in $G$. The double coset $P \backslash G/Q$ is bijective to the double coset of the Weyl group $W_P \backslash W_G/W_Q$. By [Car72, p. 64], each double coset of $W_P \backslash W_G/W_Q$ has unique element of minimal length, and they are $\{1, c_3(23), c_2c_3(13)\}$, where $c_i$ is the Weyl group element attached to $2e_i$, and $(ij)$ is the Weyl group element attached to $e_i - e_j$. Then $G = P\xi_2Q \cup P\xi_1Q \cup P\xi_0Q$, and $P\xi_0Q$ is the unique open cell, where $\xi_2 = 1$, $\xi_1 = c_3(23)$, and $\xi_0 = c_2c_3(13)$. In terms of the notation in [Bai70, p. 517], $\xi_2 = 1$, $\xi_1 = t_{e_3} \varphi_{23}$, and $\xi_0 = t_{e_2} t_{e_3} \varphi_{13}$. Let $\varphi_{ij} = m_{e_{ij}} m_{-e_{ji}} m_{e_{ij}}$ for $i \neq j$.

**Lemma 7.2.** For any $q \in Q$, $q$ normalizes $Z(\mathbb{A})$, and if $\gamma \in G$ is not contained in the open cell $P\xi_0Q$, then $\psi_S$ is non-trivial on $\gamma^{-1}P(\mathbb{A}) \gamma \cap Z(\mathbb{A})$.

**Proof.** Let $q = lv$ for $l \in L(\mathbb{Q})$ and $v \in V(\mathbb{Q})$, and $p_z \in Z(\mathbb{A})$. Then, since $Z$ is the center of $V$, $qp_zq^{-1} = (lv)p_z(lv)^{-1} = lp_zl^{-1}$. If $l$ is in the central torus of $L$, then $lp_zl^{-1} = p_z$. Otherwise, $l \in L'(\mathbb{Q})$. Here $L' = H \times \text{Spin}(9, 1)$, where $\text{Spin}(9, 1)$ is spanned by the unipotent subgroups $m_{xe_{12}}$ and $m_{xe_{21}}$. If $l \in H(\mathbb{Q})$, then by Lemma 3.3, $lp_zl^{-1} = p_z$. Suppose $l = m_{xe_{12}}$. Then by Lemma 3.2, $m_{xe_{12}}p_z = p_B m_{xe_{12}}$ for $B = m_{xe_{12}}z = p_z \in Z(\mathbb{A})$. Similarly, $m_{xe_{21}}p_z = p_{z'} \in Z(\mathbb{A})$. Hence we have proved $qZ(\mathbb{A})q^{-1} = Z(\mathbb{A})$.

We may assume that $\gamma = \xi_1, \xi_2$. If $\gamma = \xi_2 = 1$, $P(\mathbb{A}) \cap Z(\mathbb{A}) = Z(\mathbb{A})$. So $\psi_S$ is not trivial on $P(\mathbb{A}) \cap Z(\mathbb{A})$. Let $\gamma = \xi_1$. Using (3.7), we can compute easily that $\gamma^{-1}m_{xe_{31}} \gamma = p_{xe_{12}} \in Z(\mathbb{A})$. Hence $\gamma^{-1}P(\mathbb{A}) \gamma \cap Z(\mathbb{A})$ contains the subgroup $\{p_{xe_{12}} \mid x \in C_\mathbb{A}\}$. So $\psi_S$ is not trivial on $\gamma^{-1}P(\mathbb{A}) \gamma \cap Z(\mathbb{A})$.

Let $P_H$ be the Borel subgroup of $H$ consisting of upper triangular matrices.

**Lemma 7.3.** The right coset can be written as $P \backslash P\xi_0Q = \xi_0 \cdot (Y(\mathbb{Q}) \backslash V(\mathbb{Q})) \cdot (P_H(\mathbb{Q}) \backslash H(\mathbb{Q}))$.

**Proof.** We can write $q \in Q$ as $slv$ with $s$ in the central torus, $l \in \text{Spin}(9, 1)(\mathbb{Q})$, $v \in V(\mathbb{Q})$, and $h \in H(\mathbb{Q})$. It is easy to show that $\xi_0l^{-1} \in M'(\mathbb{Q})$, and $\xi_0v(y)l^{-1} \in M'(\mathbb{Q})$, and
Cusp forms on the exceptional group of type $E_7$

$\xi_0 pae_3 \xi_0^{-1} = pae_3$. By direct computation, $\xi_0 m_{x_2} \xi_0^{-1} = m_{x_2} $, and $\xi_0 m_{x_{12}} \xi_0^{-1} = m_{x_{23}} $. Also, $\xi_0 p_{y_2} \xi_0^{-1} = m_{y_{23}} $, and $\xi_0 p_{y_3} \xi_0^{-1} = m_{y_{3}} $. Note that $h(a)$ is identified with $pae_3^{-1} \xi_0^{-1} pae_3 \xi_0^{-1}$. Hence $\xi_0 h(a) \xi_0^{-1} = pae_3 p_{a-1} pae_3^{-1} \xi_0^{-1} \in M'(Q)$, giving the claim. □

We have in analogy to [Ike94, p. 630] the following lemma.

**Lemma 7.4.** It is true that:

1. $v_1(0, y, z) \iota e_3 p_{b_{e_3}} = p_{b_{e_3}} k \cdot \iota v_1(0, y, z + by' y') \iota e_3$ where $k = m_{b_{01} e_3} m_{b_{02} e_3}$ with $\nu(k) = 1$;
2. $v_1(0, y, z) \iota e_3 h(a) = h(a) \cdot \iota v_1(0, ay, z) \iota e_3$ with $\nu(h(a)) = a$;
3. $\varphi_{13} \xi_0 \iota e_3 = \iota$ with $\nu(\varphi_{13}) = 1$ and $\iota e \iota = -1$.

**Proof.** Note that $v(0, y, z) = v_1(0, y, z)$. The proof of result (2) is straightforward by using result (1), and $\iota e_3 h(a) \iota e_3^{-1} = h(a)$ for $i = 1, 2$, and $\iota e_3 h(a) \iota e_3^{-1} = h(a^{-1})$. For result (1), use $\iota e_3 p_{b_{e_3}} \iota e_3 = p_{b_{e_3}}$ for $i = 1, 2$, and $\iota e_3 \cdot \iota e_3 = m_{-x_{31}}$ for $i = 1, 2$. Result (3) follows from the fact that $\varphi_{13} \in M'$, and $\iota e_3 = \varphi_{13}^{-1} \varphi_{13}$.

**Proof of Theorem 7.1.** We first prove Theorem 7.1(2), namely,

$$E_{\psi, \varphi}(h) = \sum_{\gamma \in \text{Par}(Q) \setminus H(Q)} R(\gamma h; f, \varphi).$$

This series will be convergent for $\text{Re}(s) \gg 0$ provided that the first assertion holds [Lan76]. In fact, one has

$$E_{\psi, \varphi}(h) = \int_{V(Q) \setminus V(A)} E_S(v_1 h, f) \Theta^{\psi_s}(v_1 h; \varphi) dv_1 = \int_{V(Q) \setminus V(A)} E(v_1 h, f) \Theta^{\psi_s}(v_1 h; \varphi) dv_1$$

$$= \sum_{i=1, 2} \sum_{\gamma \in \text{Par} \setminus P_{\xi_0} Q} \int_{V(Q) \setminus V(A)} f(\gamma v_1 h) \Theta^{\psi_s}(v_1 h; \varphi) dv_1$$

$$+ \sum_{\gamma \in \text{Par} \setminus P_{\xi_0} Q} \int_{V(Q) \setminus V(A)} f(\gamma v_1 h) \Theta^{\psi_s}(v_1 h; \varphi) dv_1.$$

In the first integral above, by Lemma 7.2, there exists an element $z_0 = \gamma^{-1} p_{\gamma} \in Z(A) \cap \gamma^{-1} P(A) \gamma$ such that $\psi_S(z_0) \neq 1$. Clearly $\nu(p) = 1$. Then one has

$$\int_{V(Q) \setminus V(A)} f(\gamma v_1 h) \Theta^{\psi_s}(v_1 h; \varphi) dv_1 = \int_{V(Q) \setminus V(A)} f(\gamma z_0 v_1 h) \Theta^{\psi_s}(z_0 v_1 h; \varphi) d(z_0 v_1)$$

$$= \psi_S(z_0) \int_{V(Q) \setminus V(A)} f(p_{\gamma} v_1 h) \Theta^{\psi_s}(v_1 h; \varphi) dv_1$$

$$= \psi_S(z_0) \int_{V(Q) \setminus V(A)} f(\gamma v_1 h) \Theta^{\psi_s}(v_1 h; \varphi) dv_1,$$
which claims the vanishing of \( \int_{V(Q)} f(\gamma v_1 h) \Theta^{\psi s}(v_1 h; \varphi) \, dv_1 \). By Lemma 7.3,

\[
E_{\psi_s, \varphi}(h) = \sum_{\gamma \in Y(Q)} \sum_{\gamma \in H(Q)} f(\xi_0 v_1 h) \Theta^{\psi s}(v_1 h; \varphi) \, dv_1
\]

(see the proof of Lemma 7.3), \( v_1(2x, 0, 0) v_1(-x, y, z) = v_1(x, y, z), \) and \( \nu(X(\bar{A})) = 1. \) Hence

\[
E_{\psi_s, \varphi}(h) = \sum_{\gamma \in H(Q)} f(\xi_0 v_1 h) \Theta^{\psi s}(v_1 h; \varphi(0)) \, dv_1
\]

By Lemma 7.4(3),

\[
E_{\psi_s, \varphi}(h) = \sum_{\gamma \in H(Q)} f(v_1 \gamma h) \Theta^{\psi s}(v_1 \gamma h; \varphi(0)) \, dv_1 = \sum_{\gamma \in H(Q)} R(\gamma h; f, \varphi).
\]
and \( \nu(v_1(0,-\bar{x},0)) = 1 \), one has

\[
R(h; f, \varphi) = \int_{Y(\mathbb{A})} \int_{Z(\mathbb{A})} f(t \cdot v_1(0, y, z) t_{\mathfrak{e}_3}, h)(\omega_S(t_{\mathfrak{e}_3}, h) \varphi(x) \psi S(\frac{1}{2}(S, \lambda_S(x,y))) dy dz
\]

We now compute the actions of \( p_{\mathfrak{e}_3}, b \in \mathbb{A} \) and \( h(a), a \in \mathbb{A}^\times \) respectively. By Lemma 7.4, one has

\[
R(p_{\mathfrak{e}_3}; f, \varphi) = \int_{Y(\mathbb{A})} \int_{Z(\mathbb{A})} f(t \cdot v_1(0, y, z + by^t \bar{y}) t_{\mathfrak{e}_3}, h)(\omega_S(p_{\mathfrak{e}_3}, h) \varphi(y) \psi S(\frac{1}{2}(S, z + by^t \bar{y})) dy dz
\]

By Lemma 7.4 again, one has

\[
R(h(a); f, \varphi) = \int_{Y(\mathbb{A})} \int_{Z(\mathbb{A})} f(h(a) \cdot t \cdot v_1(0, ay, z) t_{\mathfrak{e}_3}, h)(\omega_S(h(a), h) \varphi(ay) \psi S(\frac{1}{2}(S, z)) dy dz
\]

The smoothness and \( \mathbb{K} \)-finiteness follow from those of \( f \) and \( \varphi \). Hence \( R(h; f, \varphi) \in I_1(\omega, s) \).  

8. **Compatible family of Eisenstein series**

**Definition 8.1.** Let \( k \) be a positive integer. Let \( h(\tau) \) be an elliptic modular form of weight \( k \) with respect to \( \text{SL}_2(\mathbb{Z}) \). We denote by \( \mathcal{V}(h) \), the \( \mathbb{C} \)-vector space spanned by \( \{ h_k[\gamma], \gamma \in \text{GL}_2(\mathbb{Q})^+ \} \) where \( h_k[\gamma](\tau) := j(\gamma, \tau)^{-k} h(\gamma \tau) \).

Let \( \Phi(X) = \Phi(\{ X_p \}_p) = \bigotimes'_p \Phi_p(X_p) \in \bigotimes'_p \mathbb{C}[X_p, X_p^{-1}] \) where \( p \) runs over all prime numbers. Denote by \( \mathcal{R} \) the set of all \( \Phi(X) = \Phi(\{ X_p \}_p) \) such that \( \Phi_p(X_p) = \Phi_p(X_p^{-1}) \) for any prime \( p \). For each non-zero sequence of complex numbers \( \{ a_p \}_p \) indexed by all primes, the value of \( \Phi(X) \) at \( \{ X_p \}_p = \{ a_p \}_p \) is denoted by \( \Phi(\{ a_p \}) \). For each positive even integer \( k \geq 4 \), let

\[
E^1_k(\tau) = \sum_{(c,d) \in \mathbb{Z}^2 \setminus (0,0)} (c\tau + d)^{-k},
\]

which is the Eisenstein series of weight \( k \) with respect to \( \text{SL}_2(\mathbb{Z}) \).
**Definition 8.2.** For a sufficiently large $k_0$, a compatible family of Eisenstein series is a family of elliptic modular forms, for even integer $k' \geq k_0$

$$g_{k'}(\tau) = b_{k'}(0) + \sum_{N \in \mathbb{Q}_{>0}} N^{(k'-1)/2}b_{k'}(N)q^N, \quad q = e(\tau),$$

satisfying the following three conditions.

1. For all $k' \geq k_0$, $g_{k'} \in \mathcal{V}(E_{k'}^1)$.
2. For each $N \in \mathbb{Q}_+^*$, there exists $\Phi_N \in \mathcal{R}$ such that $b_{k'}(N) = \Phi_N(\{p^{(k'-1)/2}\}_p)$.
3. There exists a congruence subgroup $\Gamma \subset \text{SL}_2(\mathbb{Z})$ such that $g_{k'} \in M_{k'}(\Gamma)$ for all $k' \geq k_0$. Here $M_{k'}(\Gamma)$ stands for the space of elliptic modular forms of weight $k$ with respect to $\Gamma$.

Then by [Ike08, Lemma 10.2], we have the following lemma.

**Lemma 8.3.** Let $f(\tau) = \sum_{n=1}^{\infty} c(n)q^n$ be a Hecke eigenform of weight $k$ with respect to $\text{SL}_2(\mathbb{Z})$ with $c(p) = p^{(k-1)/2}(\alpha_p + \alpha_p^{-1})$. Assume that there is a finite-dimensional representation $(\mathcal{U}, \mathbb{C}^d)$ of $\text{SL}_2(\mathbb{Z})$ and

$$\Phi_N := t(\Phi_{1,N}, \ldots, \Phi_{d,N}) \in \mathcal{R}^d, \quad N \in \mathbb{Q}_{>0}$$

satisfying the following two conditions.

1. There exists a vector valued modular form $\tilde{g}_{k'} = t(g_{1,k'}, \ldots, g_{d,k'})$ which has

$$\tilde{g}_{k'}(\tau) = \tilde{b}_{k'}(0) + \sum_{N \in \mathbb{Q}_{>0}} N^{(k'-1)/2}\tilde{b}_{k'}(N)q^N, \quad (\tilde{b}_{k'}(N) = t(b_{1,k'}(N), \ldots, b_{d,k'}(N)), N \in \mathbb{Q}_{>0})$$

of weight $k'$ with type $\mathcal{U}$ for each sufficiently large even integers $k'$, and hence this means that

$$\tilde{g}_{k'}(\tau)|_{k'}[\gamma] := t(g_{1,k'}|_{k'}[\gamma], \ldots, g_{d,k'}|_{k'}[\gamma]) = u(\gamma)\tilde{g}_{k'}(\tau) \quad \text{for any } \gamma \in \text{SL}_2(\mathbb{Z}).$$

2. Each component $g_{i,k'}$, $(1 \leq i \leq d)$ of $\tilde{g}_{k'}(\tau)$ is a compatible family of Eisenstein series such that

$$b_{i,k'}(N) = \Phi_{i,N}(\{p^{(k'-1)/2}\}_p).$$

Then $	ilde{h}(\tau) := \sum_{N \in \mathbb{Q}_{>0}} N^{(k-1)/2}\tilde{\Phi}_N(\{\alpha_p\}_p)q^N$ is a vector valued modular form of weight $k$ with type $\mathcal{U}$, and hence it satisfies

$$\tilde{h}(\tau)|_k[\gamma] = u(\gamma)\tilde{h} \quad \text{for any } \gamma \in \text{SL}_2(\mathbb{Z}).$$

**9. Construction of cusp forms on the exceptional domain**

In this section we shall prove our main theorem. The strategy is the same as in [Ike01, Ike08, Yam10].

For any positive integer $k \geq 10$, let $f(\tau) = \sum_{n=1}^{\infty} c(n)q^n$ be a Hecke eigenform of weight $2k - 8$ with respect to $\text{SL}_2(\mathbb{Z})$ with $c(p) = p^{(2k-9)/2}(\alpha_p + \alpha_p^{-1})$. Let us formally define a function on $\mathcal{E}$ by

$$F(Z) = \sum_{T \in \mathcal{E}(\mathbb{Z})} A_F(T)e((T, Z)), \quad A_F(T) = \det(T)^{(2k-9)/2} \prod_p \tilde{f}_T^p(\alpha_p), \quad Z \in \mathcal{E}.$$ 

Then we prove the following theorem.

**Theorem 9.1.** $F(Z)$ is a non-zero cusp form of weight $2k$ with respect to $\Gamma$. 

Remark 9.2. If $f$ has integer Fourier coefficients, then $F$ also has integer Fourier coefficients. Just observe from Corollary 6.2 that $\tilde{f}_{T}^{p}(X) = X^{d} + X^{-d} + a_{1}(X^{d-2} + X^{-(d-2)}) + \cdots + a_{d}(X^{d-2} + X^{-(d-2)})$ if $d$ is odd, and $\tilde{f}_{T}^{p}(X) = X^{d} + X^{-d} + a_{1}(X^{d-2} + X^{-(d-2)}) + \cdots + a_{d/2}$ if $d$ is even, where $d = \text{ord}_{p}(\det(T))$ and $a_{i}$’s are integers.

First of all we shall prove the convergence of $F(Z)$.

Lemma 9.3. The series $F(Z)$ is absolutely and uniformly convergent on any compact domain of $\mathfrak{K}$.

Proof. It is well known that $|\alpha_{p}| = 1$. By definition, $\tilde{f}_{T}^{p}(X) = X^{d}f_{T}^{p}(X^{-2})$, and $f_{T}^{p}$ is a monic polynomial with integer coefficients of degree $d = \text{ord}_{p}(\det(T))$, i.e., $f_{T}^{p}(X) = X^{d} + a_{1}X^{d-1} + \cdots + a_{d-1}X + a_{d}$. Let $M = \max\{|a_{1}|, \ldots, |a_{d}|\}$. We use the identity from [Kar74, p. 187],

$$ (1 - p^{-s})^{-1}S_{p}(T) = \sum_{m=0}^{\infty} \alpha_{m}(T)p^{-ms}, \quad \alpha_{m}(T) = \sum_{X} \omega_{m}(T,X), $$

where $X \in \Lambda(3)_{p}/p^{m}\Lambda(3)_{p}$ and $\tau_{i}(X) \equiv 0 \mod p^{m(i-1)}$ for $2 \le i \le 3$, and $2m \le d$. Hence $|\alpha_{m}(T)| \le p^{27m}$. We also have [Kar74, p. 197]

$$ S_{p}(T) = (1 - p^{-s})(1 - p^{4-s})(1 - p^{8-s})f_{T}^{p}(p^{9-s}). $$

Hence

$$ f_{T}^{p}(X) = (1 - p^{-5}X)^{-1}(1 - p^{-1}X)^{-1} \sum_{m=0}^{\infty} \alpha_{m}(T)p^{-9m}X^{m}. $$

So $M \le (d + 1)^{2}p^{18m}$. By the trivial estimate, $d + 1 \le p^{d}$, hence we have $M \le p^{2d}p^{9d} = p^{11d}$. Therefore, $|f_{T}^{p}(\alpha_{p})| \le (d + 1)M \le p^{12d}$. Hence

$$ |A_{F}(T)| \le \det(T)^{k+12-1/2}. $$

Now we use the fact from [Bai70, p. 538], for $l > 8$,

$$ \int_{R_{3}^{+}(\mathbb{R})} \det(X)^{-9}e^{2\pi(X,Y)}dX = \pi^{12}(2\pi i)^{-3d} \prod_{n=0}^{2} \Gamma(l - 4n)\det(Y)^{-l}, $$

where $dX$ is the ordinary Euclidean measure. Hence

$$ \left| \sum_{T \in \mathfrak{H}(\mathbb{Z})_{+}} A_{F}(T)e^{2\pi i(T,Z)} \right| \le \sum_{T \in \mathfrak{H}(\mathbb{Z})_{+}} \det(T)^{k+12-1/2}e^{2\pi i(T,Y)} \le \int_{R_{3}^{+}(\mathbb{R})} \det(X)^{k+12-1/2}e^{2\pi(X,Y)}dX $$

converges. \hfill \Box

Clearly $F(Z + N) = F(Z)$ for $N \in \mathbb{N}(\mathbb{Z})$. Also $F(\gamma Z) = F(Z)$ for $\gamma \in \mathfrak{M}(\mathbb{Z})$. Thanks to Lemma 5.2, it is enough to prove

$$ F(t_{e_{3}}Z) = j(t_{e_{3}}, Z)^{2k}F(Z). \quad (9.1) $$

We prove (9.1) by using results of previous sections. Fix $S \in \mathfrak{H}(\mathbb{Z})_{+}$. Since $F(Z)$ is invariant under $T \cap \mathbb{P}(\mathbb{Q})$ as mentioned above and is holomorphic by Lemma 9.3, then by Remark 5.7, one has the Fourier–Jacobi expansion:

$$ F_{S}(\tau, u) = \sum_{S \in \mathfrak{H}(\mathbb{Z})_{+}} F_{S}(\tau, u)e((S, W)), \quad (9.2) $$

$$ F_{S}(\tau, u) = \sum_{\xi \in \Xi(S)} F_{S, \xi}(\tau, u)\theta_{\varphi_{\xi}}^{S}(\tau, u), \quad (9.3) $$

Page 23 of 32
Then by Lemma 5.9, Corollary 6.2, and Theorem 7.1,

\[ F_{S,\xi}(\tau) = \sum_{N \in \mathbb{Z}, N - \sigma_S(\xi, \xi) \geq 0} A_F(S, N) e((N - \sigma_S(\xi, \xi)) \tau), S_{\xi, N} := \begin{pmatrix} S & S \xi \\ \tau \xi S & N \end{pmatrix} \]

\[ = \sum_{N \in \mathbb{Z}, N - \sigma_S(\xi, \xi) \geq 0} \det(S, N)^{(2k-9)/2} \prod_p \tilde{f}_p^9(\alpha_p) e((N - \sigma_S(\xi, \xi)) \tau) \]

\[ = \det(S)^{(2k-9)/2} \sum_{N \in \mathbb{Z}, N - \sigma_S(\xi, \xi) \geq 0} (N - \sigma_S(\xi, \xi))^{(2k-9)/2} \prod_p \tilde{f}_p^9(\alpha_p) e((N - \sigma_S(\xi, \xi)) \tau). \quad (9.4) \]

For the last equality above, we used the formula \( \det(S, N) = \det(S)(N - \sigma_S(\xi, \xi)) \) by using (2.3). The condition (9.1) is equivalent to claiming that \( F_S(\tau, u) \in J_{k, S}(\Gamma, J) \) for any \( S \in \mathfrak{I}_2(\mathbb{Z})_+ \). Therefore for each fixed \( S \in \mathfrak{I}_2(\mathbb{Z})_+ \), we have only to check the condition

\[ F_S|_{k, S}(w_1) = F_S(\tau, u) \text{ for } w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (9.5) \]

By [Tak96, p. 124, (2.1)], for each \( \gamma \in \text{SL}_2(\mathbb{Z}) \), there exists a unitary matrix \( u_S(\gamma) = (u_S(\gamma)_{\xi, \eta})_{\xi, \eta \in \Xi(S)} \) such that

\[ \theta^S_{\varphi_{\xi}}|_{k, S}[\gamma](\tau, u) = \sum_{\eta \in \Xi(S)} u_S(\gamma)_{\xi, \eta} \theta^S_{\varphi_{\eta}}(\tau, u). \quad (9.6) \]

Further there exists a positive integer \( \Delta_S \) depending on \( S \) such that \( u_S \) is trivial on \( \Gamma(\Delta_S) \subset \text{SL}_2(\mathbb{Z}) \). Since the elements of \( \{ \theta^S_{\varphi_{\xi}} | \xi \in \Xi(S) \} \) are linearly independent over \( \mathbb{C} \), to check (9.5) with (9.6) it suffices to prove that \( \{ F_{S, \xi} \}_{\xi \in \Xi(S)} \) is a vector valued modular form of weight \( 2k \) with type \( u_S \).

For a sufficiently large positive integer \( k' \), we now turn to consider \((S, \xi)\)-component \((\mathcal{E}_{2k', 0})_{S, \xi}\) of the classical Eisenstein series

\[ \mathcal{E}_{2k', 0}(Z) := \frac{1}{C_{2k'}} E_{2k', 0}(Z) = \sum_{T \in \mathfrak{I}_2(\mathbb{Z})_+} \bar{a}_{2k'}(T) e((T, Z)), \]

\[ \bar{a}_{2k'}(T) = \det(T)^{(2k' - 9)/2} \prod_{p | \det(T)} \tilde{f}_p^9(p^{(2k' - 9)/2}), \]

on \( \bar{\mathfrak{T}} \), where \( C_{2k'} \) is the constant in Theorem 6.1. Then one has

\[ \det(S)^{-\left(2k' - 9\right)/2} (\mathcal{E}_{2k', 0})_{S, \xi}(\tau) \]

\[ = \det(S)^{-\left(2k' - 9\right)/2} \sum_{N \in \mathbb{Z}, N - \sigma_S(\xi, \xi) \geq 0} \bar{a}_{2k'} \left( \begin{pmatrix} S & S \xi \\ \tau \xi S & N \end{pmatrix} \right) e\left( (N - \sigma_S(\xi, \xi)) \tau \right) \]

\[ = \sum_{N \in \mathbb{Z}, N - \sigma_S(\xi, \xi) \geq 0} (N - \sigma_S(\xi, \xi))^{-\left(2k' - 9\right)/2} \prod_{p | \det(S_{\xi, N})} \tilde{f}_p^9(p^{(2k' - 9)/2}) e\left( (N - \sigma_S(\xi, \xi)) \tau \right). \]

Then by Lemma 5.9, Corollary 6.2, and Theorem 7.1, \( \{ \det(S)^{-\left(2k' - 9\right)/2} (\mathcal{E}_{2k', 0})_{S, \xi} \}_{k' \geq 0} \) makes up a family of Eisenstein series. Here we use \( u_S|_{\Gamma(\Delta_S)} = 1 \) to check the third condition of Definition 8.2. Applying Lemma 8.3 with (9.4), one can conclude that

\[ F_{S, \xi} = \det(S)^{(2k - 9)/2} \sum_{n \in \mathbb{Z}, n \geq 0} n^{(2k - 9)/2} \prod_{p | \det(S_{\xi, N})} \tilde{f}_p^9(n^{(2k - 9)/2}) \]

\[ q^n, \]

is a vector-valued modular form of weight \( 2k \) with type \( u_S \). Since \( A_F(1) = 1 \), \( F(Z) \) is not identically zero. This completes the proof of Theorem 9.1.
10. Hecke operators

Karel [Kar72] defined Hecke operators and showed that the Eisenstein series are eigenfunctions. We review his results, and show that the cusp form on $\mathcal{F}$ constructed in the previous section is a Hecke eigenform.

Let $I_W$ be the identity operator on $W$. Let $Z$ be the central torus of $\text{GL}(W)$, i.e., for any field $K$,

$$Z_K = \{ \lambda I_W \mid \lambda \in K, \lambda \neq 0 \}.$$

Let $G = Z \cdot G$. Then $G$ is a $\mathbb{Q}$-group. Define a rational character $\mu$ on $G$ by

$$\{ gw_1, gw_2 \} = \mu(g)\{ w_1, w_2 \} \quad \text{for all} \quad w_1, w_2 \in W.$$

Then $\mu$ is defined over $\mathbb{Q}$, and $Q(gw) = \mu(g)^2Q(w).$ Let $S$ be the connected component of the Lie group $G(\mathbb{R})$ containing the identity element element of $G(\mathbb{R})$. Define

$$\Psi = \{ g \in S \mid gW_\circ \subset W_\circ \}.$$

Since $S$ is a connected component containing the identity element, $\mu(g) > 0$ for all $g \in \Psi$. Recall that $e = (0, 1, 0, 0)$ and $e' = (0, 0, 0, 1)$ are elements of $W_\circ$, and $\{ w_1, w_2 \} \in Z$ for all $w_1, w_2 \in W_\circ$. Hence $\mu(g) = \{ ge, ge' \} \in Z$. Hence we can define, for each $m \in \mathbb{Z}, m > 0$,

$$\Psi_m = \{ g \in \Psi \mid \mu(g) = m \},$$

and $\Psi = \bigcup_{m=1}^\infty \Psi_m$.

Fix $k$. If $\rho = z\rho' \in G(\mathbb{R}) = Z(\mathbb{R})_+ \cdot G(\mathbb{R})$ and $F$ is a function on $\mathcal{F}$, let $F(Z)[[\rho]]_k = F(\rho'Z)j(\rho', Z)^{-k}$. If $F$ is holomorphic, then $F[[\rho]]_k = F$ for all $\rho \in \Gamma$ precisely when $F$ is a modular form of weight $k$. Let $F$ be a modular form on $\mathcal{F}$ of weight $k$, and define

$$T(m) \cdot F = \sum_{\rho \in \Gamma \setminus \Psi_m \Gamma} \rho \cdot F.$$

Actually, in [Kar72] Karel used $J(g, Z) = j(g, Z)^{-18}$ as an automorphy factor. However, his result works for $j(g, Z)$ in the exactly same way.

For later purpose in connection with representation theory, we shall modify Hecke operators for $G(\mathbb{Q})$. For any element $H \in G(\mathbb{Q})$, we define a modified action of $H$ on $F$ by

$$H \ast F = v_H(\Gamma)^{-k/36} \sum_{\rho \in \Gamma \setminus \Gamma H \Gamma} \rho \cdot F, \quad v_H(\Gamma) := [H \Gamma H^{-1} : \Gamma],$$

where $[H \Gamma H^{-1} : \Gamma] = [H \Gamma H^{-1} : \Gamma \cap H \Gamma H^{-1}] / [\Gamma : \Gamma \cap H \Gamma H^{-1}]$.

Then we have the following proposition.

**Proposition 10.1.** The Eisenstein series $E_{l,0}(Z)$ is a Hecke eigenform for each Hecke operator $T(m)$. In particular, it is also an eigenform for any $H \in G(\mathbb{Q})$ with respect to the modified action $\ast$. 

For any positive integer $k \geq 10$, let $f(\tau) = \sum_{n=1}^\infty c(n)q^n$ be a Hecke eigenform of weight $2k - 8$ with respect to $\text{SL}_2(\mathbb{Z})$ with $c(p) = p^{(2k-8)/2}(\alpha_p + \alpha_p^{-1})$. Let $F(Z) = \sum_{T \in \mathcal{O}(2)} A_F(T)e((T, Z)), Z \in \mathcal{F}$ be the modular form on $\mathcal{F}$ which is constructed from $f$ in previous section. Then by
imitating Ikeda’s idea in [Ike94, §11] we will prove that $F(Z)$ is a Hecke eigenform for any $H \in G(\mathbb{Q})$. Recall the normalized Eisenstein series

$$\mathcal{E}_{2k',0}(Z) = \sum_{T \in \mathfrak{I}(\mathbb{Z})_+} \tilde{a}_{2k'}(T) e((T, Z)), \quad \tilde{a}_{2k'}(T) = \det(T)^{(2k'-9)/2} \prod_{p \mid \det(T)} \tilde{f}^p_T(p^{(2k'-9)/2})$$

of weight $2k'$ which is also an eigenform for any $H \in G(\mathbb{Q})$ by Proposition 10.1. For each $H \in G(\mathbb{Q})$, by using $G(\mathbb{Q}) = P(\mathbb{Q})G(\mathbb{Z})$ (see [Bai70, p. 532, line –4]), one can choose \{$p_{n_i} \cdot m_i\}_{i=1}^r$, $n_i \in \mathfrak{I}(\mathbb{Q}), m_i \in M(\mathbb{Q})_+$ as the complete representatives of $\Gamma \backslash \mathfrak{H}T$. Here $M(\mathbb{Q})_+$ is the subset of $M(\mathcal{Q})$ consisting of $g$ with $\nu(g) > 0$. Then it is easy to see that $\nu_H(\Gamma)^{-1/36} = \nu(m_i)^{1/2}$ for each $i$. Henceforth we settle the convention that for each $T \in \mathfrak{I}(\mathbb{Z})_+$ and each $m \in M(\mathbb{Q})$, put $\tilde{f}_{mT}(\{X_p\}_p) = 0$ if $mT \not\in \mathfrak{I}(\mathbb{Z})_+$. Then one has

$$H \ast \mathcal{E}_{2k',0}(Z) = v_H(\Gamma)^{-2k'/36} \sum_{i=1}^r (p_{n_i} \cdot m_i) \cdot \mathcal{E}_{2k',0}(Z)$$

$$= \sum_{T \in \mathfrak{I}(\mathbb{Z})_+} \sum_{i=1}^r \nu(m_i)^{9/2} \det(T)^{(2k'-9)/2} \prod_{p \mid \det(T)} \tilde{f}^p_T(p^{(2k'-9)/2}) e((T, m_iZ + n_i))$$

(use $(T, m_iZ) = ((m_i)^{-1}T, Z)$ by (3.1) and $\det(m_iT) = \nu(m_i)^{-1} \det(T)$.)

$$= \sum_{T \in \mathfrak{I}(\mathbb{Z})_+} \sum_{i=1}^r \nu(m_i)^{9/2} \det((m_i^*T, n_i)) \det(T)^{(2k'-9)/2} \prod_{p \mid \det(m_i^*T)} \tilde{f}^p_{m_i^*T}(p^{(2k'-9)/2}) e((T, Z)).$$

From this, the $T$th Fourier coefficient of $H \ast \mathcal{E}_{2k',0}$ is

$$\sum_{i=1}^r \nu(m_i)^{9/2} e((m_i^*E, n_i)) \prod_{p \mid \det(m_i^*E)} \tilde{f}^p_{m_i^*E}(X_p), X = \{X_p\}_p$$

which defines an element of $\bigotimes'_p \mathbb{C} [X_p, X_p^{-1}]$. Here $E = \text{diag}(1, 1, 1) \in \mathfrak{I}(\mathbb{Q})$. Noting $\tilde{a}_{2k'}(E) = 1 \neq 0$, by Proposition 10.1 one has

$$\alpha_H(X) := \sum_{i=1}^r \nu(m_i)^{9/2} e((m_i^*E, n_i)) \prod_{p \mid \det(m_i^*E)} \tilde{f}^p_{m_i^*E}(X_p), X = \{X_p\}_p$$

By [Ike01, Lemma 10.1], one has the equality in $\bigotimes'_p \mathbb{C} [X_p, X_p^{-1}]$

$$\alpha_H(X) \prod_{p \mid \det(T)} \tilde{f}^p_T(X_p) = \sum_{i=1}^r \nu(m_i)^{9/2} e((m_i^*T, n_i)) \prod_{p \mid \det(m_i^*T)} \tilde{f}^p_{m_i^*T}(X_p), X = \{X_p\}_p.$$`
11. The degree-56 standard L-function

In this section we will compute the standard L-function of Hecke eigenforms constructed in the previous section and the Eisenstein series respectively. Let \( F = F(Z) \) be the cusp form in Theorem 10.2 and \( \tilde{F} \) be the automorphic form on \( G(\mathbb{A}) \) attached to \( F \) (see (5.3)). Let \( \pi_F \) be the cuspidal representation of \( G(\mathbb{A}) \) attached to \( \tilde{F} \). Since \( F \) is a Hecke eigenform, one has the decomposition \( \pi_F = \pi_\infty \otimes \bigotimes_p \pi_p \). Then \( \pi_\infty \) is a holomorphic discrete series of the lowest weight \( 2k \) associated to \( \mathbb{A} \) in the notation of [Bou02]. We note that \(-2k\pi_7\) parametrizes a holomorphic discrete series when \( 2k > 17 \) (cf. [Kna86, p. 158]). Since \( \pi_\infty \) is unramified for each prime \( p \), it has a spherical vector whose Hecke eigenvalue for each element of \( G(\mathbb{Q}_p) \) coincides with that of a spherical vector in \( \text{Ind}_{G(\mathbb{Q}_p)} \nu(g) |^{2s_p} \) where \( p^{s_p} = \alpha_p \). (This is clear from the proof of Theorem 10.2. Notice \( 2s_p \), not \( s_p \). We can see it from Corollary 6.2 and Proposition 6.5. We are replacing \((2k - 9)/2 \) by \( s_p \) in Corollary 6.2.) Then by [Cas, Proposition 2.2.2] and Proposition 6.5, \[
\pi_p \simeq \text{Ind}_{G(\mathbb{Q}_p)} \nu(g) |^{2s_p},
\]
for any finite place \( p \).

In order to compute the standard L-function of \( \pi_F \), we use the Langlands–Shahidi method. Since \( G(\mathbb{Q}_p) \) is the split group of type \( E_7 \), we can compute its local L-factor. We follow the notation of [Kim05, § 2.7.8]. We consider the split exceptional group of type \( E_8 \), and its parabolic subgroup \( R \) whose Levi subgroup is \( GE_7 \), and its Borel subgroup \( B \). By inducing in stages \[
\text{Ind}_{E_8(\mathbb{Q}_p)}^{\text{R}(\mathbb{Q}_p)} \pi_p \otimes \exp(s\alpha, H_R(\mathbb{R})) = \text{Ind}_{B(\mathbb{Q}_p)}^{E_8(\mathbb{Q}_p)} \exp(\chi, H_B(\mathbb{R})),
\]
where \( \alpha = e_1 - e_9 \), and \( \chi = s(e_1 - e_9) + s_p(-e_1 + 2e_2 - e_9) + (8e_3 + 7e_4 + 6e_5 + 5e_6 + 4e_7 + 3e_8) \). Here \( \rho_{E_8} = 8e_3 + 7e_4 + 6e_5 + 5e_6 + 4e_7 + 3e_8 \) is the half-sum of positive roots of \( E_6 \). Then one can see that the unipotent radical of \( R \) is generated by 57 roots
\[
e_i - e_9 \quad \text{for } i = 1, \ldots, 8 \quad \text{and} \quad e_i - e_j \quad \text{for } j = 2, \ldots, 8,
\]
\[
e_1 + e_3 + e_5 + e_k \quad \text{for } 2 \leq j < k \leq 8 \quad \text{and} \quad -(e_i + e_3 + e_9) \quad \text{for } 2 \leq i < j \leq 8.
\]
Then \( e_1 - e_9 \) gives rise to \( 1 - p^{-2s} \), and the remaining 56 roots give rise to the following local factors:
\[
e_1 - e_3, 1 - \alpha_p p^{8-s}; e_1 + e_2 + e_8, 1 - \alpha_p^{-1} p^{8-s};
\]
\[
e_3 - e_9, 1 - \alpha_p^{-1} p^{8-s}; -(e_2 + e_8 + e_9), 1 - \alpha_p p^{8-s};
\]
\[
e_1 - e_4, 1 - \alpha_p p^{7-s}; e_1 + e_2 + e_7, 1 - \alpha_p^{-1} p^{7-s};
\]
\[
e_4 - e_9, 1 - \alpha_p^{-1} p^{7-s}; -(e_2 + e_7 + e_9), 1 - \alpha_p p^{-7-s};
\]
\[
e_1 - e_5, 1 - \alpha_p p^{6-s}; e_1 + e_2 + e_6, 1 - \alpha_p^{-1} p^{6-s};
\]
\[
e_5 - e_9, 1 - \alpha_p^{-1} p^{6-s}; -(e_2 + e_6 + e_9), 1 - \alpha_p p^{-6-s};
\]
\[
e_1 - e_6, 1 - \alpha_p p^{5-s}; e_1 + e_2 + e_5, 1 - \alpha_p^{-1} p^{5-s};
\]
\[
e_6 - e_9, 1 - \alpha_p^{-1} p^{5-s}; -(e_2 + e_5 + e_9), 1 - \alpha_p p^{-5-s};
\]
\[
e_1 - e_7, e_1 + e_2 + e_3, 1 - \alpha_p p^{4-s}; e_1 + e_2 + e_4, -(e_3 + e_4 + e_9), 1 - \alpha_p^{-1} p^{4-s};
\]
\[
e_1 - e_8, e_1 + e_6 + e_8, 1 - \alpha_p p^{3-s}; e_1 + e_2 + e_3, -(e_3 + e_5 + e_9), 1 - \alpha_p^{-1} p^{3-s};
\]
\[
e_1 + e_6 + e_7, e_1 + e_5 + e_8, 1 - \alpha_p p^{2-s}; -(e_3 + e_6 + e_9), -(e_4 + e_5 + e_9), 1 - \alpha_p^{-1} p^{2-s};
\]
\[
e_1 + e_5 + e_7, e_1 + e_4 + e_8, 1 - \alpha_p p^{1-s}; -(e_3 + e_7 + e_9), -(e_4 + e_6 + e_9), 1 - \alpha_p^{-1} p^{1-s}
\]
\[ e_1 + e_3 + e_8, e_1 + e_4 + e_7, 1 - \alpha p^{-s}; -(e_3 + e_8 + e_9), -(e_4 + e_7 + e_9), 1 - \alpha_p^{-1} p^{-s} \]
\[ e_1 + e_3 + e_7, e_1 + e_4 + e_6, 1 - \alpha p^{-1-s}; -(e_4 + e_7 + e_9), -(e_5 + e_7 + e_9), 1 - \alpha_p^{-1} p^{-1-s} \]
\[ e_1 + e_3 + e_6, e_1 + e_4 + e_5, 1 - \alpha p^{-2-s}; -(e_5 + e_8 + e_9), -(e_6 + e_7 + e_9), 1 - \alpha_p^{-1} p^{-2-s} \]
\[ e_1 + e_3 + e_5, -(e_2 + e_3 + e_9), 1 - \alpha p^{-3-s}; e_8 - e_9, -(e_6 + e_8 + e_9), 1 - \alpha_p^{-1} p^{-3-s} \]
\[ e_1 + e_3 + e_4, -(e_2 + e_4 + e_9), 1 - \alpha p^{-4-s}; e_7 - e_9, -(e_7 + e_8 + e_9), 1 - \alpha_p^{-1} p^{-4-s} \]
\[ e_1 - e_2, 1 - \alpha_3 p^{-s}; e_2 - e_9, 1 - \alpha_3^{-1} p^{-s}; e_1 + e_5 + e_6, 1 - \alpha p^{-s}; -(e_5 + e_6 + e_9), 1 - \alpha_3^{-1} p^{-s}. \]

Hence we have the degree-56 local L-function:
\[
(1 - \alpha p^{-s})^2 (1 - \alpha p^{-1} p^{-s}) \prod_{i=0}^{3} (1 - \alpha_p^{-3} p^{-s}) \]
\[
\cdot \prod_{i=5}^{8} (1 - \alpha p^{-i-s}) (1 - \alpha p^{-1} p^{-i-s}) \prod_{i=1}^{4} (1 - \alpha p^{-i-s}) (1 - \alpha_p^{-1} p^{-i-s})^2.
\]

Therefore, we have proved the following theorem.

**Theorem 11.1.** The degree-56 standard L-function \( L(s, \pi_F, St) \) of \( \pi_F \) is given by

\[
L(s, \pi_F, St) = L(s, \text{Sym}^3 \pi_f) L(s, \pi_f)^2 \prod_{i=1}^{4} L(s \pm i, \pi_f)^2 \prod_{i=5}^{8} L(s \pm i, \pi_f),
\]

where \( L(s, \text{Sym}^3 \pi_f) \) is the third symmetric power L-function.

Let \( \Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s) \). Then the local L-factor at infinity is given by

\[
L(s, \nu, St) = \Gamma_C(s + \frac{3(2k-9)}{2}) \Gamma_C(s + \frac{2k-9}{2}) \Gamma_C(s + \frac{2k-9}{2}) \sum_{i=1}^{4} \Gamma_C(s + \frac{2k-9}{2} \pm i)^2 \prod_{i=5}^{8} \Gamma_C(s + \frac{2k-9}{2} \pm i),
\]

and the completed L-function satisfies the functional equation

\[
\Lambda(s, \pi_F, St) = L(s, \nu, St) L(s, \pi_F, St) = -\Lambda(1-s, \pi_F, St).
\]

Note that the root number is \(-1\), since the root number of \( L(s, \text{Sym}^3 \pi_f) \) is \(-1\) \([CM04]\).

We have also proved the following theorem.

**Theorem 11.2.** The standard L-function \( L(s, E_{2l,0}(Z), St) \) of \( E_{2l,0}(Z) \) is

\[
L(s, E_{2l,0}(Z), St) = \zeta(s + l - \frac{9}{2}) \zeta(s + l + \frac{9}{2}) \zeta(s - l + \frac{3}{2} + \frac{27}{2}) \zeta(s - l - \frac{3}{2} + \frac{27}{2}) \]
\[
\cdot \prod_{i=5}^{8} \zeta(s \pm i - l + \frac{9}{2}) \zeta(s \pm i + l + \frac{9}{2}) \prod_{i=1}^{4} \zeta(s \pm i - l + \frac{9}{2}) \zeta(s \pm i + l + \frac{9}{2})^2.
\]

**Remark 11.3.** We write the degree-56 standard L-function of \( \pi_F \) as

\[
L(s, \pi_F, St) = L(s, \text{Sym}^3 \pi_f) \prod_{i=-4}^{4} L(s + i, \pi_f) \prod_{i=-8}^{8} L(s + i, \pi_f).
\]

This suggests the following parametrization of \( \pi_F \). Let \( \mathcal{L} \) be the (hypothetical) Langlands group over \( \mathbb{Q} \), and let \( \rho_F : \mathcal{L} \rightarrow \text{SL}_2(\mathbb{C}) \) be the two-dimensional irreducible representation of \( \mathcal{L} \).
Cusp forms on the exceptional group of type $E_7$

corresponding to $\pi_f$. Let $\text{Sym}^n$ be the irreducible $(n + 1)$-dimensional representation of $\text{SL}_2(\mathbb{C})$. Note that if $n = 2m - 1$, $\text{Im}(\text{Sym}^n) \subset \text{Sp}_{2m}(\mathbb{C})$, and if $n = 2m$, $\text{Im}(\text{Sym}^n) \subset \text{SO}_{2m+1}(\mathbb{C})$. We have the tensor product maps $\text{SL}_2(\mathbb{C}) \times \text{Sp}_{2m}(\mathbb{C}) \rightarrow \text{Sp}_{4m}(\mathbb{C})$ and $\text{SL}_2(\mathbb{C}) \times \text{SO}_{2m+1}(\mathbb{C}) \rightarrow \text{Sp}_{4m+2}(\mathbb{C})$. Hence $\rho_f \otimes \text{Sym}^m : \mathcal{L} \times \text{SL}_2(\mathbb{C}) \rightarrow \text{Sp}_{34}(\mathbb{C})$, and $\rho_f \otimes \text{Sym}^8 : \mathcal{L} \times \text{SL}_2(\mathbb{C}) \rightarrow \text{Sp}_{18}(\mathbb{C})$. Let $\text{Sym}^3 \rho_f : \mathcal{L} \times \text{SL}_2(\mathbb{C}) \rightarrow \text{Sp}(\mathbb{C})$ be the parameter of $\text{Sym}^3 \pi_f$, where it is trivial on $\text{SL}_2(\mathbb{C})$. Consider the parameter

$$\rho = \text{Sym}^3 \rho_f \oplus (\rho_f \otimes \text{Sym}^m) \oplus (\rho_f \otimes \text{Sym}^8) : \mathcal{L} \times \text{SL}_2(\mathbb{C}) \rightarrow \text{Sp}_4(\mathbb{C}) \times \text{Sp}_{34}(\mathbb{C}) \times \text{Sp}_{18}(\mathbb{C}) \subset \text{Sp}_{56}(\mathbb{C}).$$

Note that $E_7(\mathbb{C}) \subset \text{Sp}_{56}(\mathbb{C})$. We expect that $\rho$ will factor through $E_7(\mathbb{C})$, and give rise to a parameter $\rho : \mathcal{L} \times \text{SL}_2(\mathbb{C}) \rightarrow E_7(\mathbb{C})$, which parametrizes $\pi_f$.

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Appendix

In this appendix we will compute the discriminant of some quadratic forms from §4.2 and prove the orthogonal relation of theta functions in the proof of Lemma 5.9.

Let $S = \begin{pmatrix} a & u \\ \bar{u} & b \end{pmatrix} \in \mathfrak{m}_2(K)$ where $K$ is a field whose characteristic is different from 2,3. Recall $\det(S) = ab - N(u)$ and the quadratic form $\lambda_S(x, y) = \frac{1}{2}(S, x^t\bar{y} + y^t\bar{x})$ on $X(K)$ (see (4.1)).

**Lemma A.1.** Let $\text{disc}(\lambda_S)$ be the discriminant of the quadratic form $\lambda_S$, i.e., the determinant of the representation matrix of $\lambda_S$. Then $\text{disc}(\lambda_S) = \det(S)^8$.

**Proof.** Let $S = \begin{pmatrix} a & u \\ \bar{u} & b \end{pmatrix}$, where $a, b \in K$ and $u \in \mathcal{C}_K$. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ where $x_1, x_2, y_1, y_2 \in \mathcal{C}_K$. Let $\lambda_S(x, y) = \frac{1}{2}(S, x^t\bar{y} + y^t\bar{x}) = \frac{1}{2}(a(x_1\bar{y}_1 + y_1\bar{x}_1) + b(x_2\bar{y}_2 + y_2\bar{x}_2) + u(x_2\bar{y}_1 + y_2\bar{x}_1))$ be the bilinear form given by $S$.

For $x \in \mathcal{C}_K$, let $x = x_0 e_0 + \cdots + x_7 e_7$. Then with respect to the basis, the matrix of $\lambda_S$ is

$$\begin{pmatrix} aI_8 & X \\ t^t X & bI_8 \end{pmatrix} \in M_{16}(K), \quad X = (\text{Tr}(e_i((-e_j)\bar{u})))_{1 \leq i, j \leq 8}.$$

Then the discriminant of the bilinear form is the determinant of the above matrix, which is given by $\text{disc}(\lambda_S) = \det(abI_8 - t^tXX)$. Now we claim that $t^tXX$ is a diagonal matrix. Clearly, for each $j$, we have

$$\sum_{k=0}^{7} (\text{Tr}(e_k(-e_j)\bar{u}))^2 = N(u).$$

Let $i \neq j$, and consider

$$\sum_{k=0}^{7} (\text{Tr}(e_k(-e_i))\bar{u})(\text{Tr}(e_k(-e_j))\bar{u}). \quad (A.1)$$

For a given $e_k$, let $e_k(-e_i) = e_l$ and $e_k(-e_j) = e_l$. Then we claim that there exists $e_a$ such that $e_a(-e_i) = e_l$ and $e_a(-e_j) = -e_l$. This implies that $(A.1) = 0$. Now, from $e_l e_i = e_l e_j$, we have

$$e_l = (-e_j e_l)(-e_i) = (e_j e_l)e_i = -e_j(e_l e_i),$$

by non-associativity. So $-e_l e_j = e_j e_l = e_l e_i$. Let $e_a = e_l(-e_j) = e_l e_i$. Therefore, $\text{disc}(\lambda_S) = \det(S)^8$. \qed
In order to prove the orthogonal relation in the proof of Lemma 5.9, by the above lemma, we need to consider the following. Let $n$ be a positive integer and $T$ be a positive definite symmetric matrix of size $n$. Assume $T = (t_{ij})_{1 \leq i \leq j \leq n}$ is even integral, i.e., $t_{ij} \in \mathbb{Z}$ for $i = 1, \ldots, n$ and $t_{ij} \in \frac{1}{2}\mathbb{Z}$ for $1 \leq i < j \leq n$. For $\lambda \in \mathbb{Q}^n$, we define the theta function on $\mathbb{H} \times \mathbb{C}^n$ by

$$
\theta_{[\lambda]}(T; \tau, z) = \sum_{x \in \mathbb{Z}^n} \mathbf{e}(t(x + \lambda)T(x + \lambda)\tau + 2t(x + \lambda)Tz), \quad (\tau, z) \in \mathbb{H} \times \mathbb{C}^n, \quad \mathbf{e}(*) = e^{2\pi i *},$

where $[\lambda]$ stands for the image of $\lambda$ under the natural projection $\mathbb{Q}^n \to \mathbb{Q}^n/\mathbb{Z}^n$ and the definition of the above theta function depends only on $[\lambda]$. Let $\Lambda_T$ be a complete representative of $(2T)^{-1}\mathbb{Z}^n/\mathbb{Z}^n$.

**Lemma A.2.** For any $\lambda, \mu \in \Lambda_T$, the following orthogonal relation holds:

$$
\int_{(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z})^n} \theta_{[\lambda]}(T; \tau, z)\overline{\theta_{[\mu]}(T; \tau, z)} e^{-4\pi i (\text{Im } \tau)^{-1}T[\text{Im } z]} \, dz = \begin{cases} 2^{-n} \det(T)^{-1/2} (\text{Im } \tau)^{n/2} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}
$$

**Proof.** Put $z = a + \tau b, a, b \in \mathbb{R}^n$. Then we have

$$
\int_{(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z})^n} \theta_{[\lambda]}(T; \tau, z)\overline{\theta_{[\mu]}(T; \tau, z)} e^{-4\pi i (\text{Im } \tau)^{-1}T[\text{Im } z]} \, dz
$$

$$
= (\text{Im } \tau)^n \int_{(\mathbb{R}/\mathbb{Z})^n} \sum_{x,y \in \mathbb{Z}^n} \left\{ \int_{(\mathbb{R}/\mathbb{Z})^n} \mathbf{e}(2t(x + \lambda)Ta - 2t(y + \mu)Ta) \, da \right\}
$$

$$
\cdot e(2\sqrt{-1}(t(x + \lambda)Tb + t(y + \mu)Tb))
$$

$$
\cdot e(t(x + \lambda)T(x + \lambda)\tau - t(y + \mu)T(y + \mu)\tau) e^{-4\pi i (\text{Im } \tau)^{-1}T[\text{Im } z]} \, db,
$$

where $T[\text{Im } z] = t(\text{Im } z)T(\text{Im } z)$. Note that, for given $x, y \in \mathbb{Z}^n$, $2t(x - y + \lambda - \mu)T \in \mathbb{Z}^n$ if and only if $\lambda = \mu$ by the definition. Therefore

$$
\int_{(\mathbb{R}/\mathbb{Z})^n} \mathbf{e}(2t(x + \lambda)Ta - 2t(y + \mu)Ta) \, da = \begin{cases} 1 & \text{if } x = y \text{ and } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}
$$

If $\lambda = \mu$, we have

$$
\int_{(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z})^n} \theta_{[\lambda]}(T; \tau, z)\overline{\theta_{[\lambda]}(T; \tau, z)} e^{-4\pi i (\text{Im } \tau)^{-1}T[\text{Im } z]} \, dz
$$

$$
= (\text{Im } \tau)^n \int_{(\mathbb{R}/\mathbb{Z})^n} \sum_{x \in \mathbb{Z}^n} e^{-4\pi i (\text{Im } \tau)t(b+x+\lambda)T(b+x+\lambda)} \, db = (\text{Im } \tau)^n \int_{\mathbb{R}^n} e^{-4\pi i (\text{Im } \tau)bTb} \, db.
$$

Since $T$ is diagonalizable by an orthogonal matrix over $\mathbb{R}$, we may assume that $T = \text{diag}(t_1, \ldots, t_n), t_i \in \mathbb{R}_{>0}$. Hence

$$
\int_{\mathbb{R}^n} e^{-4\pi i (\text{Im } \tau)bTb} \, db = \prod_{i=1}^n \int_{\mathbb{R}} e^{-4\pi i (\text{Im } \tau)t_i b^2} \, db = \prod_{i=1}^n \frac{1}{\sqrt{4(\text{Im } \tau)t_i}} = 2^{-n} \det(T)^{-1/2} (\text{Im } \tau)^{-n/2}.
$$

Hence we have the claim. \qed
Cusp forms on the exceptional group of type $E_7$

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Cusp forms on the exceptional group of type $E_7$


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