Compositio Math. (Page 1 of 32)
doi:10.1112/S0010437X15007538

Cusp forms on the exceptional group of type E_7

Henry H. Kim and Takuya Yamauchi

Abstract

Let $\mathbf{G}$ be the connected reductive group of type $E_{7,3}$ over $\mathbb{Q}$ and $\mathfrak{T}$ be the corresponding symmetric domain in $\mathbb{C}^{27}$. Let $\Gamma = \mathbf{G}(\mathbb{Z})$ be the arithmetic subgroup defined by Baily. In this paper, for any positive integer $k \geqslant 10$, we will construct a (non-zero) holomorphic cusp form on $\mathfrak{T}$ of weight $2k$ with respect to Γ from a Hecke cusp form in $S_{2k-8}(\mathrm{SL}_2(\mathbb{Z}))$. We follow Ikeda's idea of using Siegel's Eisenstein series, their Fourier–Jacobi expansions, and the compatible family of Eisenstein series.

Contents

1. Introduction

Let $\mathbf{G}$ be the exceptional Lie group of type $E_{7,3}$ over $\mathbb{Q}$ and $\mathfrak{T} \subset \mathbb{C}^{27}$ the corresponding bounded symmetric domain. The purpose of this paper is to construct holomorphic cusp forms on $\mathfrak{T}$ from cusp forms for SL_2 over $\mathbb{Q}$. In [Ike01], Ikeda originally gave a (functorial) construction of a Siegel cusp form for Sp_{2n} (rank $2n$) from a normalized Hecke eigenform on the upper half-plane $\mathbb{H}$ with respect to $\mathrm{SL}_2(\mathbb{Z})$ which has been conjectured by Duke and Imamoglu. (Independently Ibukiyama formulated a conjecture in terms of Koecher–Maass series.) He made use of the uniform property of the Fourier coefficients of Siegel Eisenstein series for Sp_{2n} over $\mathbb{Q}$ and together with various deep facts established in [Ike01] to prove the Duke–Imamoglu conjecture. After this work, his construction was generalized to unitary groups $U(n,n)$

Received 19 May 2014, accepted in final form 27 March 2015.
2010 Mathematics Subject Classification 11F55 (primary), 11F70, 22E55, 20G41 (secondary).
Keywords: exceptional group of type E_7, Ikeda lift, Eisenstein series, Langlands functoriality.
The first author is partially supported by NSERC. The second author is partially supported by JSPS Grant-in-Aid for Scientific Research No. 23740027 and JSPS Postdoctoral Fellowships for Research Abroad No. 378.

[Ike08], quaternion unitary groups $\mathrm{Sp}(n,n)$ [Yam10], and symplectic groups Sp_{2n} over totally real fields [Ike, IH13]. Historically, in the case of Sp_2, the resulting cusp form is called Saito–Kurokawa lift, which has been studied thoroughly [Kur78, Pia83, CP88]. Our method follows his construction. The main obstruction is the hugeness of $E_{7,3}$. In the aforementioned works, the theory of Jacobi forms has been understood well, since the Heisenberg group inside the group in consideration is easy to handle. On the other hand, much less is known in the case of $E_{7,3}$. Therefore we have to consider a suitable Heisenberg subgroup in $E_{7,3}$ which has not been studied. To do this we analyze it in terms of roots.

We now explain our main theorem. We refer to the next section for the several notations which appear below. Let $\Gamma = \mathbf{G}(\mathbb{Z})$ be the arithmetic subgroup defined by Baily in [Bai70] which is constructed by using the integral Cayley numbers $\mathfrak{o}$. For a positive integer $k \geqslant 10$, let E_{2k} be the Siegel Eisenstein series on $\mathfrak{T}$ of weight $2k$ with respect to Γ. Then it has the Fourier expansion of form

$$E_{2k}(Z) = \sum_{T \in \mathfrak{J}(\mathbb{Z})_+} a_{2k}(T) \exp(2\pi\sqrt{-1}(T,Z)), \quad Z \in \mathfrak{T},$$
$$a_{2k}(T) = C_{2k} \det(T)^{(2k-9)/2} \prod_{p \mid \det(T)} \widetilde{f}_T^p(p^{(2k-9)/2}),$$

where $C_{2k} = 2^{15} \prod_{n=0}^{2} (2k-4n)/(B_{2k-4n})$, and $\widetilde{f}_T^p(X)$ is a Laurent polynomial over $\mathbb{Q}$ in X which is depending only on T and p.

Let $S_{2k-8}(\mathrm{SL}_2(\mathbb{Z}))$ be the space of elliptic cusp forms of weight $2k-8 \geqslant 12$ with respect to $\mathrm{SL}_2(\mathbb{Z})$. For each normalized Hecke eigenform $f = \sum_{n=1}^{\infty} c(n)q^n, q = \exp(2\pi\sqrt{-1}\tau), \tau \in \mathbb{H}$ in $S_{2k-8}(\mathrm{SL}_2(\mathbb{Z}))$ and each rational prime p, we define the Satake p-parameter α_p by $c(p) = p^{(2k-9)/2}(\alpha_p + \alpha_p^{-1})$. For such f, consider the following formal series on $\mathfrak{T}$:

$$F(Z) = \sum_{T \in \mathfrak{J}(\mathbb{Z})_+} A(T) \exp(2\pi\sqrt{-1}(T,Z)), \quad Z \in \mathfrak{T}, \quad A(T) = \det(T)^{(2k-9)/2} \prod_{p \mid \det(T)} \widetilde{f}_T^p(\alpha_p).$$

Then we will show the following theorem.

THEOREM 1.1. *The function $F(Z)$ is a non-zero Hecke eigen cusp form on $\mathfrak{T}$ of weight $2k$ with respect to Γ.*

If f has integer Fourier coefficients, then F also has integer Fourier coefficients (Remark 9.2). By virtue of Theorem 1.1, $F = F(Z)$ gives rise to a cuspidal automorphic representation $\pi_F = \pi_\infty \otimes \bigotimes_p' \pi_p$ of $\mathbf{G}(\mathbb{A})$. Then π_∞ is a holomorphic discrete series of the lowest weight $2k$ associated to $-2k\varpi_7$ in the notation of [Bou02] (cf. [Kna86, p. 158]). For each prime p, π_p is unramified. In fact, π_p turns out to be a degenerate principal series $\mathrm{Ind}_{\mathbf{P}(\mathbb{Q}_p)}^{\mathbf{G}(\mathbb{Q}_p)} |\nu(g)|^{2s_p}$, where $p^{s_p} = \alpha_p$. Then for each local component π_p, one can associate the local L-factor $L(s, \pi_p, St)$ of the standard L-function of π_F by using the Langlands–Shahidi method. Put $L(s, \pi_F, St) = \prod_p L(s, \pi_p, St)$ and let $L(s, \pi_f) = \prod_p (1-\alpha_p p^{-s})(1-\alpha_p^{-1}p^{-s})$ be the automorphic L-function of the cuspidal representation π_f attached to f. Then we have the following theorem.

THEOREM 1.2. *The degree-56 standard L-function $L(s, \pi_F, St)$ of π_F is given by*

$$L(s, \pi_F, St) = L(s, \mathrm{Sym}^3 \pi_f) L(s, \pi_f)^2 \prod_{i=1}^{4} L(s \pm i, \pi_f)^2 \prod_{i=5}^{8} L(s \pm i, \pi_f),$$

where $L(s, \mathrm{Sym}^3 \pi_f)$ is the symmetric cube L-function.

This paper is organized as follows. In § 2, we fix notations on Cayley numbers and exceptional Jordan algebras and review their properties. In § 3, we review the exceptional group of type $E_{7,3}$ and prove many facts which are not available in the literature. In § 4, we define the Jacobi group inside the exceptional group using the root subgroups, and recall Weil representations and theta functions. In § 5, we review modular forms on the exceptional domain and define Jacobi forms of matrix indices and study the Fourier–Jacobi coefficients of a modular form both in classical setting and in adelic setting. In § 6, we review the result of Karel on Fourier coefficients of Eisenstein series and interpret Eisenstein series in terms of degenerate principal series, following [Kud08]. Section 7 is the main technical part, where we prove the analogue of Ikeda's result [Ike94], namely, the Fourier–Jacobi coefficients of Eisenstein series are a sum of products of theta functions and Eisenstein series. In § 9, by following Ikeda [Ike01, Ike08], we construct a holomorphic cusp form on the exceptional group of type $E_{7,3}$. Our situation is similar to unitary group case, in that we do not need to consider half-integral modular forms. In § 10, we review the Hecke operators from Karel's thesis [Kar72] and modify it to fit into representation theory. Then we prove that our cusp form is a Hecke eigenform with respect to this modified action. The degree-56 standard L-function helps us to speculate on the Arthur parameter of π_F. We make a brief remark on it at the end of § 11. In the Appendix, we compute the discriminant of some quadratic forms and prove the orthogonal relation of theta functions we need.

2. Cayley numbers and exceptional Jordan algebras

In this section we will recall the Cayley numbers and the exceptional Jordan algebras. We refer to [Bai70, Cox46, Kim93]. For any field K whose characteristic is different from 2 and 3, the Cayley numbers $\mathfrak{C}_K$ over K is an eight-dimensional vector space over K with basis $\{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ satisfying the following rules for multiplication:

(1) $xe_0 = e_0x = x$ for all $x \in \mathfrak{C}_K$;
(2) $e_i^2 = -e_0$ for $i = 1, \dots, 7$;
(3) $e_i(e_{i+1}e_{i+3}) = (e_ie_{i+1})e_{i+2} = -e_0$ for any i (modulo 7).

For each $x = \sum_{i=0}^{7} x_ie_i \in \mathfrak{C}_K$, the map $x \mapsto \bar{x} = x_0e_0 - \sum_{i=1}^{7} x_ie_i$ defines an anti-involution of $\mathfrak{C}_K$. The trace and the norm on $\mathfrak{C}_K$ are defined by

$$\mathrm{Tr}(x) := x + \bar{x} = 2x_0, \quad N(x) := x\bar{x} = \sum_{i=0}^{7} x_i^2.$$

The space of Cayley numbers $\mathfrak{C}_K$ is neither commutative nor associative. In spite of this, we have

$$\mathrm{Tr}(xy) = \mathrm{Tr}(yx), \quad \mathrm{Tr}(x\bar{y}) = \mathrm{Tr}(\bar{x}y), \quad \mathrm{Tr}((xy)z) = \mathrm{Tr}(x(yz)).$$

We denote by $\mathfrak{o}$ the space of integral Cayley numbers which is a $\mathbb{Z}$-submodule of $\mathfrak{C}_K$ given by the following basis:

$$\alpha_0 = e_0, \quad \alpha_1 = e_1, \quad \alpha_2 = e_2, \quad \alpha_3 = -e_4,$$
$$\alpha_4 = \tfrac{1}{2}(e_1 + e_2 + e_3 - e_4), \quad \alpha_5 = \tfrac{1}{2}(-e_0 - e_1 - e_4 + e_5),$$
$$\alpha_6 = \tfrac{1}{2}(-e_0 + e_1 - e_2 + e_6), \quad \alpha_7 = \tfrac{1}{2}(-e_0 + e_2 + e_4 + e_7).$$

As shown in [Cox46], $\mathfrak{o}$ is stable under the operations of the anti-involution, multiplication, and addition. Further we have $\mathrm{Tr}(x), N(x) \in \mathbb{Z}$ if $x \in \mathfrak{o}$. By using this integral structure, for any $\mathbb{Z}$-algebra R, one can consider $\mathfrak{C}_R = \mathfrak{o} \otimes_{\mathbb{Z}} R$.

Let $\mathfrak{J}_K$ be the exceptional Jordan algebra consisting of the element

$$X = (x_{ij})_{1\leqslant i,j\leqslant 3} = \begin{pmatrix} a & x & y \\ \bar{x} & b & z \\ \bar{y} & \bar{z} & c \end{pmatrix}, \tag{2.1}$$

where $a, b, c \in Ke_0 = K$ and $x, y, z \in \mathfrak{C}_K$. In general, the matrix multiplication $X \cdot Y$ for two elements $X, Y \in \mathfrak{J}_K$ does not belong to $\mathfrak{J}_K$, but the square $X^2 = X \cdot X$ always does. The composition of $\mathfrak{J}_K$ is given by

$$X \circ Y = \tfrac{1}{2}(X \cdot Y + Y \cdot X).$$

For the above X, we define the trace by $\mathrm{Tr}(X) := a + b + c$, and define an inner product on $\mathfrak{J}_K \times \mathfrak{J}_K$ by $(X, Y) := \mathrm{Tr}(X \circ Y)$. Moreover we define

$$\det(X) := abc - aN(z) - bN(y) - cN(x) + \mathrm{Tr}((xz)\bar{y})$$

and a symmetric tri-linear form $(*, *, *)$ on $\mathfrak{J}_K \times \mathfrak{J}_K \times \mathfrak{J}_K$ by

$$\begin{aligned}(X, Y, Z) := \tfrac{1}{6}\{&\det(X + Y + Z) - \det(X + Y) - \det(Y + Z) - \det(Z + X) \\ &+ \det(X) + \det(Y) + \det(Z)\}.\end{aligned}$$

Then we define a bilinear pairing $\mathfrak{J}_K \times \mathfrak{J}_K \longrightarrow \mathfrak{J}_K, (X, Y) \mapsto X \times Y$ by requiring the identity

$$3(X, Y, Z) = (X \times Y, Z) = \mathrm{Tr}((X \times Y) \circ Z) \quad \text{for any } Z \in \mathfrak{J}_K.$$

In particular, for X_i, $i = 1, 2$ with entries as in (2.1), we have

$$X_1 \times X_2 = \begin{pmatrix} \frac{b_1c_2+c_1b_2}{2} - \frac{\bar{z}_1z_2+z_2\bar{z}_1}{2} & A & B \\ \overline{A} & \frac{a_1c_2+c_1a_2}{2} - \frac{\bar{y}_1y_2+y_2\bar{y}_1}{2} & C \\ \overline{B} & \overline{C} & \frac{a_1b_2+b_1a_2}{2} - \frac{\bar{x}_1x_2+x_2\bar{x}_1}{2} \end{pmatrix}, \tag{2.2}$$

where $A = (-c_1x_2 - c_2x_1)/2 + (y_1\bar{z}_2 + y_2\bar{z}_1)/2$, $B = (-b_1y_2 - b_2y_1)/2 + (x_1z_2 + x_2z_1)/2$, and $C = (-a_1z_2 - a_2z_1)/2 + (\bar{x}_1y_2 + \bar{x}_2y_1)/2$.

By using integral Cayley numbers, we define a lattice

$$\mathfrak{J}(\mathbb{Z}) := \{X = (x_{ij}) \in \mathfrak{J}_{\mathbb{Q}} \mid x_{ii} \in \mathbb{Z}, \text{ and } x_{ij} \in \mathfrak{o} \text{ for } i \neq j\},$$

and put $\mathfrak{J}(R) = \mathfrak{J}(\mathbb{Z}) \otimes_{\mathbb{Z}} R$ for any $\mathbb{Z}$-algebra R. Although the composition '$\circ$' does not preserve the integral structure, the inner product $(*, *)$ does. Hence $(\mathfrak{J}(R), \mathfrak{J}(R)) \in R$. Then one can show that the lattice $\mathfrak{J}(\mathbb{Z})$ in $\mathfrak{J}_{\mathbb{Q}}$ is the self-dual with respect to $(*, *)$, namely

$$\widetilde{\mathfrak{J}(\mathbb{Z})} := \{X \in \mathfrak{J}_{\mathbb{Q}} \mid (X, Y) \in \mathbb{Z}\ \forall Y \in \mathfrak{J}(\mathbb{Z})\} = \mathfrak{J}(\mathbb{Z}).$$

We also define $\mathfrak{J}_2(R)$ as the set of all matrices of forms

$$X = \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix}, \quad a, b \in R, \quad x \in \mathfrak{C}_R.$$

Similarly we define the inner product on $\mathfrak{J}_2(R) \times \mathfrak{J}_2(R)$ by $(X, Y) := \frac{1}{2}\mathrm{Tr}(XY + YX)$. For any such X, we define $\det(X) := ab - N(x)$. For X as above, $r \in R$, and $\xi = \binom{\xi_1}{\xi_2}, \xi_i \in \mathfrak{C}_R$ $(i = 1, 2)$, it is easy to see that

$$\det\begin{pmatrix} X & X\xi \\ {}^t\bar{\xi}X & r \end{pmatrix} = \det(X)(r - {}^t\bar{\xi}X\xi) = \det(X)(r - (X, {}^t\bar{\xi}\xi)) \tag{2.3}$$

which will be used later (§ 9). Henceforth we identify $\mathfrak{J}_2(R)$ with a subspace of $\mathfrak{J}(R)$ by $\left(\begin{smallmatrix} a & x \\ \bar{x} & b \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & x & 0 \\ \bar{x} & b & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$.

We define

$$R_3(K) = \{X \in \mathfrak{J}_K \mid \det(X) \neq 0\}$$

and define the set $R_3^+(K)$ consisting of squares of elements in $R_3(K)$. It is known that $R_3^+(\mathbb{R})$ is an open, convex cone in $\mathfrak{J}_{\mathbb{R}}$. We denote by $\overline{R_3^+(\mathbb{R})}$ the closure of $R_3^+(\mathbb{R})$ in $\mathfrak{J}_{\mathbb{R}} \simeq \mathbb{R}^{27}$ with respect to Euclidean topology. For any subring A of $\mathbb{R}$, set

$$\mathfrak{J}(A)_+ := \mathfrak{J}(A) \cap R_3^+(\mathbb{R}), \quad \mathfrak{J}(A)_{\geqslant 0} := \mathfrak{J}(A) \cap \overline{R_3^+(\mathbb{R})}.$$

We also define

$$\mathfrak{J}_2(A)_+ = \left\{ \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \in \mathfrak{J}_2(A) \,\middle|\, a, b \in A \cap \mathbb{R}_{>0}, ab - N(x) > 0 \right\},$$

and

$$\mathfrak{J}_2(A)_{\geqslant 0} = \left\{ \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \in \mathfrak{J}_2(A) \,\middle|\, a, b \in A \cap \mathbb{R}_{\geqslant 0}, ab - N(x) \geqslant 0 \right\}.$$

We define the exceptional domain as follows:

$$\mathfrak{T} := \{Z = X + Y\sqrt{-1} \in \mathfrak{J}_{\mathbb{C}} \mid X, Y \in \mathfrak{J}_{\mathbb{R}}, Y \in R_3^+(\mathbb{R})\}$$

which is a complex analytic subspace of $\mathbb{C}^{27}$. We also define

$$\mathfrak{T}_2 := \{X + Y\sqrt{-1} \in \mathfrak{J}_2(\mathbb{C}) \mid X, Y \in \mathfrak{J}_2(\mathbb{R}), Y \in \mathfrak{J}_2(\mathbb{R})_+\}.$$

3. Exceptional group of type $E_{7,3}$

In this section we recall the exceptional group of type $E_{7,3}$. Put $\mathfrak{J} = \mathfrak{J}_K$ where K is a field whose characteristic is different from 2 and 3. Define two subgroups of $\mathrm{GL}(\mathfrak{J})$ by

$$\mathbf{M} = \{g \in \mathrm{GL}(\mathfrak{J}) \mid \det(gX) = \nu(g)\det(X), \text{for } \nu(g) \neq 0\}$$
$$\mathbf{M}' = \{g \in \mathbf{M} \mid \nu(g) = 1\}.$$

Then $\mathbf{M}$ is an algebraic group over $\mathbb{Q}$ of type GE_6, and $\mathbf{M}'$ is the derived group of $\mathbf{M}$, which is a simple group of type $E_{6,2}$. The center of $\mathbf{M}'$ is the group of cube roots of unity.

There is an automorphism $g \longmapsto g^*$ of $\mathbf{M}$ of order 2 by the identity

$$(gX, g^*Y) = (g^*X, gY) = (X, Y). \tag{3.1}$$

Then g^* is the inverse adjoint of g. It satisfies $g(X \times Y) = (g^*X) \times (g^*Y)$.

Let $\mathbf{G}$ be the algebraic group over $\mathbb{Q}$ as in [Bai70]. Let $\mathbf{X}, \mathbf{X}'$ be two K-vector spaces, each isomorphic to $\mathfrak{J}$, and Ξ, Ξ' be copies of K. Let $\mathbf{W} = \mathbf{X} \oplus \Xi \oplus \mathbf{X}' \oplus \Xi'$, and for $w = (X, \xi, X', \xi') \in \mathbf{W}$, define a quartic form Q on $\mathbf{W}$ by

$$Q(w) = (X \times X, X' \times X') - \xi\det(X) - \xi'\det(X') - \tfrac{1}{4}((X, X') - \xi\xi')^2,$$

and a skew-symmetric bilinear form $\{\,,\,\}$ by

$$\{w_1, w_2\} = (X_1, X_2') - (X_2, X_1') + \xi_1\xi_2' - \xi_2\xi_1'.$$

Then

$$\mathbf{G}(K) = \{g \in \mathrm{GL}(\mathbf{W}_K) \mid Qg = Q, g\{\ ,\ \} = \{\ ,\ \}\}.$$

This defines a connected algebraic $\mathbb{Q}$-group of type $E_{7,3}$. The center of $\mathbf{G}(\mathbb{R})$ is $\{\pm\mathrm{id}\}$ and the quotient of $\mathbf{G}(\mathbb{R})$ by its center is the group of holomorphic automorphisms of $\mathfrak{T}$. The real rank of $\mathbf{G}$ is 3, and it is split over $\mathbb{Q}_p$ for any prime p.

The group $\mathbf{M}$ can be considered as a subgroup of $\mathbf{G}$ by defining the action

$$g(X, \xi, X', \xi') = (gX, \nu(g)\xi, g^*X', \nu(g)^{-1}\xi').$$

Let $\mathbf{N}$ be the subgroup of all transformations p_B for $B \in \mathfrak{J}$ as in [Bai70]. Recall the definition.

$$p_B \begin{pmatrix} X \\ \xi \\ X' \\ \xi' \end{pmatrix} = \begin{pmatrix} X + \xi' B \\ \xi + (B, X') + (B \times B, X) + \xi' \det(B) \\ X' + 2B \times X + \xi' B \times B \\ \xi' \end{pmatrix}.$$

The relative root system of $\mathbf{G}$ over $\mathbb{Q}$ is of type C_3, and we denote the positive roots by $\{e_1 \pm e_2, e_1 \pm e_3, e_2 \pm e_3, 2e_1, 2e_2, 2e_3\}$, and let $\Delta = \{e_1 - e_2, e_2 - e_3, 2e_3\}$ be the set of simple roots. We describe their root spaces: for a positive root α, let $\mathbf{U}_\alpha$ be the root subspace. For $1 \leqslant i \leqslant j \leqslant 3$, let e_{ij} is the 3×3 matrix with a 1 in the intersection of the ith row and jth column and zeros elsewhere, and let $e_i = e_{ii}$. Then for $a, b, c \in K$, $x, y, z \in \mathfrak{C}_K$,

$$\begin{gathered} \mathbf{U}_{2e_1} = \{p_{ae_1}\}, \quad \mathbf{U}_{2e_2} = \{p_{ae_2}\}, \quad \mathbf{U}_{2e_3} = \{p_{ae_3}\} \\ \mathbf{U}_{e_1+e_2} = \{p_{xe_{12}}\}, \quad \mathbf{U}_{e_1+e_3} = \{p_{ye_{13}}\}, \quad \mathbf{U}_{e_2+e_3} = \{p_{ze_{23}}\} \\ \mathbf{U}_{e_1-e_2} = \{m_{\bar{x}e_{21}} \in \mathrm{GL}(\mathfrak{J}) : m_{\bar{x}e_{21}} X = (I + xe_{12}) X (I + \bar{x} e_{21}), X \in \mathfrak{J}\} \\ \mathbf{U}_{e_1-e_3} = \{m_{\bar{y}e_{31}} \in \mathrm{GL}(\mathfrak{J}) : m_{\bar{y}e_{31}} X = (I + ye_{13}) X (I + \bar{y} e_{31}), X \in \mathfrak{J}\} \\ \mathbf{U}_{e_2-e_3} = \{m_{\bar{z}e_{32}} \in \mathrm{GL}(\mathfrak{J}) : m_{\bar{z}e_{32}} X = (I + ze_{23}) X (I + \bar{z} e_{32}), X \in \mathfrak{J}\}. \end{gathered}$$

Remark 3.1. Note that we are using different ordering of roots from [Bai70]. In [Bai70], $\mathbf{N}$ consists of root spaces of negative non-compact roots. However, it is more convenient to make it correspond to positive roots so that it may correspond to the upper triangular matrices of the form $\left(\begin{smallmatrix} I_n & B \\ O_n & I_n \end{smallmatrix}\right)$ in the Sp_{2n} case.

Note the following:

$$m^*_{xe_{ij}} = m_{-\bar{x}e_{ji}}.$$

Let $\mathbf{H}$ be the group generated by $\mathbf{U}_{2e_3}$ and ι_{e_3}, where ι_{e_i} is the Weyl group element of $2e_i$, which is given by $\iota_{e_i} = p_{e_i} p'_{-e_i} p_{e_i}$, where p'_{e_i} generates the root subspace of $-2e_i$. Then $\mathbf{H} \simeq \mathrm{SL}_2$. Let $\iota = \iota_{e_1}\iota_{e_2}\iota_{e_3}$. Then $\iota^{-1} = -\iota$, and $p'_B = \iota p_{-B} \iota^{-1}$ will generate the opposite unipotent subgroup $\overline{\mathbf{N}}$ of $\mathbf{N}$. This ι plays the role of $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ is Sp_{2n}. Its action is given by

$$\iota(X, \xi, X', \xi') = (-X', -\xi', X, \xi).$$

We define two maximal parabolic $\mathbb{Q}$-subgroups:

$$\mathbf{P} = \mathbf{M}\mathbf{N}, \quad \mathbf{Q} = \mathbf{L}\mathbf{V},$$

where $\mathbf{V}$ is generated by $\mathbf{U}_\alpha$ for $\alpha = e_1 \pm e_3, e_2 \pm e_3, e_1 + e_2, 2e_1, 2e_2$. Then $\mathbf{P}$ is the Siegel parabolic subgroup associated to $\Delta - \{2e_3\}$, and $\mathbf{Q}$ is the parabolic subgroup associated to $\Delta - \{e_2 - e_3\}$. Then $\mathbf{V}$ is the Heisenberg group, and the derived group of $\mathbf{L}$ is $\mathbf{L}' = \mathbf{H} \times \mathrm{Spin}(9, 1)$.

LEMMA 3.2. *For $g \in \mathbf{M}$ and $p_B \in \mathbf{N}$,*

$$gp_B = p_{B_1}g, \quad B_1 = \nu(g)gB.$$

Proof. By explicit computation, we see that

$$gp_B \begin{pmatrix} X \\ \xi \\ X' \\ \xi' \end{pmatrix} = \begin{pmatrix} gX + \xi' gB \\ \nu(g)(\xi + (B, X') + (B \times B, X) + \xi' \det(B)) \\ g^*X' + 2g^*(B \times X) + \xi' g^*(B \times B) \\ \nu(g)^{-1}\xi' \end{pmatrix},$$

$$p_{B_1}g \begin{pmatrix} X \\ \xi \\ X' \\ \xi' \end{pmatrix} = \begin{pmatrix} gX + \nu(g)^{-1}\xi' B_1 \\ \nu(g)\xi + (B_1, g^*X') + (B_1 \times B_1, gX) + \nu(g)^{-1}\xi' \det(B_1) \\ g^*X' + 2B_1 \times gX + \nu(g)^{-1}\xi' B_1 \times B_1 \\ \nu(g)^{-1}\xi' \end{pmatrix}.$$

By comparing coefficients, we see that $B_1 = \nu(g)gB$. $\square$

Denote the element of $V = \mathbf{V}(K)$ by

$$v(x, y, z) = m_{\bar{x}_1 e_{31}} m_{\bar{x}_2 e_{32}} \cdot p_{y_1 e_{13}} p_{y_2 e_{23}} \cdot p_z,$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad z = \begin{pmatrix} a & w \\ \bar{w} & b \end{pmatrix},$$

where $x_1, x_2, y_1, y_2, w \in \mathfrak{C}_K$ and $a, b \in K$. We identified z with $\left(\begin{smallmatrix} z & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in the definition of p_z. Then by using the above lemma, we can show that

$$v(x, y, z)v(x', y', z') = v(x + x', y + y', z + z' - y({}^t\bar{x'}) - x'({}^t\bar{y})). \tag{3.2}$$

Now let

$$\begin{aligned} X &= X(K) = \{m_{\bar{x}_1 e_{31}} m_{\bar{x}_2 e_{32}} \in V \mid x_1, x_2 \in \mathfrak{C}_K\}, \\ Y &= Y(K) = \{p_{y_1 e_{13}} p_{y_2 e_{23}} \in V \mid y_1, y_2 \in \mathfrak{C}_K\}, \\ Z &= Z(K) = \{p_z \in V \mid z \in \mathfrak{J}_2(K)\} \simeq \mathfrak{J}_2(K). \end{aligned} \tag{3.3}$$

We identify X (respectively Y) with $\mathfrak{C}_K^2$ by $m_{\bar{x}_1 e_{31}} m_{\bar{x}_2 e_{32}} \mapsto x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ (respectively by $p_{y_1 e_{13}} p_{y_2 e_{23}} \mapsto y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$). Then we have the decomposition

$$V = \mathbf{V}(K) = X \cdot Y \cdot Z. \tag{3.4}$$

We hope that it is clear from the context when X, Y, Z denote the sets and when they denote the elements of $\mathfrak{J} = \mathfrak{J}_K$.

For any $S \in \mathfrak{J}_2(K)$, define $\mathrm{tr}_S : Z = \{v(0, 0, z)\} \longrightarrow K$, $\mathrm{tr}_S(v(0, 0, z)) = \frac{1}{2}(S, z)$. Since Z is the center of V, $\mathrm{Ker}(\mathrm{tr}_S)$ is a normal subgroup of V, and we may consider the quotient $V_0 = V/\mathrm{Ker}(\mathrm{tr}_S)$.

Define the alternating form on $X \oplus Y$ by

$$\langle (x, y), (x', y') \rangle_S = \mathrm{Tr}(S(x({}^t\bar{y'}) + y'({}^t\bar{x}) - x'({}^t\bar{y}) - y({}^t\bar{x'}))).$$

Consider the map $g_S : V \longrightarrow X \oplus Y \oplus K$ defined by

$$v(x, y, z) = v(x)v(y)v(z) \longmapsto \left(x, y, \mathrm{Tr}\left(\frac{1}{2}Sz\right) + \mathrm{Tr}\left(\frac{S}{2}(y\,{}^t\bar{x} + x\,{}^t\bar{y})\right)\right). \tag{3.5}$$

From (3.2), we see that $g_S(v(x,y,z)v(x',y',z')) = (x+x', y+y', z'')$ where

$$z'' = \mathrm{Tr}\left(\frac{1}{2}Sz\right) + \mathrm{Tr}\left(\frac{S}{2}(y^t\bar{x} + x^t\bar{y})\right) + \mathrm{Tr}\left(\frac{1}{2}Sz'\right) + \mathrm{Tr}\left(\frac{S}{2}(y'^t\bar{x}' + x'^t\bar{y'})\right) + \frac{1}{2}\langle (x,y),(x',y')\rangle_S.$$

Since $\mathrm{Ker}(g_S) = \mathrm{Ker}(\mathrm{tr}_S)$, if $\det(S) \neq 0$ then we obtain the isomorphism

$$g_S : V_0 = V/\mathrm{Ker}(\mathrm{tr}_S) \xrightarrow{\sim} X \oplus Y \oplus K. \tag{3.6}$$

Next we compute the action of $\mathbf{H}(K)$ on $V = \mathbf{V}(K)$: recall that $\mathbf{H}(K) = \langle p_{be_3}, \iota_{e_3}\rangle \simeq \mathrm{SL}_2(K)$ for $b \in K$. We identify $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(K)$ with the corresponding element in $\mathbf{H}(K)$ under the isomorphism. Observe [Fre53] that

$$e_3 \times X = \frac{1}{2}\begin{pmatrix} b & -x & 0 \\ -\bar{x} & a & 0 \\ 0 & 0 & 0\end{pmatrix}, \quad e_3 \times (e_3 \times X) = \frac{1}{4}\begin{pmatrix} a & x & 0 \\ \bar{x} & b & 0 \\ 0 & 0 & 0\end{pmatrix}.$$

Then, for $i = 1, 2$,

$$\iota_{e_3}^{-1} p_{xe_{i3}} \iota_{e_3} = m_{-\bar{x}e_{3i}}, \quad \iota_{e_3}^{-1} m_{\bar{x}e_{3i}} \iota_{e_3} = p_{xe_{i3}}. \tag{3.7}$$

For $1 \leqslant i \leqslant j \leqslant 2$, we have $\iota_{e_3}^{-1} p_{xe_{ij}} \iota_{e_3} = p_{xe_{ij}}$. Hence

$$p_{be_3}^{-1} v(x,y,z) p_{be_3} = v(x, bx+y, z - bx^t\bar{x}), \quad \iota_{e_3}^{-1} v(x,y,z)\iota_{e_3} = v(-y, x, z + x^t\bar{y} + y^t\bar{x}).$$

Since $p'_{ce_3} = \iota_{e_3} p_{-ce_3} \iota_{e_3}^{-1}$, and $h(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)$ is identified with $p_{ae_3} p'_{-a^{-1}e_3} p_{ae_3} \iota_{e_3}^{-1}$, we see that

$${p'_{ce_3}}^{-1} v(x,y,z) p'_{ce_3} = v(x+cy, y, z - cy^t\bar{y}), \quad h(a)^{-1} v(x,y,z) h(a) = v(ax, a^{-1}y, z).$$

Here $h(a) \in \mathbf{M}$, and $\nu(h(a)) = a$; more explicitly,

$$h(a)(X, \xi, X', \xi') = (X + (a-1)(e_3, X)e_3, a\xi, X' - (1-a^{-1})(e_3, X')e_3, a^{-1}\xi').$$

Hence we have the following lemma.

Lemma 3.3. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathbf{H}(K)$. Then $\gamma^{-1} v(x,y,z)\gamma = v(ax+cy, bx+dy, z')$ where*

$$z' = z - \tfrac{1}{2}((ax+cy)^t\overline{(bx+dy)} + (bx+dy)^t\overline{(ax+cy)} - x^t\bar{y} - y^t\bar{x}).$$

4. Jacobi group in $E_{7,3}$, Weil representation, and theta functions

4.1 Jacobi group in $E_{7,3}$

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ its finite part. Let $\widehat{\mathbb{Z}}$ be the profinite completion of $\mathbb{Z}$. For $R = \mathbb{A}, \mathbb{A}_f, \widehat{\mathbb{Z}}, \mathbb{Q}, \mathbb{Q}_p, \mathbb{Z}_p$, $p \leqslant \infty$, or any field R, one can consider $X(R)$, $Y(R)$, $Z(R)$, $\mathbf{V}(R)$ (analogues of (3.3) and (3.4)) by using $\mathfrak{C}_R$ and the action of $H(R) \simeq \mathrm{SL}_2(R)$ on $\mathbf{V}(R)$ by using the calculation done in § 3. Note that we may not get the identification $H(R) \simeq \mathrm{SL}_2(R)$ for an arbitrary ring R since this map is described in terms of the root system. (The interested readers should consult the notion of Chevalley basis, cf. [Ste68].)

Now we introduce a new coordinate on $\mathbf{V}$ by modifying the group actions, and define the Jacobi group in $E_{7,3}$. For any $x, y \in X(R)$ and $S \in \mathfrak{J}_2(R)$ so that $\det(S) \neq 0$, we define

$$\sigma(x,y) := x^t\bar{y} + y^t\bar{x}, \quad \sigma_S(x,y) := (S, \sigma(x,y)) \quad \text{and} \quad \lambda_S(x,y) := \tfrac{1}{2}\sigma_S(x,y). \tag{4.1}$$

Clearly $\sigma(x,y) \in \mathfrak{J}_2(R) \simeq Z(R)$. A new coordinate $v_1(x,y,z)$ on $\mathbf{V}$ is defined by

$$v_1(x,y,z) := v(x,y,z-\sigma(x,y)), \quad x \in X(R), y \in Y(R) \text{ and } z \in Z(R).$$

Then by (3.2), one has

$$v_1(x,y,z)v_1(x',y',z') = v_1(x+x', y+y', z+z'+\sigma(x,y')-\sigma(x',y)). \tag{4.2}$$

The alternating pairing on $X(R) \oplus Y(R)$ is modified as

$$\langle (x,y),(x',y')\rangle_S := 2(\lambda_S(x,y') - \lambda_S(x',y)) = \sigma_S(x,y') - \sigma_S(x',y), \tag{4.3}$$

and $X(R) \oplus Y(R) \oplus R$ has the Heisenberg structure defined by

$$(x,y,a) * (x',y',b) = (x+x', y+y', a+b+\tfrac{1}{2}\langle (x,y),(x',y')\rangle_S).$$

For any $S \in \mathfrak{J}_2(\mathbb{Z})_+$, the Heisenberg structure on V is modified by passing to g_S (see (3.6) for this map) as

$$g_{1,S} : \mathbf{V}(R) \longrightarrow X(R) \oplus Y(R) \oplus R, \quad v_1(x,y,z) \mapsto (x,y,\tfrac{1}{2}(S,z)). \tag{4.4}$$

Noting $\frac{1}{2}(\sigma_S(x,y') - \sigma_S(x',y)) = \lambda_S(x,y') - \lambda_S(x',y)$, it is easy to see that

$$g_{1,S}(v_1(x,y,z)v_1(x',y',z')) = g_{1,S}(v_1(x,y,z)) * g_{1,S}(v_1(x',y',z')),$$

or, equivalently, $g_{1,S}$ preserves the Heisenberg structures in both sides.

The action of $\mathbf{H}(R)$ on $\mathbf{V}(R)$ with the new coordinates now turn to be much simpler by Lemma 3.3:

$$\gamma^{-1}v_1(x,y,z)\gamma = v_1(ax+cy, bx+dy, z) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{H}(R). \tag{4.5}$$

By using this action, we define the Jacobi group in $\mathbf{G}(R)$ by

$$J(R) := \mathbf{V}(R) \rtimes \mathbf{H}(R) \tag{4.6}$$

with the coordinates $(v_1(x,y,z), h(\gamma)), \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ under the identification $\mathbf{H}(R) \simeq \mathrm{SL}_2(R)$.

4.2 Weil representation and theta functions

In this section we shall recall Weil representation and theta series (cf. [Ike94, § 1–3]).

For each place $p \neq \infty$, put

$$e_p(x) = \exp(-2\pi\sqrt{-1} \cdot \mathrm{Frac}(x))$$

for $x \in \mathbb{Q}_p$ where $\mathrm{Frac}(x)$ stands for the fractional part of x. For $p = \infty$, put $\mathbf{e}(x) = e_\infty(x) := \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbb{R}$. Fix a non-trivial additive character $\psi : \mathbb{Q}\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$ and decompose it as the restricted tensor product $\psi = \bigotimes'_{p \leqslant \infty} \psi_p$. As a standard example, one can take $\psi^{\mathrm{st}} := \bigotimes'_{p \leqslant \infty} e_p(*)$ which will be used later when we translate the adelic setting into the classical setting and vice versa.

Fix $S \in \mathfrak{J}_2(\mathbb{Z})_+$. We denote by $h(a)$ (respectively $n(b)$) the element of $\mathbf{H}(\mathbb{Q}_p)$ corresponding to $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), a \in \mathbb{Q}_p^\times$ (respectively $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right), b \in \mathbb{Q}_p$) under $\mathbf{H}(\mathbb{Q}_p) \simeq \mathrm{SL}_2(\mathbb{Q}_p)$. Note that $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ corresponds to $\iota_{e_3} \in \mathbf{H}(\mathbb{Q}_p)$.

For each place $p \leqslant \infty$, the Schrödinger model $\omega_{S,p}$ on $\mathbf{V}(\mathbb{Q}_p)$ with the central character $\psi_{p,S} : z \mapsto \psi_p(\frac{1}{2}(S,z)), z \in Z(\mathbb{Q}_p)$ realized on the Schwartz space $\mathcal{S}(X(\mathbb{Q}_p))$ is given by

$$\omega_{S,p}(v_1(x,y,z))\varphi(t) = \varphi(t+x)\psi_p(\tfrac{1}{2}(S,z) + 2\lambda_S(t,y) + \lambda_S(x,y))$$

for $\varphi \in \mathcal{S}(X(\mathbb{Q}_p))$ and $v_1(x,y,z) \in \mathbf{V}(\mathbb{Q}_p)$. Noting the multiplication law (4.2), it is easy to check

$$\omega_{S,p}(v_1(x,y,z)v_1(x',y',z'))\varphi(t) = \omega_{S,p}(v_1(x,y,z))(\omega_{S,p}(v_1(x',y',z')\varphi(t))).$$

By the Stone–von Neumann theorem, $\omega_{S,p}$ is a unique irreducible unitary representation on which $Z(\mathbb{Q}_p)$ acts by $\psi_{p,S}$.

Recall the conjugate action $\mathbf{H}(\mathbb{Q}_p)$ on $\mathbf{V}(\mathbb{Q}_p)$ (see (4.5)) and the alternating pairing (4.3). This induces a homomorphism

$$\mathbf{H}(\mathbb{Q}_p) \hookrightarrow \mathrm{Sp}_{V/Z}(\mathbb{Q}_p) := \mathrm{Sp}(\mathbf{V}(\mathbb{Q}_p)/Z(\mathbb{Q}_p), \langle *, * \rangle_S). \tag{4.7}$$

Let $\widetilde{\mathrm{Sp}}_{V/Z}(\mathbb{Q}_p)$ be the metaplectic covering of $\mathrm{Sp}_{V/Z}(\mathbb{Q}_p)$. This covering does not split, but by [Kud94], one has a splitting $H(\mathbb{Q}_p) \hookrightarrow \widetilde{\mathrm{Sp}}_{V/Z}(\mathbb{Q}_p)$ so that the map (4.7) factors through it via the covering map. The Schrödinger model $\omega_{S,p}$ extends to the Weil representation of $\mathbf{V}(\mathbb{Q}_p) \rtimes \widetilde{\mathrm{Sp}}_{V/Z}(\mathbb{Q}_p)$ acting on $\mathcal{S}(X(\mathbb{Q}_p))$. Then the pullback to $\mathbf{H}(\mathbb{Q}_p)$ of this representation is given by

$$\begin{gathered}\omega_{S,p}(h(a))\varphi(t) = \chi_S(a)|a|_p^8\varphi(ta), \chi_S(a) := \langle \mathrm{disc}(\lambda_S), a \rangle_{\mathbb{Q}_p} \\ \omega_{S,p}(n(b))\varphi(t) = \psi_p(\lambda_S(t,t)b)\varphi(t) \\ \omega_{S,p}(\iota_{e_3})\varphi(t) = (F_S\varphi)(-t),\end{gathered}$$

where $\langle *, * \rangle_{\mathbb{Q}_p}$ stands for the Hilbert symbol on $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$ and

$$(F_S\varphi)(t) = \int_{X(\mathbb{Q}_p)} \varphi(x)\psi_p(\lambda_S(t,x))\,dx,$$

where dx means the Haar measure on $X(\mathbb{Q}_p)$ which is self-dual with respect to the Fourier transform F_S. Note that the index 8 of $|a|_p^8$ in the first formula comes from the fact that $\frac{1}{2}\dim_{\mathbb{Q}_p} X(\mathbb{Q}_p) = 8$ and we also use $\nu(h(a)) = a$. Furthermore, we always have $\chi_S(a) = 1$ since $\mathrm{disc}(\lambda_S)$ is a square by Lemma A.1 in the Appendix.

The global Weil representation of ω_S of $J(\mathbb{A})$ acting on the Schwartz space $\mathcal{S}(X(\mathbb{A}))$ is given by the restricted tensor product of $\omega_{S,p}$. In our setting, ω_S is much simpler than that of the case Sp_{2n} (compare with [Ike01, § 1]): for $\varphi \in \mathcal{S}(X(\mathbb{A}))$,

$$\omega_S(h(a))\varphi(t) = |a|_{\mathbb{A}}^8\varphi(ta), \quad \omega_S(n(b))\varphi(t) = \psi(\lambda_S(t,t)b)\varphi(t), \quad \omega_S(\iota_{e_3})\varphi(t) = (F_S\varphi)(-t),$$

where $(F_S\varphi)(t) = \int_{X(\mathbb{A})} \varphi(x)\psi(\lambda_S(t,x))\,dx$.

For each $\varphi \in \mathcal{S}(X(\mathbb{A}))$, the theta function $\Theta^{\psi_S}(v_1(x,y,z)h;\varphi)$ on $\mathbf{V}(\mathbb{A})$ is given by

$$\begin{aligned}\Theta^{\psi_S}(v_1(x,y,z)h;\varphi) &:= \sum_{\xi \in X(\mathbb{Q})} \omega_S(v_1(x,y,z)h)\varphi(\xi) \\ &= \sum_{\xi \in X(\mathbb{Q})} \omega_S(h)\varphi(\xi + x)\psi\left(\frac{1}{2}(S,z) + 2\lambda_S(\xi,y) + \lambda_S(x,y)\right).\end{aligned}$$

It is easy to see that this function is invariant under the action of $J(\mathbb{Q})$. By the equivalence of the Schrödinger model and the lattice model (cf. [Tak96]), for any $\varphi \in \mathcal{S}(X(\mathbb{A}))$ one has the Poisson summation formula which will be used later:

$$\Theta^{\psi_S}(v_1(x,y,z)(\iota_{e_3}h);\varphi) = \Theta^{\psi_S}(v_1(x,y,z)h;\varphi). \tag{4.8}$$

To end this section, we discuss the relation between the adelic theta function and the classical theta function. For any $\varphi' \in \mathcal{S}(X(\mathbb{A}_f))$, we extend this function to an element φ of $\mathcal{S}(X(\mathbb{A}))$ by

$$\varphi((x_p)_{p\leqslant\infty}) := \varphi_\infty(x_\infty)\varphi'((x_p)_{p<\infty}), \quad \varphi_\infty(x_\infty) = e^{-2\pi\cdot\sigma_S(x_\infty,x_\infty)}.$$

Put $\mathfrak{X} := X(\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C} \simeq \mathbb{C}^{\oplus 16}$ and let us extend the quadratic form σ_S linearly to that on $\mathfrak{X}$. For each $\varphi \in \mathcal{S}(X(\mathbb{A}_f))$, the classical theta function on $\mathfrak{D} := \mathbb{H}\times\mathfrak{X}$ is given by

$$\theta^S_\varphi(\tau,u) := \sum_{\xi\in X(\mathbb{Q})} \varphi(\xi)\mathbf{e}(\sigma_S(\xi,\xi)\tau + 2\sigma_S(\xi,u)).$$

The group $J(\mathbb{R})$ acts on $\mathfrak{D}$ by

$$\beta(\tau,u) := \left(\gamma\tau, \frac{u}{c\tau+d} + x(\gamma\tau) + y\right),$$

where $\beta = v_1(x,y,z)h$ with $v_1(x,y,z) \in \mathbf{V}(\mathbb{R})$ and $h = h(\gamma) \in \mathbf{H}(\mathbb{R})$ corresponds to $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. Here $\gamma\tau = (a\tau+b)/(c\tau+d)$ and put $j(\gamma,\tau) := c\tau+d$ for simplicity. For each positive even integer k, the automorphy factor on $J(\mathbb{R})\times\mathfrak{D}$ is defined by

$$j_{k,S}(\beta,(\tau,u)) := j(\gamma,\tau)^k\mathbf{e}\left(-(S,z) + \frac{c}{j(\gamma,\tau)}\sigma_S(u,u) - \frac{2\sigma_S(x,u)}{j(\gamma,u)} - \sigma_S(x,x)(\gamma\tau) - \sigma_S(x,y)\right),$$

for $\beta = v_1(x,y,z)h, h = h(\gamma)$ as above. After lengthy and painful calculation, one can check the cocycle relation:

$$j_{k,S}(\beta\beta',(\tau,u)) = j_{k,S}(\beta,\beta'(\tau,u))j_{k,S}(\beta',(\tau,u)).$$

For each function $f : \mathfrak{D} \longrightarrow \mathbb{C}$ and $\beta \in \mathbf{V}(\mathbb{R})$, we define the 'slash' operator $f|_{k,S}[\beta] : \mathfrak{D} \longrightarrow \mathbb{C}$ by

$$f|_{k,S}[\beta](\tau,u) := j_{k,S}(\beta,(\tau,u))^{-1}f(\beta(\tau,u)).$$

Then the following lemma is easy to deduce from the definition.

Lemma 4.1. *Keep the notation above. For each $\varphi \in \mathcal{S}(X(\mathbb{A}_f))$ and $h(\gamma) \in H(\mathbb{R}), \gamma \in \mathrm{SL}_2(\mathbb{R})$:*

(1) $\Theta^{(\psi_S^{\mathrm{st}})^2}(\beta;\varphi') = \theta^S_\varphi|_{8,S}[\beta](\sqrt{-1},0)$ *for any $\beta \in J(\mathbb{R})$;*
(2) $\theta^S_{\omega_S(\gamma^{-1})\varphi}(\tau,u) = j(\gamma,\tau)^{-8}\theta^S_\varphi(\gamma(\tau,u))$.

Lemma 4.2. *Keep the notation above. Let ξ be an element of $X(\mathbb{Q})$ so that $\sigma_S(\xi,x) \in \mathbb{Z}$ for all $x \in X(\mathfrak{o})$ and φ_ξ be the characteristic function of $\xi + X(\mathfrak{o})$. Then*

$$\Theta^{(\psi_S^{\mathrm{st}})^2}(v_1(x,y,z)h;\varphi_\xi) = \Theta^{(\psi_S^{\mathrm{st}})^2}(v_1(x_\infty,y_\infty,z_\infty)h;\varphi_\xi).$$

Proof. One can decompose any element $v_{1,f} \in J(\mathbb{A}_f)$ as $v_{1,f} = j_1 \cdot v_1'$ so that $j_1 \in J(\mathbb{Q})$ and $v_1' = v_1(x',y',z') \in \mathbf{V}(\widehat{\mathbb{Z}})$. Since the sum defining this theta function runs over $X(\mathbb{Q})\cap(\xi+X(\mathfrak{o}))$, we see that $\psi_S^{\mathrm{st}}((S,z') + 2\sigma_S(a,y') + \sigma_S(x',y')) = 1$ for any $a \in X(\mathbb{Q})\cap(\xi+X(\mathfrak{o}))$. Then the claim follows from the left invariance of the theta function under $J(\mathbb{Q})$. □

For any $(\tau,u) \in \mathfrak{D}$, there exist elements $v_1 \in \mathbf{V}(\mathbb{R})$ and $g_\infty \in H(\mathbb{R})$ such that $v_1g_\infty(\sqrt{-1},0) = (\tau,u)$ since 1 and τ are independent over $\mathbb{R}$. From this with Lemma 4.1(1), we make a bridge between the adelic theta functions and the classical theta functions which will be focused in the next section.

5. Modular forms on the exceptional domain and Jacobi forms

We review the definition of modular forms on the exceptional domain $\mathfrak{T}$ in [Bai70], and define Jacobi forms for our Jacobi group and study their basic properties.

5.1 Modular forms on the exceptional domain

Let $\Gamma = \mathbf{G}(\mathbb{Z})$ be the arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$ as in [Bai70], defined by $\mathbf{G}(\mathbb{Z}) = \{g \in \mathbf{G}(\mathbb{R}) : g\mathbf{W}_{\mathfrak{o}} = \mathbf{W}_{\mathfrak{o}}\}$, where $\mathbf{W}_{\mathfrak{o}} = \mathfrak{J}(\mathbb{Z}) \oplus \mathbb{Z}e \oplus \mathfrak{J}(\mathbb{Z}) \oplus \mathbb{Z}e'$, and $e = (0,1,0,0)$ and $e' = (0,0,0,1)$.

Lemma 5.1. *The arithmetic group Γ is generated by $\mathbf{N}(\mathbb{Z})$ and ι.*

Proof. By [Bai70, Theorem 5.2], Γ is generated by $\mathbf{N}(\mathbb{Z})$ and $\overline{\mathbf{N}}(\mathbb{Z})$, where $\overline{\mathbf{N}}$ is the opposite unipotent subgroup of $\mathbf{N}$. Since $\overline{\mathbf{N}} = \iota^{-1}\mathbf{N}\iota$, the result follows. □

Lemma 5.2. *The arithmetic group Γ is generated by $\mathbf{N}(\mathbb{Z})$, $\mathbf{M}'(\mathbb{Z})$ and $\mathbf{H}(\mathbb{Z})$, hence by $\mathbf{N}(\mathbb{Z})$, $\mathbf{M}'(\mathbb{Z})$ and ι_{e_3}.*

Proof. This follows from the above lemma and from the identities, $\iota = \iota_{e_1}\iota_{e_2}\iota_{e_3}$, and $\iota_{e_2} = \varphi_{23}\iota_{e_3}\varphi_{23}^{-1}$, and $\iota_{e_1} = \varphi_{13}\iota_{e_3}\varphi_{13}^{-1}$, where $\varphi_{ij} = m_{e_{ij}}m_{-e_{ji}}m_{e_{ij}}$ for $i \neq j$. □

In [Bai70, Kar74, Kim93], for $Z \in \mathfrak{T}$ and $g \in \mathbf{G}(\mathbb{R})$, the action is defined by the right action:

$$Z \cdot g = Z_1, \quad p'_Z g = p_A k p'_{Z_1} \quad \text{for } k \in \mathbf{M}(\mathbb{C}) \text{ and } Z_1 \in \mathcal{H}.$$

However, following the usual convention, it is more convenient to define the left action by

$$gZ = Z_1, \quad gp_Z = p_{Z_1} k p'_A \quad \text{for } k \in \mathbf{M}(\mathbb{C}) \text{ and } Z_1 \in \mathfrak{T}.$$

Let $j(g,Z) = \nu(k)^{-1}$ be the canonical factor of automorphy. Then $j(g,Z)$ has the following properties:

$$j(p_B, Z) = 1 \quad \text{for all } B \in \mathfrak{J}_{\mathbb{R}}, \quad j(\iota, Z) = \det(-Z), \quad j(g_1g_2, Z) = j(g_1, g_2Z)j(g_2, Z).$$

If $J(Z,g)$ is the functional determinant of g at Z, then $J(Z,g) = j(g,Z)^{-18}$. By Lemma 3.2, if $k \in \mathbf{M}(\mathbb{R})$, kZ is just the transformation $\nu(k)(kX + kY\sqrt{-1})$, where $Z = X + Y\sqrt{-1}$, and for kX and kY, k is considered as an element of $\mathrm{GL}(\mathfrak{J})$, and $j(k,Z) = \nu(k)^{-1}$.

Definition 5.3. Let F be a holomorphic function on $\mathfrak{T}$ which for some integer $k > 0$ satisfies

$$F(\gamma Z) = F(Z)j(\gamma, Z)^k, \quad Z \in \mathfrak{T}, \gamma \in \Gamma.$$

Then F is called a modular form on $\mathfrak{T}$ of weight k. We denote by $\mathcal{M}_k(\Gamma)$ the space of such forms. For a holomorphic function $F : \mathfrak{T} \longrightarrow \mathbb{C}$, the boundary map Φ is defined by

$$\Phi F(Z') = \lim_{\tau \to \sqrt{-1}\infty} F\begin{pmatrix} Z' & * \\ {}^t\bar{*} & \tau \end{pmatrix},$$

where $Z' \in \mathfrak{T}_2$. We call $\mathcal{S}_k(\Gamma) := \mathrm{Ker}(\Phi) \cap \mathcal{M}_k(\Gamma)$ the space of cusp forms of weight k with respect to Γ. We should remark that there is only one equivalent class of cusps since $\mathbf{G}(\mathbb{Q}) = P(\mathbb{Q})\mathbf{G}(\mathbb{Z})$ [Bai70, Theorem 5.2].

Since $F(Z+B) = F(Z)$ for $B \in \mathfrak{J}(\mathbb{Z})$ and $\mathfrak{J}(\mathbb{Z})$ is self-dual, F has a Fourier expansion of the form

$$F(Z) = \sum_{T \in \mathfrak{J}(\mathbb{Z})_{\geqslant 0}} a(T)\mathbf{e}((T,Z)). \tag{5.1}$$

By the Koecher principle, we do not need the holomorphy at the cusps.

If F is a cusp form, then $a(T) = 0$ for $T \notin \mathfrak{J}(\mathbb{Z})_+$.

5.2 Jacobi forms of matrix index

We define and study Jacobi forms of matrix index on $\mathfrak{D} = \mathbb{H} \times \mathfrak{X}$ in the classical setting. Set

$$\Gamma_J := J(\mathbb{Q}) \cap \mathbf{G}(\mathbb{Z}).$$

Definition 5.4. Let k be a positive (even) integer and S be an element of $\mathfrak{J}_2(\mathbb{Z})_+$. We say a holomorphic function $\phi : \mathfrak{D} \longrightarrow \mathbb{C}$ is a Jacobi form (respectively Jacobi cusp form) of weight k and index S if ϕ satisfies the following conditions:

(1) $\phi|_{k,S}[\beta] = \phi$ for any $\beta \in \Gamma_J$;

(2) ϕ has a Fourier expansion of the form

$$\phi(\tau, u) = \sum_{\xi \in X(\mathbb{Q}), N \in \mathbb{Z}} c(N, \xi)\mathbf{e}(N\tau + \sigma_S(\xi, u)),$$

where $c(N, \xi) = 0$ unless $S_{\xi,N} := \left(\begin{smallmatrix} S & S\xi \\ {}^t\xi S & N \end{smallmatrix}\right)$ belongs to $\mathfrak{J}(\mathbb{Z})_{\geqslant 0}$ (respectively $\mathfrak{J}(\mathbb{Z})_+$).

We denote by $J_{k,S}(\Gamma_J)$ (respectively $J_{k,S}^{\mathrm{cusp}}(\Gamma_J)$) the space of Jacobi forms (respectively Jacobi cusp forms) of weight k and index S.

Define the dual of the lattice $\Lambda := X(\mathbb{Z}) = \mathfrak{o}^2$ with respect to the quadratic form σ_S by

$$\widetilde{\Lambda}(S) = \{x \in X(\mathbb{Q}) \mid \sigma_S(x, y) \in \mathbb{Z} \ \forall y \in \Lambda\}.$$

If $S \in \mathfrak{J}_2(\mathbb{Z})_+$, then the quotient $\widetilde{\Lambda}(S)/\Lambda$ is a finite group. Fix a complete representative $\Xi(S)$ of $\widetilde{\Lambda}(S)/\Lambda$ and denote by φ_ξ the characteristic function $\xi + \prod_{p<\infty} X(\mathfrak{o}_p) \in \mathcal{S}(X(\mathbb{A}_f))$. Any Jacobi form turns to be the sum of products of elliptic modular forms and theta functions by following lemma.

Lemma 5.5. *Assume $S \in \mathfrak{J}_2(\mathbb{Z})_+$. Let $\Xi(S)$ be a complete representative of $\widetilde{\Lambda}(S)/\Lambda$. Then any $\phi \in J_{k,S}(\Gamma_J)$ has an expression of the form*

$$\phi(\tau, u) = \sum_{\xi \in \Xi(S)} \phi_{S,\xi}(\tau)\theta^S_{\varphi_\xi}(\tau, u), \quad \phi_{S,\xi}(\tau) = \sum_{\substack{N \in \mathbb{Z} \\ N - \sigma_S(\xi,\xi) \geqslant 0}} c(N, \xi)\mathbf{e}((N - \sigma_S(\xi, \xi))\tau).$$

Furthermore, for each $\xi \in \Xi(S)$, $\phi_{S,\xi}(\tau)$ is an elliptic modular form of weight $k - 8$.

Proof. See [Kri96, § 2, example (iv)] and also the argument in [Ike01, p. 656]. □

Let k be a positive even integer and F be a modular form of weight k on $\mathfrak{T}$. Then we have the Fourier–Jacobi expansion

$$F\begin{pmatrix} W & u \\ {}^t\bar{u} & \tau \end{pmatrix} = \sum_{S \in \mathfrak{J}_2(\mathbb{Z})_{\geqslant 0}} F_S(\tau, u)\mathbf{e}((S, W)), \quad W \in \mathfrak{T}_2, \tau \in \mathbb{H} \text{ and } u \in X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}. \tag{5.2}$$

Lemma 5.6. *Keep the notation above. Assume $S \in \mathfrak{J}(\mathbb{Z})_+$. Then $F_S(\tau, u) \in J_{k,S}(\Gamma_J)$.*

Proof. It is easy to see that $F_S(\tau, u) = \sum_{T \in \mathfrak{J}_S^+} a(T)\mathbf{e}(c\tau)\mathbf{e}((T, {}^t\bar{v}u))$, where $\mathfrak{J}(\mathbb{Z})_+$ is the set of $T = \left(\begin{smallmatrix} S & v \\ {}^t\bar{v} & c \end{smallmatrix}\right) \in \mathfrak{J}^+$. Then the claim follows from the argument at [Ike01, p. 656]. □

Remark 5.7. Consider any holomorphic function $F(Z), Z = \begin{pmatrix} W & u \\ {}^t\bar{u} & \tau \end{pmatrix}, W \in \mathfrak{T}_2,\ \tau \in \mathbb{H}$, and $u \in X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ on $\mathfrak{T}$ which is invariant under $\Gamma \cap P(\mathbb{Q})$. Then one has the Fourier and Fourier–Jacobi expansion

$$F(Z) = \sum_{T \in \mathfrak{J}(\mathbb{Z})_{\geqslant 0}} A_F(T)\mathbf{e}((T, Z)) = \sum_{S \in \mathfrak{J}_2(\mathbb{Z})_{\geqslant 0}} F_S(\tau, u)\mathbf{e}((S, W)),$$

as in (5.2). By the proof of Lemma 5.5,

$$F_S(\tau, u) = \sum_{\xi \in \Xi(S)} F_{S,\xi}(\tau)\theta^S_{\varphi_\xi}(\tau, u), \quad F_{S,\xi}(\tau) = \sum_{\substack{N \in \mathbb{Z} \\ N - \sigma_S(\xi,\xi) \geqslant 0}} A_F(S_{\xi,N})\mathbf{e}((N - \sigma_S(\xi, \xi))\tau),$$

where $S_{\xi,N} = \begin{pmatrix} S & S\xi \\ {}^t\bar{\xi}S & N \end{pmatrix}$. In this paper, the function $F_{S,\xi}$ will be called by (S, ξ)-component of F.

We now discuss the relationship between the adelic setting and classical setting. (See [BJ77] for automorphic forms in the adelic setting.) Let ψ be a non-trivial additive character of $\mathbb{Q}\backslash\mathbb{A}$ and, for $S \in \mathfrak{J}_2(\mathbb{Z})_+$, put $\psi_S = \psi \circ \mathrm{tr}_S : Z \longrightarrow \mathbb{C}, z \mapsto \psi(\frac{1}{2}(S, z))$.

DEFINITION 5.8. Let $\tilde{F}$ be an automorphic function on $\mathbf{G}(\mathbb{A})$. For each $S \in \mathfrak{J}_2(\mathbb{Z})_+$, the Sth Fourier–Jacobi coefficient F_{ψ_S} of $\tilde{F}$ with respect to ψ is a function on $J(\mathbb{Q})\backslash J(\mathbb{A})$ given by

$$F_{\psi_S}(v_1 h) = \int_{Z(\mathbb{Q})\backslash Z(\mathbb{A})} \tilde{F}(z v_1 h)\psi_S^{-1}(z)\, dz, \quad v \in \mathbf{V}(\mathbb{A}), h \in H(\mathbb{A}).$$

Let F be a modular form in $\mathcal{M}_k(\Gamma)$, and consider its Fourier–Jacobi expansion

$$F = \sum_{S \in \mathfrak{J}_2(\mathbb{Z})_{\geqslant 0}} F_S(\tau, u)\mathbf{e}((S, W))$$

as above (see (5.2)). We are using F_{ψ_S} for the Fourier–Jacobi coefficient of $\tilde{F}$. We hope that this does not cause confusion with F_S, which is the Fourier–Jacobi coefficient of F. Let $\tilde{F}$ denote the automorphic form on $\mathbf{G}(\mathbb{A})$ corresponding to F by the strong approximation theorem, namely,

$$\tilde{F}(g) = j(g_\infty, E\sqrt{-1})^{-k} F(g_\infty E\sqrt{-1}) \quad \text{for } g = \gamma \cdot g_\infty \cdot k' \in \mathbf{G}(\mathbb{Q})\mathbf{G}(\mathbb{R})K. \tag{5.3}$$

Similarly if we write any element of $J(\mathbb{A})$ as $v_1 h = a \cdot v_{1,\infty} h_\infty \cdot k'_J \in J(\mathbb{Q})J(\mathbb{R})K_J$ where $K_J = K \cap J(\mathbb{A})$, one has

$$F_{(\psi_S^{\mathrm{st}})^2}(v_1 h) = F_S|_{k,S}[v_{1,\infty} h_\infty](\sqrt{-1}, 0),$$

by Lemma 5.6. It follows from this that $F_{(\psi_S^{\mathrm{st}})^2}(v_1 h)$ is left invariant under the action of the lattice $\Lambda = X(\mathfrak{o})$. We also identify Λ with a lattice of $Y(\mathbb{R})$ in an obvious way.

Fix $S \in \mathfrak{J}_2(\mathbb{Z})_+$. For each $\xi \in \Xi(S)$, we put

$$J^S_{\varphi_\xi}(h; F_{(\psi_S^{\mathrm{st}})^2}) := \int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} F_{(\psi_S^{\mathrm{st}})^2}(v_1 h)\overline{\Theta^{(\psi_S^{\mathrm{st}})^2}(v_1 h; \varphi_\xi)}\, dv_1.$$

Since $F_{(\psi_S^{\mathrm{st}})^2}(z v_1 h) = (\psi_S^{\mathrm{st}})^2(z) F_{(\psi_S^{\mathrm{st}})^2}(v_1 h)$, one has

$$J^S_{\varphi_\xi}(h; F_{(\psi_S^{\mathrm{st}})^2}) = \int_{(X \oplus Y)(\mathbb{Q})\backslash(X \oplus Y)(\mathbb{A})} F_{(\psi_S^{\mathrm{st}})^2}(v_1(x, y, 0)h)\overline{\Theta^{(\psi_S^{\mathrm{st}})^2}(v_1(x, y, 0)h; \varphi_\xi)}\, dv_1(x, y, 0).$$

By Lemma 4.2, $\Theta^{(\psi_S^{\mathrm{st}})^2}(v_1(x, y, 0)h; \varphi_\xi) = \Theta^{(\psi_S^{\mathrm{st}})^2}(v_1(x_\infty, y_\infty, 0)h; \varphi_\xi)$. Then one has

$$J^S_{\varphi_\xi}(h; F_{(\psi_S^{\mathrm{st}})^2}) = \int_{\Lambda\backslash X(\mathbb{R}) \oplus \Lambda\backslash Y(\mathbb{R})} F_S|_{k,S}[v_{1,\infty} h_\infty](\sqrt{-1}, 0)\overline{\Theta^{(\psi_S^{\mathrm{st}})^2}(v_{1,\infty} h; \varphi_\xi)}\, dv_{1,\infty}(x_\infty, y_\infty),$$

where $v_{1,\infty}(x_\infty, y_\infty) = v_1(x_\infty, y_\infty, 0)$. Take $h_\infty = \begin{pmatrix} y_\infty^{1/2} & x_\infty y_\infty^{-1/2} \\ 0 & y_\infty^{-1/2} \end{pmatrix} \in \mathbf{H}(\mathbb{R})$ so that $h_\infty\sqrt{-1} = x_\infty + \sqrt{-1}y_\infty$. Set $\tau = h_\infty\sqrt{-1}$ and $v_{1,\infty}h_\infty(\sqrt{-1}, 0) = (\tau, u)$. Put $L_\tau := \{\lambda_1\tau + \lambda_2 \in X(\mathbb{R})\otimes_\mathbb{R}\mathbb{C} \mid \lambda_i \in \Lambda, i = 1, 2\}$. Then by Lemma 4.1, one has

$$
\begin{aligned}
J^S_{\varphi_\xi}(h; F_{(\psi_S^{\mathrm{st}})^2}) &= \int_{\Lambda\backslash X(\mathbb{R})\oplus\Lambda\backslash Y(\mathbb{R})} F_S|_{k,S}[v_{1,\infty}h_\infty](\sqrt{-1},0)\overline{\theta^S_{\varphi_\xi}|_{8,S}[v_{1,\infty}h_\infty](\sqrt{-1},0)}\, dv_{1,\infty} \\
&\quad (\text{put } u = x_\infty + \tau y_\infty) \\
&= \frac{1}{j(g_\infty, i)^{k-8}} \int_{L_\tau\backslash X(\mathbb{R})\otimes_\mathbb{R}\mathbb{C}} F_S(\tau,u)\overline{\theta^S_{\varphi_\xi}(\tau,u)} e^{-4\pi(\mathrm{Im}(\tau))^{-1}\sigma_S(\mathrm{Im}(u),\mathrm{Im}(u))} \left|\frac{\partial(x_\infty, y_\infty)}{\partial u}\right| du \\
&= \frac{2^{-8}y_\infty^{-8}}{j(g_\infty, i)^{k-8}} \int_{L_\tau\backslash X(\mathbb{R})\otimes_\mathbb{R}\mathbb{C}} F_S(\tau,u)\overline{\theta^S_{\varphi_\xi}(\tau,u)} e^{-4\pi(\mathrm{Im}(\tau))^{-1}\sigma_S(\mathrm{Im}(u),\mathrm{Im}(u))}\, du \\
&\quad (\text{by Lemma 5.5}) \\
&= \frac{2^{-8}y_\infty^{-8}}{j(g_\infty, i)^{k-8}} \int_{L_\tau\backslash X(\mathbb{R})\otimes_\mathbb{R}\mathbb{C}} \sum_{\xi'\in\Xi(S)} F_{S,\xi'}(\tau)\theta^S_{\varphi_{\xi'}}(\tau,u)\overline{\theta^S_{\varphi_\xi}(\tau,u)} e^{-4\pi(\mathrm{Im}(\tau))^{-1}\sigma_S(\mathrm{Im}(u),\mathrm{Im}(u))}\, du \\
&= \frac{2^{-8}y_\infty^{-8}}{j(g_\infty, i)^{k-8}} F_{S,\xi}(\tau) 2^{-24}\det(S)^{-4} y_\infty^8 \\
&= 2^{-32}\det(S)^{-4} j(h_\infty, i)^{-(k-8)} F_{S,\xi}(\tau).
\end{aligned}
$$

Here we used the following formula to get the last equality: for each ξ,

$$
\int_{L_\tau\backslash X(\mathbb{R})\otimes_\mathbb{R}\mathbb{C}} \theta^S_{\varphi_{\xi'}}(\tau,u)\overline{\theta^S_{\varphi_\xi}(\tau,u)} e^{-4\pi(\mathrm{Im}(\tau))^{-1}\sigma_S(\mathrm{Im}(u),\mathrm{Im}(u))}\, du = \begin{cases} 2^{-24}\det(S)^{-4}y_\infty^8 & \text{if } \xi' = \xi, \\ 0 & \text{otherwise.} \end{cases}
$$

(Apply Lemma A.2 for $n = 16$ and combine this with $\mathrm{disc}(\sigma_S) = 2^{16}\,\mathrm{disc}(\lambda_S) = 2^{16}\det(S)^8$ by Lemma A.1.)

Summing up, we have proved the following.

Lemma 5.9. *For $\tau = h_\infty i$, $j(h_\infty, i)^{(k-8)} J^S_{\varphi_\xi}(h_\infty; F_{(\psi_S^{\mathrm{st}})^2}) = C_S F_{S,\xi}(\tau)$, where $C_S = 2^{-32}\det(S)^{-4}$.*

In the next section, we will prove $J^S_{\varphi_\xi}(h; F_{(\psi_S^{\mathrm{st}})^2})$ is a section of a degenerate principal series representation of $\mathrm{SL}_2(\mathbb{A})$ if $\tilde{F}$ is an (adelic) Eisenstein series on $\mathbf{G}(\mathbb{A})$. By Lemma 5.9 above, we will conclude that $F_{S,\xi}(\tau)$ is an Eisenstein series in the classical sense.

6. Eisenstein series and their Fourier coefficients

Recall from [Kim93] an Eisenstein series. Let $\Gamma_\infty = \Gamma\cap\mathbf{N}(\mathbb{Q})$. For l a positive integer and $s \in \mathbb{C}$,

$$
E_{2l,s}(Z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, Z)^{-2l}|j(\gamma, Z)|^{-s}.
$$

When $s = 0$ and $2l > 18$, Karel [Kar74] computed the Fourier coefficients and showed that they have bounded denominators. Let

$$
E_{2l}(Z) = E_{2l,0}(Z) = \sum_{T\in\mathfrak{J}(\mathbb{Z})_+} a_{2l}(T)\mathbf{e}((T, Z)).
$$

Theorem 6.1 [Kar74]. *For $T \in \mathfrak{J}(\mathbb{Z})_+$, we have*

$$
a_{2l}(T) = C_{2l}\det(T)^{2l-9}\prod_{p\mid\det(T)} f_T^p(p^{9-2l}),
$$

where $C_{2l} = 2^{15}\prod_{n=0}^{2}(2l-4n)/(B_{2l-4n})$, *and* f_T^p *is a monic polynomial with rational integer coefficients of degree* $d = \mathrm{ord}_p(\det(T))$. *It satisfies the functional equation*

$$X^d f_T^p(X^{-1}) = f_T^p(X).$$

Here B_{2k} is the Bernoulli number; $\zeta(2k) = (2^{2k-1}\pi^{2k}B_{2k})/(2k)!$. If n_{2l} is the numerator of $\prod_{n=0}^{2} B_{2l-4n}$, then $n_{2l}E_{2l}(Z)$ has rational integer Fourier coefficients. The functional equation of f_T^p is implicit in [Kar74], and it is stated explicitly in [Kim93, p. 185].

COROLLARY 6.2. *Keep the notation in the theorem above. Set* $\widetilde{f}_T^p(X) := X^d f_T^p(X^{-2})$, *where* $d = \mathrm{ord}_p(\det(T))$. *Then*

$$a_{2l}(T) = C_{2l}\det(T)^{(2l-9)/2}\prod_{p|\det(T)}\widetilde{f}_T^p(p^{(2l-9)/2}),$$

and $\widetilde{f}_T^p(X) = \widetilde{f}_T^p(X^{-1})$.

We can interpret this from the degenerate principal series as in the Siegel case [Kud08]. Let K_∞ be the stabilizer of $E\sqrt{-1}$ in $\mathbf{G}(\mathbb{R})$, where $E = \mathrm{diag}(1,1,1) \in \mathfrak{J}(\mathbb{Q})$. It is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$, and its complexification $K_{\infty,\mathbb{C}}$ is conjugate in $\mathbf{G}(\mathbb{C})$ to $\mathbf{M}(\mathbb{C})$ by the Cayley transform. Let $K = K_\infty\prod_p K_p$, where $K_p = \mathbf{G}(\mathbb{Z}_p)$. By the strong approximation theorem, $\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{Q})\mathbf{G}(\mathbb{R})K$.

For $s \in \mathbb{C}$, let $I(s)$ be the degenerate principal series representation of $\mathbf{G}(\mathbb{A})$ consisting of any smooth, K-finite function $f : \mathbf{G}(\mathbb{A}) \longrightarrow \mathbb{C}$ such that

$$f(pg) = \delta_{\mathbf{P}}^{1/2}(p)|\nu(p)|_{\mathbb{A}}^{s}(g)$$

for any $p \in \mathbf{P}(\mathbb{A})$ and any $g \in \mathbf{G}(\mathbb{A})$ where $\mathbf{P} = \mathbf{MN}$ is the Siegel parabolic subgroup. Note that the modulus character δ_P is given by $\delta_{\mathbf{P}}(mn) = |\nu(m)|_{\mathbb{A}}^{18}$. We denote it also by $I(s) = \mathrm{Ind}_{\mathbf{P}(\mathbb{Q}_p)}^{\mathbf{G}(\mathbb{Q}_p)}|\nu(g)|^s$.

Let $\Phi(g,s) = \Phi_\infty(g,s)\otimes\bigotimes_p\Phi_p(g,s)$ be a standard section in $I(s)$. Then one can define the Siegel Eisenstein series

$$E(g,s,\Phi) = \sum_{\gamma\in\mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})}\Phi(\gamma g,s).$$

It satisfies the functional equation

$$E(g,s,\Phi) = E(g,-s,M(s)\Phi), \quad M(s): I(s)\longrightarrow I(-s), \quad M(s)\Phi(g) = \int_{\mathbf{N}(\mathbb{A})}\Phi(ng,s)\,dn.$$

Now $\mathbf{G}(\mathbb{R}) = \mathbf{P}(\mathbb{R})K_\infty$, and hence Φ_∞ is determined by its restriction to K_∞. We choose

$$\Phi_\infty(k,s) = \nu(\mathbf{k})^{2l},$$

where $\mathbf{k} \in \mathbf{M}(\mathbb{C})$ corresponds to $k \in K_\infty$ by the Cayley transform. Hence $\Phi(mnk,s) = |\nu(m)|_{\mathbb{A}}^{s+9}\nu(\mathbf{k})^{2l}$.

By [Bai70, p. 527], given $Y\sqrt{-1} \in \mathfrak{T}$, there exists $m \in \mathbf{M}(\mathbb{R})$ such that $m(E\sqrt{-1}) = Y\sqrt{-1}$. Hence $p_X m(E\sqrt{-1}) = X + Y\sqrt{-1}$. Let $g = p_X m$.

Now for $\gamma \in \Gamma$, by Iwasawa decomposition, $\gamma g = nm'k$ with $n \in \mathbf{N}(\mathbb{R})$, $m' \in \mathbf{M}(\mathbb{R})$, and $k \in K_\infty$. Then

$$\gamma g(E\sqrt{-1}) = \gamma Z = nm'(E\sqrt{-1}) = X_1 + Y_1\sqrt{-1}.$$

Hence $m'(E\sqrt{-1}) = Y_1\sqrt{-1}$ and $n = p_{X_1}$. On the other hand,

$$j(\gamma g, E\sqrt{-1}) = j(\gamma, Z)j(g, E\sqrt{-1}) = j(m', E\sqrt{-1})j(k, E\sqrt{-1}).$$

Here $j(g, E\sqrt{-1}) = j(m, E\sqrt{-1}) = \det(Y)^{-1}$, $j(m', E\sqrt{-1}) = \det(Y_1)^{-1}$. By [BB66, p. 500],

Lemma 6.3. *For $k \in K_\infty$, $j(k, E\sqrt{-1}) = \nu(\mathbf{k})^{-1}$, and hence $|j(k, E\sqrt{-1})| = 1$.*

So

$$\det(Y_1) = \frac{\det(Y)}{|j(\gamma, Z)|}, \quad j(k, E\sqrt{-1}) = \frac{j(\gamma, Z)}{|j(\gamma, Z)|}.$$

Therefore,

$$\Phi_\infty(\gamma g, s) = \nu(m')^{-s-9}\nu(\mathbf{k})^{2l} = \det(Y)^{s+9} j(\gamma, Z)^{-2l}|j(\gamma, Z)|^{-s-9+2l}.$$

Hence as in [Kud08], for $\Phi(g, s) = \Phi_\infty(g, s) \otimes \bigotimes_p \Phi_p(g, s)$, $\Phi_\infty(g, s) = \nu(\mathbf{k})^{2l}$, and $\Phi_p(g, s) = \Phi_p^0(g, s)$, the normalized spherical section for all p,

$$E(g, s, \Phi) = \det(Y)^{s+9} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, Z)^{-2l}|j(\gamma, Z)|^{-s-9+2l}.$$

Hence

$$E(g, s, \Phi) = \det(Y)^{s+9} E_{2l, s+9-2l}(Z) = j(g, E\sqrt{-1})^{-(s+9)} E_{2l, s+9-2l}(Z).$$

Summing up, we have proved the following.

Proposition 6.4. *The adelic Eisenstein series $E(g, 2l-9, \Phi)$ on $\mathbf{G}(\mathbb{A})$ which is associated to a standard section of $I(2l-9)$ corresponds to $E_{2l,0}(Z)$ via (5.3).*

Let $I(s) = \bigotimes I_p(s)$ and $I_p(s)$ be the p-adic degenerate principal series. Then we have the following proposition.

Proposition 6.5 [Wei03]. *$I_p(s)$ is irreducible except at $s = \pm 1, \pm 5, \pm 9$.*

Remark 6.6. In terms of representation theory, the singular modular forms of weight 4 and 8 constructed in [Kim93] are subrepresentations of $I(s)$ when $s = -5, -1$, respectively.

7. Fourier–Jacobi expansion of Eisenstein series on $E_{7,3}$

As seen in § 4.2 (see Lemma 5.5 and Lemma 5.6), for each $S \in \mathfrak{J}(\mathbb{Z})_+$, the Sth Fourier–Jacobi coefficient of a modular form F on $\mathfrak{T}$ is represented by the sum of the products of theta series and elliptic modular forms. In this section we shall prove these elliptic modular forms turn to be Eisenstein series on $\mathbb{H}$ if F is an Eisenstein series. To do this we generalize the argument in [Ike94, § 3] in our setting and by virtue of Lemma 5.9 this enable us to work on the adelic setting which is much simpler than the classical setting.

Let ω be a unitary character of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ and $s \in \mathbb{C}$. Let $\mathbb{K} = \mathrm{SL}_2(\widehat{\mathbb{Z}}) \times \mathrm{SO}(2)$ be the standard maximal compact subgroup of $\mathrm{SL}_2(\mathbb{A})$. We denote by $I(\omega, s)$, the degenerate principal series representation of $\mathbf{G}(\mathbb{A})$ consisting of any function $f : \mathbf{G}(\mathbb{A}) \longrightarrow \mathbb{C}$ such that

$$f(pg) = \delta_{\mathbf{P}}^{1/2}(p)|\nu(p)|_{\mathbb{A}}^s \omega(\nu(p)) f(g)$$

for any $p \in \mathbf{P}(\mathbb{A})$ and any $g \in \mathbf{G}(\mathbb{A})$. Recall that $\delta_P^{1/2}(mn) = |\nu(m)|_{\mathbb{A}}^9$. Similarly we also define the space $I_1(\omega, s)$ consisting of any smooth, $\mathbb{K}$-finite function $f : \mathrm{SL}_2(\mathbb{A}) \longrightarrow \mathbb{C}$ such that

$$f(pg) = \delta_B^{1/2}(p)|a|_{\mathbb{A}}^s \omega(\nu(p)) f(g)$$

for any $p = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \in B(\mathbb{A})$ and any $g \in \mathrm{SL}_2(\mathbb{A})$. Here B is the Borel subgroup of SL_2 which consists of upper-triangular matrices and $\delta_B^{1/2}(p) = |a|_{\mathbb{A}}$ for $p = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \in B(\mathbb{A})$. For any section $f \in I(\omega, s)$, we define the Eisenstein series on $\mathbf{G}(\mathbb{A})$ of type (ω, s) by

$$E(g; f) := \sum_{\mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})} f(\gamma g), \quad g \in \mathbf{G}(\mathbb{A}).$$

Let ψ be a non-trivial additive character of $\mathbb{Q}\backslash\mathbb{A}$ and, for $S \in \mathfrak{J}_2(\mathbb{Z})_+$, put $\psi_S = \psi \circ \mathrm{tr}_S : Z(\mathbb{A}) \longrightarrow \mathbb{C}$. Consider the Sth Fourier–Jacobi coefficient $E_S(v_1 h; f)$ of $E_S(g; f)$ with respect to ψ_S (see Definition 5.8). For each $\varphi \in \mathcal{S}(X(\mathbb{A}))$, put

$$E_{\psi_S,\varphi}(h) := \int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} E_S(v_1 h, f)\overline{\Theta^{\psi_S}(v_1 h; \varphi)}\, dv_1, \quad h \in \mathrm{SL}_2(\mathbb{A}). \tag{7.1}$$

The main purpose in this section is to prove the following key theorem.

Theorem 7.1. *Keep the notation above. Assume that φ is $\mathbb{K}$-finite, and hence the $\mathbb{C}$-span $\langle \omega_S(k)\varphi \mid k \in \mathbb{K}\rangle_{\mathbb{C}}$ is finite-dimensional. For $\mathrm{Re}(s) \gg 0$ we have the following:*

(1) *$R(h; f, \varphi) := \int_{\mathbf{V}(\mathbb{A})} f(\iota \cdot v_1 \cdot \iota_{e_3} \cdot h)\overline{\omega_S(v_1(\iota_{e_3} \cdot h))\varphi(0)}\, dv_1$ is a section of $I(\omega, s)$;*
(2) *$E_{\psi_S,\varphi}$ is an Eisenstein series on $\mathrm{SL}_2(\mathbb{A})$ associated to $R(h; f, \varphi)$.*

To prove this, we need some lemmas. Let $P = \mathbf{P}(\mathbb{Q}), G = \mathbf{G}(\mathbb{Q}), Q = \mathbf{Q}(\mathbb{Q})$. Note that Q is the normalizer of $\mathbf{V}(\mathbb{Q})$ in G. The double coset $P\backslash G/Q$ is bijective to the double coset of the Weyl group $W_P\backslash W_G/W_Q$. By [Car72, p. 64], each double coset of $W_P\backslash W_G/W_Q$ has unique element of minimal length, and they are $\{1, c_3(23), c_2c_3(13)\}$, where c_i is the Weyl group element attached to $2e_i$, and (ij) is the Weyl group element attached to $e_i - e_j$. Then $G = P\xi_2 Q \cup P\xi_1 Q \cup P\xi_0 Q$, and $P\xi_0 Q$ is the unique open cell, where $\xi_2 = 1$, $\xi_1 = c_3(23)$, and $\xi_0 = c_2c_3(13)$. In terms of the notation in [Bai70, p. 517], $\xi_2 = 1$, $\xi_1 = \iota_{e_3}\varphi_{23}$, and $\xi_0 = \iota_{e_2}\iota_{e_3}\varphi_{13}$, where $\varphi_{ij} = m_{e_{ij}} m_{-e_{ji}} m_{e_{ij}}$ for $i \neq j$.

Lemma 7.2. *For any $q \in Q$, q normalizes $Z(\mathbb{A})$, and if $\gamma \in G$ is not contained in the open cell $P\xi_0 Q$, then ψ_S is non-trivial on $\gamma^{-1}\mathbf{P}(\mathbb{A})\gamma \cap Z(\mathbb{A})$.*

Proof. Let $q = lv$ for $l \in \mathbf{L}(\mathbb{Q})$ and $v \in \mathbf{V}(\mathbb{Q})$, and $p_z \in Z(\mathbb{A})$. Then, since Z is the center of V, $qp_zq^{-1} = (lv)p_z(lv)^{-1} = lp_zl^{-1}$. If l is in the central torus of $\mathbf{L}$, then $lp_zl^{-1} = p_z$. Otherwise, $l \in \mathbf{L}'(\mathbb{Q})$. Here $\mathbf{L}' = \mathbf{H} \times \mathrm{Spin}(9, 1)$, where $\mathrm{Spin}(9, 1)$ is spanned by the unipotent subgroups $m_{xe_{12}}$ and $m_{xe_{21}}$. If $l \in \mathbf{H}(\mathbb{Q})$, then by Lemma 3.3, $lp_zl^{-1} = p_z$. Suppose $l = m_{xe_{12}}$. Then by Lemma 3.2, $m_{xe_{12}}p_z = p_B m_{xe_{12}}$ for $B = m_{xe_{12}}z = p_{z'} \in Z(\mathbb{A})$. Similarly, $m_{xe_{21}}p_z = p_{z''} \in Z(\mathbb{A})$. Hence we have proved $qZ(\mathbb{A})q^{-1} = Z(\mathbb{A})$.

We may assume that $\gamma = \xi_1, \xi_2$. If $\gamma = \xi_2 = 1$, $P(\mathbb{A}) \cap Z(\mathbb{A}) = Z(\mathbb{A})$. So ψ_S is not trivial on $\mathbf{P}(\mathbb{A}) \cap Z(\mathbb{A})$. Let $\gamma = \xi_1$. Using (3.7), we can compute easily that $\gamma^{-1}m_{\bar{x}e_{31}}\gamma = p_{xe_{12}} \in Z(\mathbb{A})$. Hence $\gamma^{-1}\mathbf{P}(\mathbb{A})\gamma \cap Z(\mathbb{A})$ contains the subgroup $\{p_{xe_{12}} \mid x \in \mathfrak{C}_{\mathbb{A}}\}$. So ψ_S is not trivial on $\gamma^{-1}\mathbf{P}(\mathbb{A})\gamma \cap Z(\mathbb{A})$. □

Let P_H be the Borel subgroup of H consisting of upper triangular matrices.

Lemma 7.3. *The right coset can be written as $P\backslash P\xi_0 Q = \xi_0 \cdot (Y(\mathbb{Q})\backslash\mathbf{V}(\mathbb{Q})) \cdot (P_H(\mathbb{Q})\backslash H(\mathbb{Q}))$.*

Proof. We can write $q \in Q$ as $q = slvh$ with s in the central torus, $l \in \mathrm{Spin}(9, 1)(\mathbb{Q})$, $v \in \mathbf{V}(\mathbb{Q})$, and $h \in H(\mathbb{Q})$. It is easy to show that $\xi_0 l\xi_0^{-1} \in \mathbf{M}'(\mathbb{Q})$, and $\xi_0 v(y)\xi_0^{-1} \in \mathbf{M}'(\mathbb{Q})$, and

$\xi_0 p_{ae_3}\xi_0^{-1} = p_{ae_1}$. By direct computation, $\xi_0 m_{xe_{21}}\xi_0^{-1} = m_{xe_{32}}$, and $\xi_0 m_{xe_{12}}\xi_0^{-1} = m_{xe_{23}}$. Also, $\xi_0 p_{ye_{13}}\xi_0^{-1} = m_{\bar{y}e_{31}}$, and $\xi_0 p_{ye_{23}}\xi_0^{-1} = m_{\bar{y}e_{21}}$. Note that $h(a)$ is identified with $p_{ae_3}p'_{-a^{-1}e_3}p_{ae_3}\iota_{e_3}^{-1}$. Hence $\xi_0 h(a)\xi_0^{-1} = p_{ae_1}p'_{-a^{-1}e_1}p_{ae_1}\iota_{e_1}^{-1} \in \mathbf{M}'(\mathbb{Q})$, giving the claim. $\square$

We have in analogy to [Ike94, p. 630] the following lemma.

Lemma 7.4. *It is true that:*

(1) $\iota \cdot v_1(0, y, z)\iota_{e_3}p_{be_3} = p_{be_3}k \cdot \iota \cdot v_1(0, y, z + by{}^t\bar{y})\iota_{e_3}$ *where* $k = m_{by_1e_{13}}m_{by_2e_{23}}$ *with* $\nu(k) = 1$;

(2) $\iota \cdot v_1(0, y, z)\iota_{e_3}h(a) = h(a) \cdot \iota \cdot v_1(0, ay, z)\iota_{e_3}$ *with* $\nu(h(a)) = a$;

(3) $\varphi_{13}\xi_0\iota_{e_3} = \iota$ *with* $\nu(\varphi_{13}) = 1$ *and* $\iota_{e_3}^2 = -1$.

Proof. Note that $v(0, y, z) = v_1(0, y, z)$. The proof of result (2) is straightforward by using result (1), and $\iota_{e_i}h(a)\iota_{e_i}^{-1} = h(a)$ for $i = 1, 2$, and $\iota_{e_3}h(a)\iota_{e_3}^{-1} = h(a^{-1})$. For result (1), use $\iota_{e_i}p_{be_3}\iota_{e_i}^{-1} = p_{be_3}$ for $i = 1, 2$, and $\iota \cdot m_{\bar{x}e_{3i}} \cdot \iota^{-1} = m_{-xe_{i3}}$ for $i = 1, 2$. Result (3) follows from the fact that $\varphi_{13} \in \mathbf{M}'$, and $\iota_{e_1} = \varphi_{13}\iota_{e_3}\varphi_{13}$. $\square$

Proof of Theorem 7.1. We first prove Theorem 7.1(2), namely,

$$E_{\psi_S,\varphi}(h) = \sum_{\gamma \in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} R(\gamma h; f, \varphi).$$

This series will be convergent for $\mathrm{Re}(s) \gg 0$ provided that the first assertion holds [Lan76]. In fact, one has

$$\begin{aligned} E_{\psi_S,\varphi}(h) &= \int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} E_S(v_1h, f)\overline{\Theta^{\psi_S}(v_1h;\varphi)}\,dv_1 = \int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} E(v_1h, f)\overline{\Theta^{\psi_S}(v_1h;\varphi)}\,dv_1 \\ &= \sum_{i=1,2}\sum_{\gamma\in P\backslash P\xi_iQ}\int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\gamma v_1h)\overline{\Theta^{\psi_S}(v_1h;\varphi)}\,dv_1 \\ &\quad + \sum_{\gamma\in P\backslash P\xi_0Q}\int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\gamma v_1h)\overline{\Theta^{\psi_S}(v_1h;\varphi)}\,dv_1. \end{aligned}$$

In the first integral above, by Lemma 7.2, there exists an element $z_0 = \gamma^{-1}p\gamma \in Z(\mathbb{A}) \cap \gamma^{-1}P(\mathbb{A})\gamma$ such that $\psi_S(z_0) \neq 1$. Clearly $\nu(p) = 1$. Then one has

$$\begin{aligned} \int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\gamma v_1h)\overline{\Theta^{\psi_S}(v_1h;\varphi)}\,dv_1 &= \int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\gamma z_0v_1h)\overline{\Theta^{\psi_S}(z_0v_1h;\varphi)}\,d(z_0v_1) \\ &= \overline{\psi_S(z_0)}\int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(p\gamma v_1h)\overline{\Theta^{\psi_S}(v_1h;\varphi)}\,dv_1 \\ &= \overline{\psi_S(z_0)}\int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\gamma v_1h)\overline{\Theta^{\psi_S}(v_1h;\varphi)}\,dv_1, \end{aligned}$$

which claims the vanishing of $\int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\gamma v_1 h)\overline{\Theta^{\psi_S}(v_1 h;\varphi)}\, dv_1$. By Lemma 7.3,

$$\begin{aligned}
E_{\psi_S,\varphi}(h) &= \sum_{\gamma_1\in Y(\mathbb{Q})\backslash\mathbf{V}(\mathbb{Q})}\ \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\xi_0\gamma_1\gamma v_1 h)\overline{\Theta^{\psi_S}(v_1 h;\varphi)}\, dv_1\\
&= \sum_{\gamma_1\in Y(\mathbb{Q})\backslash\mathbf{V}(\mathbb{Q})}\ \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\xi_0\gamma_1\gamma v_1 h)\overline{\Theta^{\psi_S}(v_1 h;\varphi)}\, dv_1\\
&\qquad \text{(transform } v_1 \text{ into } \gamma^{-1}v_1\gamma)\\
&= \sum_{\gamma_1\in Y(\mathbb{Q})\backslash\mathbf{V}(\mathbb{Q})}\ \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\xi_0\gamma_1 v_1\gamma h)\overline{\Theta^{\psi_S}((\gamma^{-1}v_1\gamma)h;\varphi)}\, dv_1\\
&\qquad \text{(use } J(\mathbb{Q})\text{-invariance of } \Theta^{\psi_S})\\
&= \sum_{\gamma_1\in Y(\mathbb{Q})\backslash\mathbf{V}(\mathbb{Q})}\ \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{\mathbf{V}(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\xi_0\gamma_1 v_1\gamma h)\overline{\Theta^{\psi_S}(\gamma_1 v_1(\gamma h);\varphi)}\, dv_1\\
&= \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{Y(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\xi_0 v_1\gamma h)\overline{\Theta^{\psi_S}(v_1(\gamma h);\varphi)}\, dv_1\\
&\qquad \text{(Poisson summation formula (4.8))}\\
&= \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{Y(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\xi_0 v_1\gamma h)\sum_{\ell\in Y(\mathbb{Q})}\overline{F_S(\omega_S((-\ell)v_1\gamma h)\varphi(0))}\, dv_1.
\end{aligned}$$

Transforming v_1 into $(-\ell)^{-1}v_1$, one has $f(\xi_0(-\ell)^{-1}v_1\gamma h) = f(\xi_0\gamma h)$, since ξ_0 and ℓ (hence $(-\ell)^{-1}$) are commutative up to the multiplication by an element of $P(\mathbb{Q})$, and $|\nu(P(\mathbb{Q}))|_{\mathbb{A}} = 1$. Hence

$$\begin{aligned}
E_{\psi_S,\varphi}(h) &= \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{Y(\mathbb{Q})\backslash\mathbf{V}(\mathbb{A})} f(\xi_0 v_1\gamma h)\sum_{\ell\in Y(\mathbb{Q})}\overline{F_S(\omega_S(v_1\gamma h)\varphi(0))}\, dv_1\\
&= \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{\mathbf{V}(\mathbb{A})} f(\xi_0 v_1\gamma h)\overline{F_S(\omega_S(v_1\gamma h)\varphi(0))}\, dv_1\\
&= \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{\mathbf{V}(\mathbb{A})} f(\xi_0 v_1(x,y,z)\gamma h)\overline{\omega_S(\iota_{e_3} v_1(-x,y,z)\gamma h)\varphi(0)}\, dv_1(x,y,z).
\end{aligned}$$

It is easy to see that ξ_0 commutes with $X(\mathbb{A})$ (see the proof of Lemma 7.3), $v_1(2x,0,0)v_1(-x,y,z) = v_1(x,y,z)$, and $\nu(X(\mathbb{A})) = 1$. Hence

$$\begin{aligned}
E_{\psi_S,\varphi}(h) &= \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{\mathbf{V}(\mathbb{A})} f(\xi_0 v_1\gamma h)\overline{\omega_S(\iota_{e_3} v_1\gamma h)\varphi(0)}\, dv_1\\
&\qquad \text{(transform } v_1 \text{ into } {\iota_{e_3}}^{-1}v_1\iota_{e_3})\\
&= \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{\mathbf{V}(\mathbb{A})} f(\xi_0\iota_{e_3}^{-1}v_1\iota_{e_3}\gamma h)\overline{\omega_S(v_1\iota_{e_3}\gamma h)\varphi(0)}\, dv_1.
\end{aligned}$$

By Lemma 7.4(3),

$$E_{\psi_S,\varphi}(h) = \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} \int_{\mathbf{V}(\mathbb{A})} f(\iota v_1\iota_{e_3}\gamma h)\overline{\omega_S(v_1\iota_{e_3}\gamma h)\varphi(0)}\, dv_1 = \sum_{\gamma\in P_H(\mathbb{Q})\backslash H(\mathbb{Q})} R(\gamma h; f,\varphi).$$

We now prove Theorem 7.1(1). Noting that

$$\iota\cdot v_1(x,y,z) = \iota\cdot v_1(x,0,0)\iota^{-1}\cdot\iota\cdot v_1(0,y,z) = v_1(0,-\bar{x},0)\cdot\iota\cdot v_1(0,y,z),$$

and $\nu(v_1(0,-\bar{x},0))=1$, one has

$$\begin{aligned}
&R(h;f,\varphi)\\
&=\int_{\mathbf{V}(\mathbb{A})} f(\iota\cdot v_1(x,y,z)\iota_{e_3}h)\overline{\omega_S(\iota_{e_3}h)\varphi(x)\psi_S\left(\frac{1}{2}(S,z)+\lambda_S(x,y)\right)}\,dv_1\\
&=\int_{X(\mathbb{A})}\int_{Y(\mathbb{A})}\int_{Z(\mathbb{A})} f(\iota\cdot v_1(0,y,z)\iota_{e_3}h)\overline{\omega_S(\iota_{e_3}h)\varphi(x)\psi_S\left(\frac{1}{2}(S,z)+\lambda_S(x,y)\right)}\,dv_1\\
&=\int_{Y(\mathbb{A})}\int_{Z(\mathbb{A})} f(\iota\cdot v_1(0,y,z)\iota_{e_3}h)\overline{\left(\int_{X(\mathbb{A})}\omega_S(\iota_{e_3}h)\varphi(x)\psi_S(\lambda_S(x,y))\,dx\right)\psi_S\left(\frac{1}{2}(S,z)\right)}\,dy\,dz\\
&=\int_{Y(\mathbb{A})}\int_{Z(\mathbb{A})} f(\iota\cdot v_1(0,y,z)\iota_{e_3}h)\overline{(F_S(\omega_S(\iota_{e_3}h)\varphi)(-y))\psi_S\left(\frac{1}{2}(S,z)\right)}\,dy\,dz\\
&=\int_{Y(\mathbb{A})}\int_{Z(\mathbb{A})} f(\iota\cdot v_1(0,y,z)\iota_{e_3}h)\overline{(\omega_S(h)\varphi)(y)\psi_S\left(\frac{1}{2}(S,z)\right)}\,dy\,dz.
\end{aligned}$$

We now compute the actions of $p_{be_3}, b\in\mathbb{A}$ and $h(a), a\in\mathbb{A}^\times$ respectively. By Lemma 7.4, one has

$$\begin{aligned}
R(p_{be_3}h;f,\varphi)&=\int_{Y(\mathbb{A})}\int_{Z(\mathbb{A})} f(\iota\cdot v_1(0,y,z+by^t\bar{y})\iota_{e_3}h)\overline{(\omega_S(p_{be_3}h)\varphi)(y)\psi_S\left(\frac{1}{2}(S,z+by^t\bar{y})\right)}\,dy\,dz\\
&\qquad\text{(transform } z \text{ into } z+by^t\bar{y} \text{ and } \omega_S(p_{be_3}h)=\omega_S(h))\\
&=R(h;f,\varphi).
\end{aligned}$$

By Lemma 7.4 again, one has

$$\begin{aligned}
&R(h(a)h;f,\varphi)\\
&=\int_{Y(\mathbb{A})}\int_{Z(\mathbb{A})} f(h(a)\cdot\iota\cdot v_1(0,ay,z)\iota_{e_3}h)\overline{(\omega_S(h(a)h)\varphi)(y)\psi_S\left(\frac{1}{2}(S,z)\right)}\,dy\,dz\\
&=\delta_{\mathbf{P}}^{1/2}(h(a))|a|_{\mathbb{A}}^s\omega(a)|a|_{\mathbb{A}}^8\int_{Y(\mathbb{A})}\int_{Z(\mathbb{A})} f(\iota\cdot v_1(0,ay,z)\iota_{e_3}h)\overline{(\omega_S(h)\varphi)(ay)\psi_S\left(\frac{1}{2}(S,z)\right)}\,dy\,dz.
\end{aligned}$$

Transform y into y/a and note that $d(y/a)=|a|_{\mathbb{A}}^{-16}\,dy$ and $\delta_{\mathbf{P}}^{1/2}(h(a))=|a|_{\mathbb{A}}^9$. Then

$$R(h(a)h;f,\varphi)=|a|_{\mathbb{A}}^{1+s}\omega(a)R(h;f,\varphi)=\delta_{P_H}^{1/2}(a)|a|_{\mathbb{A}}^s\omega(a)R(h;f,\varphi).$$

The smoothness and $\mathbb{K}$-finiteness follow from those of f and φ. Hence $R(h;f,\varphi)\in I_1(\omega,s)$. □

8. Compatible family of Eisenstein series

Definition 8.1. Let k be a positive integer. Let $h(\tau)$ be an elliptic modular form of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$. We denote by $\mathcal{V}(h)$, the $\mathbb{C}$-vector space spanned by $\{h|_k[\gamma],\gamma\in\mathrm{GL}_2(\mathbb{Q})^+\}$ where $h|_k[\gamma](\tau):=j(\gamma,\tau)^{-k}h(\gamma\tau)$.

Let $\Phi(X)=\Phi(\{X_p\}_p)=\bigotimes_p'\Phi_p(X_p)\in\bigotimes_p'\mathbb{C}[X_p,X_p^{-1}]$ where p runs over all prime numbers. Denote by $\mathcal{R}$ the set of all $\Phi(X)=\Phi(\{X_p\}_p)$ such that $\Phi_p(X_p)=\Phi_p(X_p^{-1})$ for any prime p. For each non-zero sequence of complex numbers $\{a_p\}_p$ indexed by all primes, the value of $\Phi(X)$ at $\{X_p\}_p=\{a_p\}_p$ is denoted by $\Phi(\{a_p\})$. For each positive even integer $k\geqslant 4$, let

$$E_k^1(\tau)=\sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}\\(c,d)=1}}(c\tau+d)^{-k},$$

which is the Eisenstein series of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$.

Definition 8.2. For a sufficiently large k_0, a compatible family of Eisenstein series is a family of elliptic modular forms, for even integer $k' \geqslant k_0$

$$g_{k'}(\tau) = b_{k'}(0) + \sum_{N \in \mathbb{Q}_{>0}} N^{(k'-1)/2} b_{k'}(N) q^N, \quad q = \mathbf{e}(\tau),$$

satisfying the following three conditions.

(1) For all $k' \geqslant k_0$, $g_{k'} \in \mathcal{V}(E^1_{k'})$.
(2) For each $N \in \mathbb{Q}^\times_+$, there exists $\Phi_N \in \mathcal{R}$ such that $b_{k'}(N) = \Phi_N(\{p^{(k'-1)/2}\}_p)$.
(3) There exists a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that $g_{k'} \in M_{k'}(\Gamma)$ for all $k' \geqslant k_0$. Here $M_{k'}(\Gamma)$ stands for the space of elliptic modular forms of weight k with respect to Γ.

Then by [Ike08, Lemma 10.2], we have the following lemma.

Lemma 8.3. *Let $f(\tau) = \sum_{n=1}^\infty c(n)q^n$ be a Hecke eigenform of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$ with $c(p) = p^{(k-1)/2}(\alpha_p + \alpha_p^{-1})$. Assume that there is a finite-dimensional representation $(u, \mathbb{C}^d)$ of $\mathrm{SL}_2(\mathbb{Z})$ and*

$$\vec{\Phi}_N := {}^t(\Phi_{1,N}, \ldots, \Phi_{d,N}) \in \mathcal{R}^d, \quad N \in \mathbb{Q}_{>0}$$

satisfying the following two conditions.

(1) *There exists a vector valued modular form $\vec{g}_{k'} = {}^t(g_{1,k'}, \ldots, g_{d,k'})$ which has*

$$\vec{g}_{k'}(\tau) = \vec{b}_{k'}(0) + \sum_{N \in \mathbb{Q}_{>0}} N^{(k'-1)/2} \vec{b}_{k'}(N) q^n, \quad (\vec{b}_{k'}(N) = {}^t(b_{1,k'}(N), \ldots, b_{d,k'}(N)), N \in \mathbb{Q}_{\geqslant 0})$$

of weight k' with type u for each sufficiently large even integers k', and hence this means that

$$\vec{g}_{k'}(\tau)|_{k'}[\gamma] := {}^t(g_{1,k'}|_{k'}[\gamma], \ldots, g_{d,k'}|_{k'}[\gamma]) = u(\gamma)\vec{g}_k(\tau) \quad \text{for any } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

(2) *Each component $g_{i,k'}, (1 \leqslant i \leqslant d)$ of $\vec{g}_{k'}(\tau)$ is a compatible family of Eisenstein series such that*

$$b_{i,k'}(N) = \Phi_{i,N}(\{p^{(k'-1)/2}\}_p).$$

Then $\vec{h}(\tau) := \sum_{N \in \mathbb{Q}_{>0}} N^{(k-1)/2} \vec{\Phi}_N(\{\alpha_p\}_p) q^N$ is a vector valued modular form of weight k with type u, and hence it satisfies

$$\vec{h}(\tau)|_k[\gamma] = u(\gamma)\vec{h} \quad \text{for any } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

9. Construction of cusp forms on the exceptional domain

In this section we shall prove our main theorem. The strategy is the same as in [Ike01, Ike08, Yam10].

For any positive integer $k \geqslant 10$, let $f(\tau) = \sum_{n=1}^\infty c(n)q^n$ be a Hecke eigenform of weight $2k-8$ with respect to $\mathrm{SL}_2(\mathbb{Z})$ with $c(p) = p^{(2k-9)/2}(\alpha_p + \alpha_p^{-1})$. Let us formally define a function on $\mathfrak{T}$ by

$$F(Z) = \sum_{T \in \mathfrak{J}(\mathbb{Z})_+} A_F(T)\mathbf{e}((T, Z)), \quad A_F(T) = \det(T)^{(2k-9)/2} \prod_p \widetilde{f}^p_T(\alpha_p), Z \in \mathfrak{T}.$$

Then we prove the following theorem.

Theorem 9.1. *$F(Z)$ is a non-zero cusp form of weight $2k$ with respect to Γ.*

Remark 9.2. If f has integer Fourier coefficients, then F also has integer Fourier coefficients. Just observe from Corollary 6.2 that $\widetilde{f_T^p}(X) = X^d + X^{-d} + a_1(X^{d-2} + X^{-(d-2)}) + \cdots + a_{(d-1)/2}$ $(X + X^{-1})$ if d is odd, and $\widetilde{f_T^p}(X) = X^d + X^{-d} + a_1(X^{d-2} + X^{-(d-2)}) + \cdots + a_{d/2}$ if d is even, where $d = \mathrm{ord}_p(\det(T))$ and a_i's are integers.

First of all we shall prove the convergence of $F(Z)$.

Lemma 9.3. *The series $F(Z)$ is absolutely and uniformly convergent on any compact domain of $\mathfrak{T}$.*

Proof. It is well known that $|\alpha_p| = 1$. By definition, $\widetilde{f_T^p}(X) = X^d f_T^p(X^{-2})$, and f_T^p is a monic polynomial with integer coefficients of degree $d = \mathrm{ord}_p(\det(T))$, i.e., $f_T^p(X) = X^d + a_1 X^{d-1} + \cdots + a_{d-1}X + a_d$. Let $M = \max\{|a_1|, \dots, |a_d|\}$. We use the identity from [Kar74, p. 187],

$$(1 - p^{-s})^{-1} S_p(T) = \sum_{m=0}^{\infty} \alpha_m(T) p^{-ms}, \quad \alpha_m(T) = \sum_X \omega_m^{(T,X)},$$

where $X \in \Lambda(3)_p / p^m \Lambda(3)_p$ and $\tau_i(X) \equiv 0 \pmod{p^{m(i-1)}}$ for $2 \leqslant i \leqslant 3$, and $2m \leqslant d$. Hence $|\alpha_m(T)| \leqslant p^{27m}$. We also have [Kar74, p. 197]

$$S_p(T) = (1 - p^{-s})(1 - p^{4-s})(1 - p^{8-s}) f_T^p(p^{9-s}).$$

Hence

$$f_T^p(X) = (1 - p^{-5}X)^{-1}(1 - p^{-1}X)^{-1} \sum_{m=0}^{\infty} \alpha_m(T) p^{-9m} X^m.$$

So $M \leqslant (d+1)^2 p^{18m}$. By the trivial estimate, $d + 1 \leqslant p^d$, hence we have $M \leqslant p^{2d} p^{9d} = p^{11d}$. Therefore, $|\widetilde{f_T^p}(\alpha_p)| \leqslant (d+1)M \leqslant p^{12d}$. Hence

$$|A_F(T)| \leqslant \det(T)^{k+12-1/2}.$$

Now we use the fact from [Bai70, p. 538], for $l > 8$,

$$\int_{R_3^+(\mathbb{R})} \det(X)^{l-9} e^{2\pi(X,Y)} \, dX = \pi^{12} (2\pi i)^{-3l} \prod_{n=0}^{2} \Gamma(l - 4n) \det(Y)^{-l},$$

where dX is the ordinary Euclidean measure. Hence

$$\left| \sum_{T \in \mathfrak{J}(\mathbb{Z})_+} A_F(T) e^{2\pi i (T,Z)} \right| \leqslant \sum_{T \in \mathfrak{J}(\mathbb{Z})_+} \det(T)^{k+12-1/2} e^{2\pi(T,Y)} \leqslant \int_{R_3^+(\mathbb{R})} \det(X)^{k+12-1/2} e^{2\pi(X,Y)} \, dX$$

converges. □

Clearly $F(Z + N) = F(Z)$ for $N \in \mathbf{N}(\mathbb{Z})$. Also $F(\gamma Z) = F(Z)$ for $\gamma \in \mathbf{M}'(\mathbb{Z})$. Thanks to Lemma 5.2, it is enough to prove

$$F(\iota_{e_3} Z) = j(\iota_{e_3}, Z)^{2k} F(Z). \tag{9.1}$$

We prove (9.1) by using results of previous sections. Fix $S \in \mathfrak{J}_2(\mathbb{Z})_+$. Since $F(Z)$ is invariant under $\Gamma \cap \mathbf{P}(\mathbb{Q})$ as mentioned above and is holomorphic by Lemma 9.3, then by Remark 5.7, one has the Fourier–Jacobi expansion:

$$F \begin{pmatrix} W & u \\ {}^t\bar{u} & \tau \end{pmatrix} = \sum_{S \in \mathfrak{J}_2(\mathbb{Z})_+} F_S(\tau, u) \mathbf{e}((S, W)), \tag{9.2}$$

$$F_S(\tau, u) = \sum_{\xi \in \Xi(S)} F_{S,\xi}(\tau) \theta^S_{\varphi_\xi}(\tau, u), \tag{9.3}$$

and

$$\begin{aligned} F_{S,\xi}(\tau) &= \sum_{\substack{N\in\mathbb{Z}\\ N-\sigma_S(\xi,\xi)\geqslant 0}} A_F(S_{\xi,N})\mathbf{e}((N-\sigma_S(\xi,\xi))\tau),\ S_{\xi,N} := \begin{pmatrix} S & S\xi \\ {}^t\bar{\xi}S & N\end{pmatrix} \\ &= \sum_{\substack{N\in\mathbb{Z}\\ N-\sigma_S(\xi,\xi)\geqslant 0}} \det(S_{\xi,N})^{(2k-9)/2}\prod_p \widetilde{f}^p_{S_{\xi,N}}(\alpha_p)\mathbf{e}((N-\sigma_S(\xi,\xi))\tau) \\ &= \det(S)^{(2k-9)/2}\sum_{\substack{N\in\mathbb{Z}\\ N-\sigma_S(\xi,\xi)\geqslant 0}} (N-\sigma_S(\xi,\xi))^{(2k-9)/2}\prod_p \widetilde{f}^p_{S_{\xi,N}}(\alpha_p)\mathbf{e}((N-\sigma_S(\xi,\xi))\tau). \end{aligned} \tag{9.4}$$

For the last equality above, we used the formula $\det(S_{\xi,N}) = \det(S)(N-\sigma_S(\xi,\xi))$ by using (2.3). The condition (9.1) is equivalent to claiming that $F_S(\tau,u)\in J_{k,S}(\Gamma_J)$ for any $S\in\mathfrak{J}_2(\mathbb{Z})_+$. Therefore for each fixed $S\in\mathfrak{J}_2(\mathbb{Z})_+$, we have only to check the condition

$$F_S|_{k,S}[w_1](\tau,u) = F_S(\tau,u) \quad \text{for } w_1 = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}. \tag{9.5}$$

By [Tak96, p. 124, (2.1)], for each $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, there exists a unitary matrix $u_S(\gamma) = (u_S(\gamma)_{\xi\eta})_{\xi,\eta\in\Xi(S)}$ such that

$$\theta^S_{\varphi_\xi}|_{k,S}[\gamma](\tau,u) = \sum_{\eta\in\Xi(S)} u_S(\gamma)_{\xi\eta}\theta^S_{\varphi_\eta}(\tau,u). \tag{9.6}$$

Further there exists a positive integer Δ_S depending on S such that u_S is trivial on $\Gamma(\Delta_S)\subset\mathrm{SL}_2(\mathbb{Z})$. Since the elements of $\{\theta^S_{\varphi_\xi}\mid \xi\in\Xi(S)\}$ are linearly independent over $\mathbb{C}$, to check (9.5) with (9.6) it suffices to prove that $\{F_{S,\xi}\}_{\xi\in\Xi(S)}$ is a vector valued modular form of weight $2k$ with type u_S.

For a sufficiently large positive integer k', we now turn to consider (S,ξ)-component $(\mathcal{E}_{2k',0})_{S,\xi}$ of the classical Eisenstein series

$$\mathcal{E}_{2k',0}(Z) := \frac{1}{C_{2k'}}E_{2k',0}(Z) = \sum_{T\in\mathfrak{J}(\mathbb{Z})_+}\widetilde{a}_{2k'}(T)\mathbf{e}((T,Z)),$$
$$\widetilde{a}_{2k'}(T) = \det(T)^{(2k'-9)/2}\prod_{p|\det(T)}\widetilde{f}^p_T(p^{(2k'-9)/2}),$$

on $\mathfrak{T}$, where $C_{2k'}$ is the constant in Theorem 6.1. Then one has

$$\begin{aligned} &\det(S)^{-(2k'-9)/2}(\mathcal{E}_{2k',0})_{S,\xi}(\tau) \\ &= \det(S)^{-(2k'-9)/2}\sum_{\substack{N\in\mathbb{Z}\\ N-\sigma_S(\xi,\xi)\geqslant 0}}\widetilde{a}_{2k'}\begin{pmatrix} S & S\xi\\ {}^t\bar{\xi}S & N\end{pmatrix}\mathbf{e}((N-\sigma_S(\xi,\xi))\tau) \\ &= \sum_{\substack{N\in\mathbb{Z}\\ N-\sigma_S(\xi,\xi)\geqslant 0}}(N-\sigma_S(\xi,\xi))^{(2k'-9)/2}\prod_{p|\det(S_{\xi,N})}\widetilde{f}^p_{S_{\xi,N}}(p^{(2k'-9)/2})\mathbf{e}((N-\sigma_S(\xi,\xi))\tau). \end{aligned}$$

Then by Lemma 5.9, Corollary 6.2, and Theorem 7.1, $\{\det(S)^{-(2k'-9)/2}(\mathcal{E}_{2k',0})_{S,\xi}\}_{k'\gg 0}$ makes up a family of Eisenstein series. Here we use $u_S|_{\Gamma(\Delta_S)} = 1$ to check the third condition of Definition 8.2. Applying Lemma 8.3 with (9.4), one can conclude that

$$F_{S,\xi} = \det(S)^{(2k-9)/2}\sum_{\substack{n\in\mathbb{Z}_{>0}\\ n=N-\sigma_S(\xi,\xi),N\in\mathbb{Z}}} n^{(2k-9)/2}\prod_{p|\det(S_{\xi,N})}\widetilde{f}^p_{S_{\xi,N}}(\alpha_p)q^n,$$

is a vector-valued modular form of weight $2k$ with type u_S. Since $A_F(1) = 1$, $F(Z)$ is not identically zero. This completes the proof of Theorem 9.1.

10. Hecke operators

Karel [Kar72] defined Hecke operators and showed that the Eisenstein series are eigenfunctions. We review his results, and show that the cusp form on $\mathfrak{T}$ constructed in the previous section is a Hecke eigenform.

Let $I_{\mathbf{W}}$ be the identity operator on $\mathbf{W}$. Let $\mathbf{Z}$ be the central torus of $\mathrm{GL}(\mathbf{W})$, i.e., for any field K,

$$\mathbf{Z}_K = \{\lambda I_{\mathbf{W}} \mid \lambda \in K,\, \lambda \neq 0\}.$$

Let $\widetilde{\mathbf{G}} = Z \cdot \mathbf{G}$. Then $\widetilde{\mathbf{G}}$ is a $\mathbb{Q}$-group. Define a rational character μ on $\widetilde{\mathbf{G}}$ by

$$\{gw_1, gw_2\} = \mu(g)\{w_1, w_w\} \quad \text{for all } w_1, w_2 \in \mathbf{W}.$$

Then μ is defined over $\mathbb{Q}$, and $Q(gw) = \mu(g)^2 Q(w)$. Let S be the connected component of the Lie group $\widetilde{\mathbf{G}}(\mathbb{R})$ containing the identity element of $\mathbf{G}(\mathbb{R})$. Define

$$\Psi = \{g \in S \mid g\mathbf{W}_{\mathfrak{o}} \subset \mathbf{W}_{\mathfrak{o}}\}.$$

Since S is a connected component containing the identity element, $\mu(g) > 0$ for all $g \in \Psi$. Recall that $e = (0,1,0,0)$ and $e' = (0,0,0,1)$ are elements of $\mathbf{W}_{\mathfrak{o}}$, and $\{w_1, w_2\} \in \mathbb{Z}$ for all $w_1, w_2 \in \mathbf{W}_{\mathfrak{o}}$. Hence $\mu(g) = \{ge, ge'\} \in \mathbb{Z}$. Hence we can define, for each $m \in \mathbb{Z}$, $m > 0$,

$$\Psi_m = \{g \in \Psi \mid \mu(g) = m\},$$

and $\Psi = \bigcup_{m=1}^{\infty} \Psi_m$.

Fix k. If $\rho = z\rho' \in \widetilde{\mathbf{G}}(\mathbb{R}) = Z(\mathbb{R})_+ \cdot \mathbf{G}(\mathbb{R})$ and F is a function on $\mathfrak{T}$, let $F(Z)|[\rho]_k = F(\rho' Z) j(\rho', Z)^{-k}$. If F is holomorphic, then $F|[\rho]_k = F$ for all $\rho \in \Gamma$ precisely when F is a modular form of weight k. Let F be a modular form on $\mathfrak{T}$ of weight k, and define

$$T(m) \cdot F = \sum_{\rho \in \Gamma \backslash \Gamma \Psi_m \Gamma} \rho \cdot F.$$

Actually, in [Kar72] Karel used $J(g, Z) = j(g, Z)^{-18}$ as an automorphy factor. However, his result works for $j(g, Z)$ in the exactly same way.

For later purpose in connection with representation theory, we shall modify Hecke operators for $\mathbf{G}(\mathbb{Q})$. For any element $H \in \mathbf{G}(\mathbb{Q})$, we define a modified action of H on F by

$$H \star F = v_H(\Gamma)^{-k/36} \sum_{\rho \in \Gamma \backslash \Gamma H \Gamma} \rho \cdot F, \quad v_H(\Gamma) := [H\Gamma H^{-1} : \Gamma],$$

where $[H\Gamma H^{-1} : \Gamma] = [H\Gamma H^{-1} : \Gamma \cap H\Gamma H^{-1}]/[\Gamma : \Gamma \cap H\Gamma H^{-1}]$.

Then we have the following proposition.

Proposition 10.1. *The Eisenstein series $E_{l,0}(Z)$ is a Hecke eigenform for each Hecke operator $T(m)$. In particular, it is also an eigenform for any $H \in \mathbf{G}(\mathbb{Q})$ with respect to the modified action $\star$.*

For any positive integer $k \geqslant 10$, let $f(\tau) = \sum_{n=1}^{\infty} c(n) q^n$ be a Hecke eigenform of weight $2k-8$ with respect to $\mathrm{SL}_2(\mathbb{Z})$ with $c(p) = p^{(2k-9)/2}(\alpha_p + \alpha_p^{-1})$. Let $F(Z) = \sum_{T \in \mathfrak{J}(\mathbb{Z})_+} A_F(T)\mathbf{e}((T, Z))$, $Z \in \mathfrak{T}$ be the modular form on $\mathfrak{T}$ which is constructed from f in previous section. Then by imitating Ikeda's idea in [Ike94, § 11] we will prove that $F(Z)$ is a Hecke eigenform for any $H \in \mathbf{G}(\mathbb{Q})$. Recall the normalized Eisenstein series

$$\mathcal{E}_{2k',0}(Z) = \sum_{T \in \mathfrak{J}(\mathbb{Z})_+} \widetilde{a}_{2k'}(T)\mathbf{e}((T,Z)), \quad \widetilde{a}_{2k'}(T) = \det(T)^{(2k'-9)/2} \prod_{p \mid \det(T)} \widetilde{f}_T^p(p^{(2k'-9)/2})$$

of weight $2k'$ which is also an eigenform for any $H \in \mathbf{G}(\mathbb{Q})$ by Proposition 10.1. For each $H \in \mathbf{G}(\mathbb{Q})$, by using $\mathbf{G}(\mathbb{Q}) = P(\mathbb{Q})\mathbf{G}(\mathbb{Z})$ (see [Bai70, p. 532, line -4]), one can choose $\{p_{n_i} \cdot m_i\}_{i=1}^r$, $n_i \in \mathfrak{J}(\mathbb{Q}), m_i \in M(\mathbb{Q})_+$ as the complete representatives of $\Gamma\backslash\Gamma H \Gamma$. Here $M(\mathbb{Q})_+$ is the subset of $M(\mathbb{Q})$ consisting of g with $\nu(g) > 0$. Then it is easy to see that $v_H(\Gamma)^{-1/36} = \nu(m_i)^{1/2}$ for each i. Henceforth we settle the convention that for each $T \in \mathfrak{J}(\mathbb{Z})_+$ and each $m \in M(\mathbb{Q})$, put $\widetilde{f}_{mT}(\{X_p\}_p) = 0$ if $mT \notin \mathfrak{J}(\mathbb{Z})_+$. Then one has

$$\begin{aligned}
&H \star \mathcal{E}_{2k',0}(Z) \\
&= v_H(\Gamma)^{-2k'/36} \sum_{i=1}^r (p_{n_i} \cdot m_i) \cdot \mathcal{E}_{2k',0}(Z) \\
&= \sum_{T \in \mathfrak{J}(\mathbb{Z})_+} \sum_{i=1}^r \nu(m_i)^{k'} \det(T)^{(2k'-9)/2} \prod_{p \mid \det(T)} \widetilde{f}_T^p(p^{(2k'-9)/2})\mathbf{e}((T, m_i Z + n_i)) \\
&\quad (\text{use } (T, m_i Z) = ((m_i^*)^{-1}T, Z) \quad \text{by (3.1) and } \det(m_i^* T) = \nu(m_i)^{-1}\det(T).) \\
&= \sum_{T \in \mathfrak{J}(\mathbb{Z})_+} \sum_{i=1}^r \nu(m_i)^{9/2}\mathbf{e}((m_i^* T, n_i)) \det(T)^{(2k'-9)/2} \prod_{p \mid \det(m_i^* T)} \widetilde{f}_{m_i^* T}^p(p^{(2k'-9)/2})\mathbf{e}((T,Z)).
\end{aligned}$$

From this, the Tth Fourier coefficient of $H \star \mathcal{E}_{2k',0}$ is

$$\sum_{i=1}^r \nu(m_i)^{9/2}\mathbf{e}((m_i^* T, n_i)) \det(T)^{(2k'-9)/2} \prod_{p \mid \det(m_i^* T)} \widetilde{f}_{m_i^* T}^p(p^{(2k'-9)/2}).$$

Put

$$\alpha_H(X) := \sum_{i=1}^r \nu(m_i)^{9/2}\mathbf{e}((m_i^* E, n_i)) \prod_{p \mid \det(m_i^* E)} \widetilde{f}_{m_i^* E}^p(X_p), X = \{X_p\}_p$$

which defines an element of $\bigotimes_p' \mathbb{C}[X_p, X_p^{-1}]$. Here $E = \mathrm{diag}(1,1,1) \in \mathfrak{J}(\mathbb{Q})$. Noting $\widetilde{a}_{2k'}(E) = 1 \neq 0$, by Proposition 10.1 one has

$$\alpha_H(\{p^{(2k'-9)/2}\}_p) \prod_{p \mid \det(T)} \widetilde{f}_T^p(p^{(2k'-9)/2}) = \sum_{i=1}^r \nu(m_i)^{9/2}\mathbf{e}((m_i^* T, n_i)) \prod_{p \mid \det(m_i^* T)} \widetilde{f}_{m_i^* T}^p(p^{(2k'-9)/2}).$$

By [Ike01, Lemma 10.1], one has the equality in $\bigotimes_p' \mathbb{C}[X_p, X_p^{-1}]$

$$\alpha_H(X) \prod_{p \mid \det(T)} \widetilde{f}_T^p(X_p) = \sum_{i=1}^r \nu(m_i)^{9/2}\mathbf{e}((m_i^* T, n_i)) \prod_{p \mid \det(m_i^* T)} \widetilde{f}_{m_i^* T}^p(X_p), X = \{X_p\}_p.$$

Then one has

$$\begin{aligned}
H \star F(Z) &= \sum_{T \in \mathfrak{J}(\mathbb{Z})_+} \sum_{i=1}^r \nu(m_i)^{9/2}\mathbf{e}((m_i^* T, n_i)) \det(T)^{(2k'-9)/2} \prod_{p \mid \det(m_i^* T)} \widetilde{f}_{m_i^* T}^p(\alpha_p)\mathbf{e}((T,Z)) \\
&= \sum_{T \in \mathfrak{J}(\mathbb{Z})_+} \alpha_H(\{\alpha_p\}_p) \det(T)^{(2k'-9)/2} \prod_{p \mid \det(T)} \widetilde{f}_T^p(\alpha_p)\mathbf{e}((T,Z)) = \alpha_H(\{\alpha_p\}_p)F(Z).
\end{aligned}$$

Hence we have proved the following theorem.

THEOREM 10.2. *$F(Z)$ is a Hecke eigenform for $\mathbf{G}(\mathbb{Q})$ with respect to the action $\star$.*

11. The degree-56 standard L-function

In this section we will compute the standard L-function of Hecke eigenforms constructed in the previous section and the Eisenstein series respectively. Let $F = F(Z)$ be the cusp form in Theorem 10.2 and $\widetilde{F}$ be the automorphic form on $\mathbf{G}(\mathbb{A})$ attached to F (see (5.3)). Let π_F be the cuspidal representation of $\mathbf{G}(\mathbb{A})$ attached to $\widetilde{F}$. Since F is a Hecke eigenform, one has the decomposition $\pi_F = \pi_\infty \otimes \bigotimes'_p \pi_p$. Then π_∞ is a holomorphic discrete series of the lowest weight $2k$ associated to $-2k\varpi_7$ in the notation of [Bou02]. We note that $-2k\varpi_7$ parametrizes a holomorphic discrete series when $2k > 17$ (cf. [Kna86, p. 158]). Since π_p is unramified for each prime p, it has a spherical vector whose Hecke eigenvalue for each element of $\mathbf{G}(\mathbb{Q}_p)$ coincides with that of a spherical vector in $\mathrm{Ind}_{\mathbf{P}(\mathbb{Q}_p)}^{\mathbf{G}(\mathbb{Q}_p)} |\nu(g)|^{2s_p}$ where $p^{s_p} = \alpha_p$. (This is clear from the proof of Theorem 10.2. Notice $2s_p$, not s_p. We can see it from Corollary 6.2 and Proposition 6.4. We are replacing $(2k-9)/2$ by s_p in Corollary 6.2.) Then by [Cas, Proposition 2.2.2] and Proposition 6.5,

$$\pi_p \simeq \mathrm{Ind}_{\mathbf{P}(\mathbb{Q}_p)}^{\mathbf{G}(\mathbb{Q}_p)} |\nu(g)|^{2s_p},$$

for any finite place p.

In order to compute the standard L-function of π_F, we use the Langlands–Shahidi method. Since $\mathbf{G}(\mathbb{Q}_p)$ is the split group of type E_7, we can compute its local L-factor. We follow the notation of [Kim05, § 2.7.8]. We consider the split exceptional group of type E_8, and its parabolic subgroup R whose Levi subgroup is GE_7, and its Borel subgroup B. By inducing in stages

$$\mathrm{Ind}_{R(\mathbb{Q}_p)}^{E_8(\mathbb{Q}_p)} \pi_p \otimes \exp(s\tilde{\alpha}, H_R(\,)) = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{E_8(\mathbb{Q}_p)} \exp(\chi, H_B(\,)),$$

where $\tilde{\alpha} = e_1 - e_9$, and $\chi = s(e_1 - e_9) + s_p(-e_1 + 2e_2 - e_9) + (8e_3 + 7e_4 + 6e_5 + 5e_6 + 4e_7 + 3e_8)$. Here $\rho_{E_6} = 8e_3 + 7e_4 + 6e_5 + 5e_6 + 4e_7 + 3e_8$ is the half-sum of positive roots of E_6. Then one can see that the unipotent radical of R is generated by 57 roots

$$\begin{gathered} e_i - e_9 \quad \text{for } i = 1, \dots, 8 \quad \text{and} \quad e_1 - e_j \quad \text{for } j = 2, \dots, 8, \\ e_1 + e_j + e_k \quad \text{for } 2 \leqslant j < k \leqslant 8 \quad \text{and} \quad -(e_i + e_j + e_9) \quad \text{for } 2 \leqslant i < j \leqslant 8. \end{gathered}$$

Then $e_1 - e_9$ gives rise to $1 - p^{-2s}$, and the remaining 56 roots give rise to the following local factors:

$e_1 - e_3, 1 - \alpha_p p^{8-s};\ e_1 + e_2 + e_8, 1 - \alpha_p^{-1} p^{8-s};$
$\quad e_3 - e_9, 1 - \alpha_p^{-1} p^{-8-s};\ -(e_2 + e_8 + e_9), 1 - \alpha_p p^{-8-s}$
$e_1 - e_4, 1 - \alpha_p p^{7-s};\ e_1 + e_2 + e_7, 1 - \alpha_p^{-1} p^{7-s};$
$\quad e_4 - e_9, 1 - \alpha_p^{-1} p^{-7-s};\ -(e_2 + e_7 + e_9), 1 - \alpha_p p^{-7-s}$
$e_1 - e_5, 1 - \alpha_p p^{6-s};\ e_1 + e_2 + e_6, 1 - \alpha_p^{-1} p^{6-s};$
$\quad e_5 - e_9, 1 - \alpha_p^{-1} p^{-6-s};\ -(e_2 + e_6 + e_9), 1 - \alpha_p p^{-6-s}$
$e_1 - e_6, 1 - \alpha_p p^{5-s};\ e_1 + e_2 + e_5, 1 - \alpha_p^{-1} p^{5-s};$
$\quad e_6 - e_9, 1 - \alpha_p^{-1} p^{-5-s};\ -(e_2 + e_5 + e_9), 1 - \alpha_p p^{-5-s}$
$e_1 - e_7, e_1 + e_7 + e_8, 1 - \alpha_p p^{4-s};\ e_1 + e_2 + e_4, -(e_3 + e_4 + e_9), 1 - \alpha_p^{-1} p^{4-s}$
$e_1 - e_8, e_1 + e_6 + e_8, 1 - \alpha_p p^{3-s};\ e_1 + e_2 + e_3, -(e_3 + e_5 + e_9), 1 - \alpha_p^{-1} p^{3-s}$
$e_1 + e_6 + e_7, e_1 + e_5 + e_8, 1 - \alpha_p p^{2-s};\ -(e_3 + e_6 + e_9), -(e_4 + e_5 + e_9), 1 - \alpha_p^{-1} p^{2-s}$
$e_1 + e_5 + e_7, e_1 + e_4 + e_8, 1 - \alpha_p p^{1-s};\ -(e_3 + e_7 + e_9), -(e_4 + e_6 + e_9), 1 - \alpha_p^{-1} p^{1-s}$

$e_1+e_3+e_8, e_1+e_4+e_7, 1-\alpha_p p^{-s};\ -(e_3+e_8+e_9), -(e_4+e_7+e_9), 1-\alpha_p^{-1}p^{-s}$
$e_1+e_3+e_7, e_1+e_4+e_6, 1-\alpha_p p^{-1-s};\ -(e_4+e_8+e_9), -(e_5+e_7+e_9), 1-\alpha_p^{-1}p^{-1-s}$
$e_1+e_3+e_6, e_1+e_4+e_5, 1-\alpha_p p^{-2-s};\ -(e_5+e_8+e_9), -(e_6+e_7+e_9), 1-\alpha_p^{-1}p^{-2-s}$
$e_1+e_3+e_5, -(e_2+e_3+e_9), 1-\alpha_p p^{-3-s};\ e_8-e_9, -(e_6+e_8+e_9), 1-\alpha_p^{-1}p^{-3-s}$
$e_1+e_3+e_4, -(e_2+e_4+e_9), 1-\alpha_p p^{-4-s};\ e_7-e_9, -(e_7+e_8+e_9), 1-\alpha_p^{-1}p^{-4-s}$
$e_1-e_2, 1-\alpha_p^3 p^{-s};\ e_2-e_9, 1-\alpha_p^{-3}p^{-s};\ e_1+e_5+e_6, 1-\alpha_p p^{-s};\ -(e_5+e_6+e_9), 1-\alpha_p^{-1}p^{-s}.$

Hence we have the degree-56 local L-function:

$$(1-\alpha_p p^{-s})^2(1-\alpha_p^{-1}p^{-s})^2\prod_{i=0}^{3}(1-\alpha_p^{3-2i}p^{-s})$$
$$\cdot\prod_{i=5}^{8}(1-\alpha_p p^{\pm i-s})(1-\alpha_p^{-1}p^{\pm i-s})\prod_{i=1}^{4}(1-\alpha_p p^{\pm i-s})^2(1-\alpha_p^{-1}p^{\pm i-s})^2.$$

Therefore, we have proved the following theorem.

Theorem 11.1. *The degree-56 standard L-function $L(s,\pi_F,St)$ of π_F is given by*

$$L(s,\pi_F,St)=L(s,\mathrm{Sym}^3\pi_f)L(s,\pi_f)^2\prod_{i=1}^{4}L(s\pm i,\pi_f)^2\prod_{i=5}^{8}L(s\pm i,\pi_f),$$

where $L(s,\mathrm{Sym}^3\pi_f)$ is the third symmetric power L-function.

Let $\Gamma_{\mathbb{C}}(s)=2(2\pi)^{-s}\Gamma(s)$. Then the local L-factor at ∞ is given by

$$L(s,\pi_\infty,St)=\Gamma_{\mathbb{C}}(s+\tfrac{3(2k-9)}{2})\Gamma_{\mathbb{C}}(s+\tfrac{2k-9}{2})\Gamma_{\mathbb{C}}(s+\tfrac{2k-9}{2})^2\prod_{i=1}^{4}\Gamma_{\mathbb{C}}(s+\tfrac{2k-9}{2}\pm i)^2\prod_{i=5}^{8}\Gamma_{\mathbb{C}}(s+\tfrac{2k-9}{2}\pm i),$$

and the completed L-function satisfies the functional equation

$$\Lambda(s,\pi_F,St)=L(s,\pi_\infty,St)L(s,\pi_F,St)=-\Lambda(1-s,\pi_F,St).$$

Note that the root number is -1, since the root number of $L(s,\mathrm{Sym}^3\pi_f)$ is -1 [CM04].

We have also proved the following theorem.

Theorem 11.2. *The standard L-function $L(s,E_{2l,0}(Z),St)$ of $E_{2l,0}(Z)$ is*

$$\begin{aligned}&L(s,E_{2l,0}(Z),St)\\&\quad=\zeta(s+l-\tfrac{9}{2})^2\zeta(s-l+\tfrac{9}{2})^2\zeta(s-3l+\tfrac{27}{2})\zeta(s-l+\tfrac{9}{2})\zeta(s+l-\tfrac{9}{2})\zeta(s+3l-\tfrac{27}{2})\\&\qquad\cdot\prod_{i=5}^{8}\zeta(s\pm i-l+\tfrac{9}{2})\zeta(s\pm i+l-\tfrac{9}{2})\prod_{i=1}^{4}\zeta(s\pm i-l+\tfrac{9}{2})^2\zeta(s\pm i+l-\tfrac{l}{2})^2.\end{aligned}$$

Remark 11.3. We write the degree-56 standard L-function of π_F as

$$L(s,\pi_F,St)=L(s,\mathrm{Sym}^3\pi_f)\prod_{i=-4}^{4}L(s+i,\pi_f)\prod_{i=-8}^{8}L(s+i,\pi_f).$$

This suggests the following parametrization of π_F. Let $\mathcal{L}$ be the (hypothetical) Langlands group over $\mathbb{Q}$, and let $\rho_f:\mathcal{L}\longrightarrow\mathrm{SL}_2(\mathbb{C})$ be the two-dimensional irreducible representation of $\mathcal{L}$

corresponding to π_f. Let Sym^n be the irreducible $(n+1)$-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. Note that if $n=2m-1$, $\mathrm{Im}(\mathrm{Sym}^n)\subset \mathrm{Sp}_{2m}(\mathbb{C})$, and if $n=2m$, $\mathrm{Im}(\mathrm{Sym}^n)\subset \mathrm{SO}_{2m+1}(\mathbb{C})$. We have the tensor product maps $\mathrm{SL}_2(\mathbb{C})\times \mathrm{Sp}_{2m}(\mathbb{C})\longrightarrow \mathrm{Sp}_{4m}(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})\times \mathrm{SO}_{2m+1}(\mathbb{C})\longrightarrow \mathrm{Sp}_{4m+2}(\mathbb{C})$. Hence $\rho_f\otimes\mathrm{Sym}^{16}:\mathcal{L}\times\mathrm{SL}_2(\mathbb{C})\longrightarrow \mathrm{Sp}_{34}(\mathbb{C})$, and $\rho_f\otimes\mathrm{Sym}^{8}:\mathcal{L}\times\mathrm{SL}_2(\mathbb{C})\longrightarrow \mathrm{Sp}_{18}(\mathbb{C})$. Let $\mathrm{Sym}^3\rho_f:\mathcal{L}\times\mathrm{SL}_2(\mathbb{C})\longrightarrow \mathrm{Sp}_4(\mathbb{C})$ be the parameter of $\mathrm{Sym}^3\pi_f$, where it is trivial on $\mathrm{SL}_2(\mathbb{C})$. Consider the parameter

$$\rho=\mathrm{Sym}^3\rho_f\oplus(\rho_f\otimes\mathrm{Sym}^{16})\oplus(\rho_f\otimes\mathrm{Sym}^{8}):\mathcal{L}\times\mathrm{SL}_2(\mathbb{C})\longrightarrow \mathrm{Sp}_4(\mathbb{C})\times\mathrm{Sp}_{34}(\mathbb{C})\times\mathrm{Sp}_{18}(\mathbb{C})\subset\mathrm{Sp}_{56}(\mathbb{C}).$$

Note that $E_7(\mathbb{C})\subset\mathrm{Sp}_{56}(\mathbb{C})$. We expect that ρ will factor through $E_7(\mathbb{C})$, and give rise to a parameter $\rho:\mathcal{L}\times\mathrm{SL}_2(\mathbb{C})\longrightarrow E_7(\mathbb{C})$, which parametrizes π_F.

Acknowledgements

We would like to thank T. Ibukiyama, T. Ikeda, H. Katsurada, M. Nevins, K. Takase, and S. Yamana for helpful discussions. We thank the referees for helpful remarks and encouragement.

Appendix

In this appendix we will compute the discriminant of some quadratic forms from § 4.2 and prove the orthogonal relation of theta functions in the proof of Lemma 5.9.

Let $S=\left(\begin{smallmatrix} a & u\\ \bar u & b\end{smallmatrix}\right)\in\mathfrak{J}_2(K)$ where K is a field whose characteristic is different from 2,3. Recall $\det(S)=ab-N(u)$ and the quadratic form $\lambda_S(x,y)=\frac12(S,x\,{}^t\bar y+y\,{}^t\bar x)$ on $X(K)$ (see (4.1)).

Lemma A.1. *Let $\mathrm{disc}(\lambda_S)$ be the discriminant of the quadratic form λ_S, i.e., the determinant of the representation matrix of λ_S. Then $\mathrm{disc}(\lambda_S)=\det(S)^8$.*

Proof. Let $S=\left(\begin{smallmatrix} a & u\\ \bar u & b\end{smallmatrix}\right)$, where $a,b\in K$ and $u\in\mathfrak{C}_K$. Let $x=\left(\begin{smallmatrix} x_1\\ x_2\end{smallmatrix}\right)$, $y=\left(\begin{smallmatrix} y_1\\ y_2\end{smallmatrix}\right)$ where $x_1,x_2,y_1,y_2\in\mathfrak{C}_K$. Let

$$\lambda_S(x,y)=\tfrac12(S,x\,{}^t\bar y+y\,{}^t\bar x)=\tfrac12\big(a(x_1\bar y_1+y_1\bar x_1)+b(x_2\bar y_2+y_2\bar x_2)+u(x_2\bar y_1+y_2\bar x_1)+(x_1\bar y_2+y_1\bar x_2)\bar u\big)$$

be the bilinear form given by S.

For $x\in\mathfrak{C}_K$, let $x=x_0e_0+\cdots+x_7e_7$. Then with respect to the basis, the matrix of λ_S is

$$\begin{pmatrix} aI_8 & X\\ {}^tX & bI_8\end{pmatrix}\in M_{16}(K),\quad X=(\mathrm{Tr}(e_i((-e_j)\bar u)))_{1\leqslant i,j\leqslant 8}.$$

Then the discriminant of the bilinear form is the determinant of the above matrix, which is given by $\mathrm{disc}(\lambda_S)=\det(abI_8-{}^tXX)$. Now we claim that ${}^tXX$ is a diagonal matrix. Clearly, for each j, we have

$$\sum_{k=0}^{7}(\mathrm{Tr}(e_k(-e_j)\bar u))^2=N(u).$$

Let $i\neq j$, and consider

$$\sum_{k=0}^{7}(\mathrm{Tr}(e_k(-e_i))\bar u)(\mathrm{Tr}(e_k(-e_j))\bar u).\tag{A.1}$$

For a given e_k, let $e_k(-e_i)=e_l$ and $e_k(-e_j)=e_{l'}$. Then we claim that there exists e_a such that $e_a(-e_i)=e_{l'}$ and $e_a(-e_j)=-e_l$. This implies that (A.1) $=0$. Now, from $e_le_i=e_{l'}e_j$, we have

$$e_l=(-e_je_{l'})(-e_i)=(e_je_{l'})e_i=-e_j(e_{l'}e_i),$$

by non-associativity. So $-e_le_j=e_je_l=e_{l'}e_i$. Let $e_a=e_l(-e_j)=e_{l'}e_i$. Therefore, $\mathrm{disc}(\lambda_S)=\det(S)^8$. □

In order to prove the orthogonal relation in the proof of Lemma 5.9, by the above lemma, we need to consider the following. Let n be a positive integer and T be a positive definite symmetric matrix of size n. Assume $T = (t_{ij})_{1\leqslant i\leqslant j\leqslant n}$ is even integral, i.e., $t_{ii} \in \mathbb{Z}$ for $i = 1, \dots, n$ and $t_{ij} \in \frac{1}{2}\mathbb{Z}$ for $1 \leqslant i < j \leqslant n$. For $\lambda \in \mathbb{Q}^n$, we define the theta function on $\mathbb{H} \times \mathbb{C}^n$ by

$$\theta_{[\lambda]}(T;\tau,z) = \sum_{x\in\mathbb{Z}^n} \mathbf{e}({}^t(x+\lambda)T(x+\lambda)\tau + 2{}^t(x+\lambda)Tz), \quad (\tau,z) \in \mathbb{H}\times\mathbb{C}^n, \quad \mathbf{e}(*) = e^{2\pi\sqrt{-1}*},$$

where $[\lambda]$ stands for the image of λ under the natural projection $\mathbb{Q}^n \longrightarrow \mathbb{Q}^n/\mathbb{Z}^n$ and the definition of the above theta function depends only on $[\lambda]$. Let Λ_T be a complete representative of $(2T)^{-1}\mathbb{Z}^n/\mathbb{Z}^n$.

LEMMA A.2. *For any $\lambda, \mu \in \Lambda_T$, the following orthogonal relation holds:*

$$\int_{(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z})^n} \theta_{[\lambda]}(T;\tau,z)\overline{\theta_{[\mu]}(T;\tau,z)}e^{-4\pi(\operatorname{Im}\tau)^{-1}T[\operatorname{Im} z]}\,dz = \begin{cases} 2^{-n}\det(T)^{-1/2}(\operatorname{Im}\tau)^{n/2} & \textit{if } \lambda=\mu, \\ 0 & \textit{otherwise.}\end{cases}$$

Proof. Put $z = a + \tau b, a, b \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
&\int_{(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z})^n} \theta_{[\lambda]}(T;\tau,z)\overline{\theta_{[\mu]}(T;\tau,z)}e^{-4\pi(\operatorname{Im}\tau)^{-1}T[\operatorname{Im} z]}\,dz \\
&= (\operatorname{Im}\tau)^n \int_{(\mathbb{R}/\mathbb{Z})^n} \sum_{x,y\in\mathbb{Z}^n} \left\{ \int_{(\mathbb{R}/\mathbb{Z})^n} \mathbf{e}(2{}^t(x+\lambda)Ta - 2{}^t(y+\mu)Ta)\,da \right\} \\
&\quad \cdot \mathbf{e}(2\sqrt{-1}({}^t(x+\lambda)Tb + {}^t(y+\mu)Tb)) \\
&\quad \cdot \mathbf{e}({}^t(x+\lambda)T(x+\lambda)\tau - {}^t(y+\mu)T(y+\mu)\overline{\tau})e^{-4\pi(\operatorname{Im}\tau)^{-1}T[\operatorname{Im} z]}\,db,
\end{aligned}$$

where $T[\operatorname{Im} z] = {}^t(\operatorname{Im} z)T(\operatorname{Im} z)$. Note that, for given $x, y \in \mathbb{Z}^n$, $2{}^t(x - y + \lambda - \mu)T \in \mathbb{Z}^n$ if and only if $\lambda = \mu$ by the definition. Therefore

$$\int_{(\mathbb{R}/\mathbb{Z})^n} \mathbf{e}(2{}^t(x+\lambda)Ta - 2{}^t(y+\mu)Ta)\,da = \begin{cases} 1 & \text{if } x = y \text{ and } \lambda = \mu, \\ 0 & \text{otherwise.}\end{cases}$$

If $\lambda = \mu$, we have

$$\begin{aligned}
&\int_{(\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z})^n} \theta_{[\lambda]}(T;\tau,z)\overline{\theta_{[\lambda]}(T;\tau,z)}e^{-4\pi(\operatorname{Im}\tau)^{-1}T[\operatorname{Im} z]}\,dz \\
&= (\operatorname{Im}\tau)^n \int_{(\mathbb{R}/\mathbb{Z})^n} \sum_{x\in\mathbb{Z}^n} e^{-4\pi(\operatorname{Im}\tau){}^t(b+x+\lambda)T(b+x+\lambda)}\,db = (\operatorname{Im}\tau)^n \int_{\mathbb{R}^n} e^{-4\pi(\operatorname{Im}\tau){}^tbTb}\,db.
\end{aligned}$$

Since T is diagonalizable by an orthogonal matrix over $\mathbb{R}$, we may assume that $T = \operatorname{diag}(t_1, \dots, t_n), t_i \in \mathbb{R}_{>0}$. Hence

$$\int_{\mathbb{R}^n} e^{-4\pi(\operatorname{Im}\tau){}^tbTb}\,db = \prod_{i=1}^n \int_{\mathbb{R}} e^{-4\pi(\operatorname{Im}\tau)t_it^2}\,dt = \prod_{i=1}^n \frac{1}{\sqrt{4(\operatorname{Im}\tau)t_i}} = 2^{-n}\det(T)^{-1/2}(\operatorname{Im}\tau)^{-n/2}.$$

Hence we have the claim. □

References

Bai70 W. L. Baily Jr., *An exceptional arithmetic group and its Eisenstein series*, Ann. of Math. (2) **91** (1970), 512–549.

BB66 W. L. Baily Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domain*, Ann. of Math. (2) **84** (1966), 442–528.

BJ77 A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, in *Automorphic forms, representations and L-functions*, Proceedings of Symposia in Pure Mathematics, vol. 33, part 1 (American Mathematics Society, Providence, RI, 1977), 189–207.

Bou02 N. Bourbaki, *Lie groups and Lie algebras* (Springer, Berlin, 2002), ch. 4–6.

Car72 R. W. Carter, *Simple groups of Lie type*, Pure and Applied Mathematics, vol. 28 (John Wiley & Sons, 1972).

Cas W. Casselman, *Introduction to admissible representations of p-adic groups*, http://www.math.ubc.ca/∼cass/.

CM04 J. Cogdell and P. Michel, *On the complex moments of symmetric power L-functions at* $s = 1$, Int. Math. Res. Not. IMRN **31** (2004), 1561–1617.

CP88 J. Cogdell and I. Piatetski-Shapiro, *Base change for the Saito–Kurokawa representations of* PGSp(4), J. Number Theory **30** (1988), 298–320.

Cox46 H. S. M. Coxeter, *Integral Cayley numbers*, Duke Math. J. **13** (1946), 561–578.

Fre53 H. Freudenthal, *Zur ebenen Oktavengeometrie*, Indag. Math. (N.S.) **15** (1953), 195–200.

Ike94 T. Ikeda, *On the theory of Jacobi forms and Fourier–Jacobi coefficients of Eisenstein series*, J. Math. Kyoto Univ. **34** (1994), 615–636.

Ike01 T. Ikeda, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. (2) **154** (2001), 641–681.

Ike08 T. Ikeda, *On the lifting of Hermitian modular forms*, Compositio Math. **144** (2008), 1107–1154.

Ike T. Ikeda, *On the lifting of automorphic representation of* $\mathrm{PGL}_2(\mathbb{A})$ *to* $\mathrm{Sp}_{2n}(\mathbb{A})$ *of* $\widetilde{\mathrm{Sp}_{2n+1}}(\mathbb{A})$ *over a totally real algebraic field*, available at: https://www.math.kyoto-u.ac.jp/∼ikeda/adelic1.pdf.

IH13 T. Ikeda and K. Hiraga, *On the Kohnen plus space for Hilbert modular forms of half-integral weight I*, Compositio Math. **149** (2013), 1963–2010.

Kar72 M. Karel, *On certain Eisenstein series and their Fourier coefficients*, PhD thesis, University of Chicago (1972).

Kar74 M. Karel, *Fourier coefficients of certain Eisenstein series*, Ann. of Math. (2) **99** (1974), 176–202.

Kim93 H. Kim, *Exceptional modular form of weight 4 on an exceptional domain contained in* $\mathbb{C}^{27}$, Rev. Mat. Iberoam. **9** (1993), 139–200.

Kim05 H. Kim, *On local L-functions and normalized intertwining operators*, Canad. J. Math. **57** (2005), 535–597.

Kna86 A. W. Knapp, *Representation theory of semisimple groups* (Princeton University Press, 1986).

Kri96 A. Krieg, *Jacobi forms of several variables and the Maass space*, J. Number Theory **56** (1996), 242–255.

Kud94 S. Kudla, *Splitting metaplectic covers of dual reductive pairs*, Israel J. Math. **87** (1994), 361–401.

Kud08 S. Kudla, *Some extensions of the Siegel–Weil formula*, in *Eisenstein series and applications*, Progress in Mathematics, vol. 258 (Birkhäuser, Boston, MA, 2008), 205–237.

Kur78 N. Kurokwa, *Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two*, Invent. Math. **49** (1978), 149–165.

Lan76 R. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, vol. 544 (Springer, 1976).

Pia83 I. I. Piatetski-Shapiro, *On the Saito–Kurokawa lifting*, Invent. Math. **71** (1983), 309–338.

Ste68 R. Steinberg, *Lectures on Chevalley groups*, Notes prepared by John Faulkner and Robert Wilson, Yale University (1968).

Tak96 K. Takase, *On two-fold covering group of* $\mathrm{Sp}(n, \mathbb{R})$ *and automorphic factor of weight 1/2*, Comment. Math. Univ. St. Pauli **45** (1996), 117–145.

Wei03 M. Weissman, *The Fourier–Jacobi map and small representations*, Represent. Theory **7** (2003), 275–299.

Yam10 S. Yamana, *On the lifting of elliptic cusp forms to cusp forms on quaternionic unitary groups*, J. Number Theory **130** (2010), 2480–2527.

Henry H. Kim henrykim@math.toronto.edu
Department of Mathematics, University of Toronto,
Toronto, Ontario M5S 2E4, Canada

and

Korea Institute for Advanced Study, Seoul, Korea

Takuya Yamauchi yamauchi@edu.kagoshima-u.ac.jp, tyama@math.toronto.edu
Department of Mathematics, Faculty of Education,
Kagoshima University, Korimoto 1-20-6, Kagoshima 890-0065, Japan

and

Department of Mathematics, University of Toronto,
Toronto, Ontario M5S 2E4, Canada