BI-LIPSCHITZ SUFFICIENCY OF JETS

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ABSTRACT. We give some theorems of bi-Lipschitz or C^1 sufficiency of jets which are expressed by means of transversality with respect to some strata of a stratification satisfying the (L) condition of T. Mostowski. This enables us to prove that the number of metric types of intersection of smooth transversals to a stratum of an (a) regular stratification of a subanalytic set is finite.

0. INTRODUCTION

In this paper we study the problem of sufficiency of jets. This is a classical problem of singularity theory. The question is to determine whether a function germ is characterized (up to some equivalence relation) by its Taylor expansion at the origin. Many authors have given versatile criteria during the three last decades [Wa, Ku1, Ku2, Ku3, KuT, TW, Ma, Ta, Y, Ko1, Ko2, Wi].

We give here several results about sufficiency of jets. We study the intersection of a stratified space with a transversal to a given stratum, the zero locus of maps and the maps themselves up to a homeomorphism. We focus on the bi-Lipschitz or C^1 equivalence in each case.

Our criteria for sufficiency of jets are proved by studying stratifications satisfying the (L) condition of T. Mostowski. This condition has been introduced to obtain the bi-Lipschitz triviality along the strata. Existence of stratifications satisfying this condition has been proved for complex analytic or real subanalytic sets [M, P].

It is important to note that for determinacy of transversal (section 4) we will *not* assume that the given stratum is a stratum of an (L) regular stratification (which would be a rather strong assumption). We will just fix an (L) regular stratification which is compatible with our given stratum. Actually such a stratification always exists for a given subanalytic set X . Then, we will put transversality conditions with respect to this stratification generalizing the results of [TW].

In the two strata case we will get some results of C^1 determinacy. It is interesting to note that these theorems will apply not only on manifolds but also on the more general class of spaces having an isolated singularity at the origin. Moreover the criteria obtained are better than the ones given in [Wa] or [Ta] in the sense that the order of determinacy is lower.

We end this paper by two further applications of the results of section 3. The first one is a finiteness result which is also a generalization of a result of [TW]. We prove that the

2000 *Mathematics Subject Classification.* 32C05, 58A20, 58A35.

Key words and phrases. Stratified sets, sufficiency of jets, bi-Lipschitz equivalence.

This paper is partially supported by the Polish KBN grant 0235/P03/2004/27.

number of Lipschitz types of intersection of smooth direct transversals at a given point is finite when the stratification satisfies the Whitney (a) condition (Theorem 6.1.1). The second one is about Kuo and Trotman's blowing up. This is a transformation of stratified spaces introduced by T. C. Kuo and D. Trotman which is known to improve the regularity.

In [TW] the authors provide explicit criteria using the (t) condition about stratifications. This condition has been introduced by R. Thom and is weaker than the classical ones (Whitney and Kuo-Verdier). It does not guarantee the topological triviality along the strata. It is proved in [Ku1, TW, Wi] that the (t) condition may induce the (w) condition of Kuo-Verdier. This observation is then used by the authors to prove some determinacy theorems for transversals to a stratum of a stratification. We emphasize in this paper that Lipschitz type conditions on stratifications have similar properties.

The paper is organized as follows. In the first section we introduce definitions, notations and show some basic results. In the second we study Lipschitz stratifications. In section 3, after recalling the notion of pull-back through a deformation of transversals we show our pull-back theorems for Lipschitz type conditions. These theorems enable us to state all the determinacy theorems of sections 4 and 5. In section 4 we deal with bi-Lipschitz sufficiency of jets and in section 5 we focus on C^1 sufficiency of jets.

Acknowledgement. The author is grateful to his wife for her very careful reading of the manuscript.

Notations. Throughout this paper n and m will be two fixed integers, $d = m + n$, Y will denote $\{0_{\mathbb{R}^n}\} \times \mathbb{R}^m$, N will denote $\mathbb{R}^n \times \{0_{\mathbb{R}^m}\}$ so that $\mathbb{R}^d = N + Y$. We denote by π the orthogonal projection onto Y .

Given $q \in \mathbb{R}^d$, we will write $d(q; Y)$ for the distance from q to Y . Let $\mathbb{G}_d = \bigcup_{l=m}^d \mathbb{G}_d^l$ where $\mathbb{G}_d^l$ is the Grassmannian of vector spaces of dimension l in $\mathbb{R}^d$.

Given a vector subspace A of $\mathbb{R}^d$ we will denote by π_A the orthogonal projection on A . We denote by $\mathbb{Z}_+$ the set of integers with the symbol $+$ as index (i. e. a_+ with $a \in \mathbb{Z}$). We also fix a subanalytic subset X of $\mathbb{R}^d$.

All considered stratifications will be assumed to be subanalytic. The letters i and j will stand for two elements of $\mathbb{Z}$ satisfying $i \leq j$.

We extend to $\mathbb{Z} \cup \mathbb{Z}_+$ the order of $\mathbb{Z}$ as follows. Let k and k' be two integers, then by convention $k_+ \leq k'_+$ if $k \leq k'$, $k_+ < k'_+$ if $k < k'$, and $k < k_+$. We also extend the addition by setting:

$$\begin{aligned} k + k'_+ &= (k + k')_+ \\ k_+ + k'_+ &= (k + k')_+. \end{aligned}$$

Let f and g be two functions defined on a subset A of $\mathbb{R}^d$. We will write $f \lesssim g$ whenever there exist a strictly positive constant C and a neighborhood U of the origin such that $f(q) \leq Cg(q)$ for all $q \in U \cap A$. We will write $f(q) \lesssim g(q)^{k_+}$ if there exists a function ψ defined at each point where f and g are defined, and a neighborhood U of the origin such that $f(q) \leq g(q)^k \psi(q)$ with $\psi(q)$ tending to 0 at the origin.

Let $k, k' \in \mathbb{Z} \cup \mathbb{Z}_+$. With the above notations if $|f(q)| \lesssim d(q; Y)^k$ and $|g(q)| \lesssim d(q; Y)^{k'}$ then $|f(q).g(q)| \lesssim d(q; Y)^{k+k'}$.

1. DEFINITIONS AND PRELIMINARIES

Definition 1.0.1. A **direct transversal** to Y is the germ of a mapping $v : N \rightarrow Y$, at least C^2 satisfying $v(0) = 0$. We will denote by Γ_v its graph.

Definition 1.0.2. We call a **horn neighborhood of order j of Γ_v** a neighborhood of type:

$$H(v; j) = \{q = (x; t) \in \mathbb{R}^n \times \mathbb{R}^m : |v_s(x) - t_s| \leq C|x|^j, s = 1, \dots, m\},$$

where C is a positive constant.

Definition 1.0.3. Let A be a subset of $\mathbb{R}^d$ containing Y and let k be an element of $\mathbb{Z} \cup \mathbb{Z}_+$. A function $w : A \rightarrow \mathbb{R}$ is called **rugose with exponent k** if for any point q of A :

$$(1.1) \quad |w(q) - w(\pi(q))| \lesssim d(q; Y)^k.$$

It is called **Lipschitz with exponent k** if it is rugose with exponent $k + 1$ and if for any couple $(q; q') \in A \times A$ such that $|q - q'| \leq \frac{1}{2c}d(q; Y)$ (for some $c > 1$) :

$$(1.2) \quad |w(q) - w(q')| \lesssim d(q; Y)^k |q - q'|.$$

Remark 1. (1) If the function is constant on Y we may omit the hypothesis “rugose with exponent $k + 1$ ” in the above definition as long as we withdraw the restriction on the couples $(q; q')$ and we change the inequality for:

$$(1.3) \quad |w(q) - w(q')| \lesssim \theta(q; q')^k |q - q'|.$$

where $\theta(q; q') = \max(d(q; Y); d(q'; Y))$.

(2) A mapping which is Lipschitz with exponent 0_+ and C^1 outside Y is C^1 in the sense of Whitney.

We recall a classical result about partitions of unity (see for instance [F](3.1.13)).

Proposition 1.0.4. *There exists a positive constant C such that for any family $(U_l)_{l \in L}$ of open subsets of $\mathbb{R}^d$ there exists a C^∞ partition of unity $(\alpha_l)_{l \in L}$ whose supports form a locally finite covering of the covering U (of $\bigcup_{l \in L} U_l$), and whose derivatives satisfy:*

$$|d_x \alpha_l| \leq \frac{C}{h(x)},$$

where $h(x) = \sup\{\min\{1, d(x; \mathbb{R}^d \setminus U_l)\} / l \in L\}$.

Proposition 1.0.5. *Let X be a closed subanalytic subset of $\mathbb{R}^d$ containing Y and let $k \leq i$. Let $w_0 \in \mathbb{R}$ and let w be a function defined on X , rugose with exponent $i + 1$ and Lipschitz with exponent k such that $w|_Y \equiv w_0$. Then w may be extended to the whole of $\mathbb{R}^d$ into a function still rugose with exponent $i + 1$ and Lipschitz with exponent k .*

Proof. First by Remark 1, (1.2) is true with no restriction on $(q; q')$ as long as we replace $d(q; Y)$ by $\theta(q; q')$. We fix a constant C such that for any couple $(q; q') \in X \times X$ the function w satisfies:

$$(1.4) \quad |w(q) - w(q')| \leq C |q - q'| \theta(q; q')^k.$$

Let $A_p = \{q \in \mathbb{R}^d / \frac{1}{3 \cdot 2^p} < d(q; Y) < \frac{1}{2^p}\}$. By (1.4), for any couple $(q; q')$ in $A_p \times A_p \cap X \times X$

$$(1.5) \quad |w(q) - w(q')| \leq C \frac{|q - q'|}{2^{kp}}.$$

Thus, using Banach's extension theorem (cf. [Ba]), we can extend $w|_{A_p \cap X}$ to the whole of A_p , to a function $\frac{C}{2^{kp}}$ Lipschitz. Let $\hat{w}$ be this extension. The function $\hat{w}$ satisfies (1.5). Hence for any couple of points $(q; q')$ in $A_p \times A_p$:

$$(1.6) \quad |\hat{w}(q) - \hat{w}(q')| \leq 3C|q - q'|\theta(q; q')^k.$$

We define the following subsets of A_p :

$$B_p = \{q \in A_p / d(q; A_p \cap X) \leq \frac{2}{3 \cdot 2^{p(i-k+1)}}\}$$

and

$$C_p = \{q \in A_p / d(q; A_p \cap X) \geq \frac{1}{3 \cdot 2^{p(i-k+1)}}\}.$$

By the preceding proposition there exists a partition of unity $(\alpha; 1 - \alpha)$ adapted to the covering of A_p given by $(B_p; C_p)$ which satisfies $|d_x \alpha| \leq \frac{C}{h(x)}$ where $h(x)$ is the function defined in the Proposition 1.0.4. But since $h(q) \geq \frac{1}{3 \cdot 2^{p(i-k+1)}}$ we get $|d_x \alpha| \leq 3C2^{p(i-k+1)}$ for all x in A_p .

Then by the Mean Value Theorem there exists a positive constant C (independent of p) such that:

$$|\alpha(q) - \alpha(q')| \leq C2^{p(i-k+1)}|q - q'| \leq 3Cd(q; Y)^{k-i-1}|q - q'|$$

for any couple $(q; q') \in A_p \times A_p$.

Moreover by definition if $q \in B_p$ then $d(q; A_p \cap X) \leq \frac{2}{3 \cdot 2^{p(i-k+1)}}$, which implies since q belongs to A_p :

$$(1.7) \quad d(q; A_p \cap X) \leq 2d(q; Y)^{i-k+1}.$$

We now set $w_p(q) := \alpha(q)\hat{w}(q) + (1 - \alpha(q))w_0$. On B_p , the function $\hat{w}$ is rugose of exponent $i + 1$. Given a point q of B_p , let q_0 be the point which realizes the distance to $A_p \cap X$. Then:

$$(1.8) \quad \begin{aligned} d(q_0; Y) &\leq d(q; Y) + |q - q_0| \\ &\leq d(q; Y) + 2d(q; Y)^{i-k+1} \quad (\text{by (1.7)}) \\ &\leq 3d(q; Y) \quad \text{for } k \leq i. \end{aligned}$$

Then we get using (1.6):

$$\begin{aligned} |\hat{w}(q) - w_0| &\leq |\hat{w}(q) - w(q_0)| + |w(q_0) - w_0| \\ &\leq C|q - q_0|\theta(q; q_0)^k + C(q_0; Y)^{i+1} \\ &\leq C|q - q_0|d(q; Y)^k + Cd(q_0; Y)^{i+1} \quad (\text{by (1.8)}) \\ &\leq d(q; A_p \cap X)d(q; Y)^k + Cd(q; Y)^{i+1} \\ &\leq d(q; Y)^{i+1} \quad (\text{by (1.7)}). \end{aligned}$$

In consequence w_p is also a function rugose with exponent $i + 1$. Let us show that w_p is a function Lipschitz with exponent k . Let $(q; q') \in A_p \times A_p$, we have:

$$\begin{aligned}
|w_p(q) - w_p(q')| &\leq |\alpha(q)\hat{w}(q) - \alpha(q')\hat{w}(q')| + |(\alpha(q) - \alpha(q'))w_0| \\
&\leq |\alpha(q)(\hat{w}(q) - w_0) - \alpha(q')(\hat{w}(q) - w_0)| \\
&\leq |\alpha(q) - \alpha(q')||\hat{w}(q) - w_0| + |\alpha(q')||\hat{w}(q) - \hat{w}(q')| \\
&\leq C|q - q'|d(q; Y)^{k-i-1}d(q; Y)^{i+1} + |q - q'|d(q; Y)^k \\
&\leq |q - q'|d(q; Y)^k.
\end{aligned}$$

This proves that $\hat{w}$ is Lipschitz with exponent k on A_p .

Again applying Proposition 1.0.4 we find a partition of unity $(\phi_p)_{p \in \mathbb{N}}$ adapted to the covering $\mathbb{R}^d \setminus Y$ by the A_p 's. Then as $d(A_p \setminus \bigcup_{q \neq p} A_q; \mathbb{R}^d \setminus A_p) \geq \frac{1}{3 \cdot 2^p}$ for any $q \in A_p$ we have: $|d_q \phi_p| \leq 3C \cdot 2^p$. This implies that $|d_q \phi_p| \leq \frac{3C}{d(q; Y)}$ for $q \in A_p$. For $q \notin A_p$ this is still true since $d\phi_p(q) = 0$.

This implies by the Mean Value Theorem:

$$(1.9) \quad |\phi_p(q) - \phi_p(q')| \lesssim \frac{|q - q'|}{d(q; Y)}$$

for any couple $(q; q') \in A_p \times A_p$ satisfying $|q - q'| \leq \frac{1}{2c}d(q; Y)$.

We thus set $w(q) = \sum_{p \in \mathbb{N}} \phi_p(q)w_p(q)$. As all the w_p 's are rugose with exponent $i + 1$, so is the function w . It remains to see that w is Lipschitz with exponent k . Let q and q' be two points of $\mathbb{R}^d$.

First remark that each point of $\mathbb{R}^d$ belongs to at most two of the A_p 's and consequently, in the sum defining w , at most two terms are nonzero at each point q .

Let $(q; q') \in \mathbb{R}^{2d}$ satisfying $|q - q'| \leq \frac{1}{2c}d(q; Y)$. Then

$$\begin{aligned}
|w(q) - w(q')| &= \left| \sum_{p \in \mathbb{N}} \phi_p(q)(w_p(q) - w_0) - \sum_{p \in \mathbb{N}} \phi_p(q')(w_p(q') - w_0) \right| \\
&\leq \left| \sum_{p \in \mathbb{N}} |\phi_p(q) - \phi_p(q')|(w_p(q) - w_0)| + \sum_{p \in \mathbb{N}} \phi_p(q')|w_p(q) - w_p(q')| \right| \\
&\leq 4C \frac{|q - q'|}{d(q; Y)} d(q; Y)^{i+1} + 4C|q - q'|d(q; Y)^k \\
&\leq 4C|q - q'|d(q; Y)^k \quad \text{for } i \geq k.
\end{aligned}$$

For $i \in \mathbb{Z}_+$ or $k \in \mathbb{Z}_+$ the proof is analogous. $\square$

Note that the Lipschitz constants of the extension may be expressed as a function of the constant initially given and the ratio may be bounded by some constants depending only on the dimension of the ambient space.

Remark 2. The extension $\hat{w}$, rugose with exponent $i + 1$ and Lipschitz with exponent k , may fulfill as well:

- (1) $\hat{w}$ is C^∞ on $\mathbb{R}^n \setminus X$.
- (2) Let $(W_l)_{l \in L}$ be a collection of open disjoint manifold of X , such that w is differentiable at any point of W_l for any l . Then $\hat{w}$ may be chosen differentiable at any point of W_l in $\mathbb{R}^d$ for all l .

To prove it, it is enough to replace in the above proof the extension provided by Banach's Theorem, by an extension which is C^∞ in the complement of X . Actually we can make it differentiable by considering its convolution with a smooth map.

1.1. Lipschitz distributions. Let X be a subset of $\mathbb{R}^d$. A l -**distribution** of $\mathbb{R}^d$ over X will be a continuous distribution $\Lambda : X \rightarrow \mathbb{G}_d^l$. We need a definition about distributions.

Given a direct transversal v , we denote by $\mathcal{D}_v$ the m -distribution generated by the vectors $(\partial_x v_s - \partial_s)$, $s = 1, \dots, m$, where $v = (v_1; \dots; v_m)$.

Definition 1.1.1. Let $\Lambda : X \rightarrow \mathbb{G}_d^l$ be an l -distribution. Then Λ is said to be **rugose with exponent i** if

$$|(P_q^\perp)|_Y| \lesssim d(q; Y)^i$$

where P_q denotes the orthogonal projection on $\Lambda(q)$ and $|\cdot|$ is the norm of linear maps.

It is said to be **Lipschitz with exponent k** if it is rugose with exponent $k + 1$ and if for any couple $(q; q') \in X \times X$ such that $|q - q'| \leq \frac{1}{2c}d(q; Y)$, we have:

$$|P_q - P_{q'}| \leq C|q - q'|d(q; Y)^k.$$

It is important to note that the integer k may be negative. Note that the case $k = 0_+$ corresponds to distributions that extend to C^1 distributions to the ambient space if the distribution is C^1 on its domain. This is a consequence of the Whitney extension theorem. This will enable us to provide theorems for C^1 sufficiency of jets.

It is well known that Lipschitz vector fields are integrable and generate a bi-Lipschitz one parameter group. Clearly a distribution is Lipschitz (with exponent 0) if and only if it admits a family of Lipschitz sections $\zeta_1, \dots, \zeta_m$ which induce a basis of Y at each point of Y . The one parameter groups generated by this family of vector fields give rise to an isotopy which induce a trivialization of X along Y (considering the variables of Y as parameters). So we can state:

Lemma 1.1.2. . Let X be a closed subset stratified by a stratification $\mathcal{S}$ and let $\Lambda : X \rightarrow \mathbb{G}_d^m$ be a Lipschitz m -distribution tangent to $\mathcal{S}$. Then X is bi-Lipschitz trivial along Y . If $X \setminus Y$ is smooth and if Λ is actually Lipschitz with exponent 0_+ and C^1 on $X \setminus Y$ then X is C^1 trivial along Y . $\square$

2. SOME REGULARITY CONDITIONS

Throughout this section X will denote a subset of $\mathbb{R}^d$, and Λ will be a mapping from X into $\mathbb{G}_d$. For instance it can be an m -distribution over X or the union of the tangent bundles of the strata of a stratification $\mathcal{S}$. In the latter case we will write $\Lambda(\mathcal{S})$ instead of Λ . Given such a subset we will denote by P_q the orthogonal projection onto Λ_q . The mapping v will be a fixed direct transversal. We recall that $\mathcal{D}_v$ denotes the distribution generated by the vectors $(\partial_x v_s - \partial_s)$, $s = 1, \dots, m$.

2.1. The (t^i) conditions. The (t^i) conditions generalize the (t) condition introduced by R. Thom which was devoted to the case of C^∞ transversals. We recall their definitions which originate in [TW] or [Wi].

Let U and V be two vector subspaces of $\mathbb{R}^d$ and let us recall the following definition [TW, Wi]:

$$(2.1) \quad \tau(U; V) = \sup_{u \in U - \{0\}} \inf_{v \in V - \{0\}} \tan \theta(u; v) = \sup \frac{|\pi_V^\perp(u)|}{|\pi_V(u)|}$$

where $\theta(u; v)$ denotes the angle between u and v .

Let $\Sigma(T) := \{v \in \text{Lin}(N; Y) / \Gamma_v \not\subset T\}$. Define also

$$\delta(A; B) = \sup_{u \in A, |u|=1} d(u; B)$$

and

$$\mu(A; B) = \inf_{u \in A, |u|=1} d(u; B^\perp)$$

for A and B vector subspaces of $\mathbb{R}^d$.

The following identity is very easy to check:

$$(2.2) \quad \delta(A \cap B; C \cap B) \leq \frac{\delta(A \cap B; C)}{\mu(B^\perp; C)}$$

for any given vector subspaces A, B, C of $\mathbb{R}^d$.

We start by recalling the definition of the (t^i) conditions (see [TW]). We will say that the sequence (y_s) is **i -flat with respect to** a sequence (x_s) if $|y_s| \ll |x_s|^i$.

Definition 2.1.1. Let v be a direct transversal. We will say that $(v; \Lambda)$ is (t^i) if no sequence $(x_s; y_s; \Lambda_{(x_s; y_s)})$ tending to 0 satisfies: $|y_s - v(x_s)|$ is i -flat with respect to (x_s) is $d(d_{x_s} v; \Sigma(\Lambda_{(x_s; y_s)}))$ is $(i - 1)$ -flat with respect to (x_s) .

In the case $\Lambda = \Lambda(\mathcal{S})$, the condition (t^i) can be characterized by the fact that any C^i direct transversal to Y is transverse to the other strata. We will write $(v; \mathcal{S})$ instead of $(v; \Lambda(\mathcal{S}))$.

Remark 3. It is proved in [TW] that:

$$\tau(Y; T) = 1/d(0; \Sigma(T)).$$

This implies in particular that the condition (t^0) , is the condition (w) of Kuo-Verdier [Ve].

Moreover, for $i \geq 1$ the (t^i) for the couple $(0; \Lambda)$ condition may be characterized by:

$$(2.3) \quad \frac{1}{\mu(Y; \Lambda_q)} \lesssim |x|^{1-i}.$$

in some horn neighborhood of the 0 map of order i . Note that in particular if $i = 1$ then the (t^i) condition asserts that $\mu(Y; T)$ is bounded below away from zero on X in a horn neighborhood of 0 of order 1.

Observe that if we work up to the mapping $\phi_v(x; y) := (x; y - v(x))$, we reduce the study to the zero transversal.

Therefore, in the case of an arbitrary transversal we get the following characterization of the condition (t^i) for $(v; \Lambda)$. The condition (t^i) holds for $(v; \Lambda)$ if and only if in a horn neighborhood of v of order i we have:

$$(2.4) \quad \frac{1}{\mu(\mathcal{D}_v(q); \Lambda_q)} \lesssim |x|^{1-i}.$$

The (t^i) condition may provide very explicit criteria for sufficiency of jets with respect to topological equivalence. In this paper we are interested in C^1 or bi-Lipschitz equivalence. For this, we need to introduce another condition which is expressed by means of another function μ_1 . Let us define for $m > 1$, A and B vector subspaces of $\mathbb{R}^d$:

$$\mu_1(A; B) = \inf\{|\pi_B(u)| \cdot |\pi_B(u')|/|u|, u' \in A, |u| = |u'| = 1, u \perp u'\}.$$

Define also μ_1 to be μ if $m = 1$.

In view of (2.4) it is natural to set the definition of the (t_1^j) conditions as follows.

Definition 2.1.2. We will say that $(v; \Lambda)$ satisfies **the (t_1^j) condition** if in some horn neighborhood of order j of Γ_v :

$$(2.5) \quad \frac{1}{\mu_1(\mathcal{D}_v(q); \Lambda_q)} \lesssim d(q; Y)^{1-j}.$$

For simplicity if $(v; \Lambda)$ is (t^i) and (t_1^j) we will **shorten it in $(t_1^{i,j})$** .

Remark 4. Observe again that if we work up to the mapping $\phi_v(x; y) := (x; y - v(x))$, we reduce the study to the zero transversal. More precisely, if we set $\Lambda'_q := d\phi_v(\Lambda_q)$ we get a distribution on $\phi_v^{-1}(X)$. Then $(0; \Lambda')$ is $(t_1^{i,j})$ if and only if so is the couple $(v; \Lambda)$.

2.2. Lipschitz type conditions. We are going to construct distributions which are tangent to Lipschitz stratifications in the sense of Mostowski. We will work with a stratification of $\mathbb{R}^d$, $\mathcal{S} = \{S_0, \dots, S_d\}$ with either $\dim S_j = j$ or $S_j = \emptyset$. We set $X_l := \cup_{j=1}^l S_j$.

We recall the definition of the condition of Mostowski [M]. For convenience, we write it for a couple $(\Lambda; \mathcal{S})$, where Λ is map $\Lambda : \mathbb{R}^d \rightarrow \mathbb{G}_d$, instead of just a stratification $\mathcal{S}$ but it comes to the same in the case where Λ_q is the tangent space to the stratum which contains q .

Let $\Lambda : \mathbb{R}^d \rightarrow \mathbb{G}_d$ with $\Lambda_q \subseteq T_q S_p$ where S_p is the stratum containing q . We denote by P_q the projection on Λ_q .

Following [M] we denote by $d_j(q)$ the distance from q to X_j .

Definition 2.2.1. Let $c > 1$ be a fixed constant. A **c -chain** of a point $q \in S_s$ relative to a stratification Σ , is a strictly decreasing sequence of indices

$$s = s_1 > s_2 > \dots > s_r = l$$

and a sequence of points $\{q_{s_p}\}$ where $q_{s_p} \in S_{s_p}$ and $q_{s_1} = q$, such that each s_p is the greatest integer ($< s_{p-1}$) for which:

$$d_k(q) \geq 2c^2 d_{s_p}(q) \quad \forall k, \quad l \leq k < s_p$$

$$\text{and } |q - q_{s_p}| \leq c d_{s_p}(q).$$

If there is no confusion, we will call the sequence of points $\{q_{s_i}\}$ a c -chain of q , or simply a chain of q . The length of such a chain is r , the number of its elements.

We will need two identities about chains that are straightforward to check from the definition. We have:

$$(2.6) \quad \begin{aligned} |q_{j_s} - q_{j_{s+1}}| &\leq 2^{n+1} c^2 (n+1) d_{j_s-1}(q) \quad \text{for } 2 \leq s+1 \leq r \\ d_{j_s-1}(q) &\leq 2 d_{j_s-1}(q_{j_s}) \quad \text{for } 2 \leq s. \end{aligned}$$

Definition 2.2.2. (see [P, M].) Let $c > 1$ be a fixed constant. We say that $(\mathcal{S}; \Lambda)$ is (L) **regular** if there is some constant $C > 0$ such that for every c -chain $q_{s_1} = q, q_{s_2}, \dots, q_{s_r}$ relative to Σ , and each p , $1 \leq p \leq r$,

$$(2.7) \quad |P_{q_{s_1}}^\perp P_{q_{s_2}} \cdots P_{q_{s_p}}| \leq C \frac{|q - q_{s_2}|}{d(q; X_{s_p-1})}$$

and, for all $q' \in S_{s_1}$ such that $|q - q'| \leq \frac{1}{2c} d(q; X_{s_1-1})$, then:

$$(2.8) \quad |(P_q - P_{q'}) P_{q_{s_2}} \cdots P_{q_{s_p}}| \leq C \frac{|q - q'|}{d(q; X_{s_p-1})}$$

where $d_{l-1} \equiv 1$. In particular,

$$(2.9) \quad |P_q - P_{q'}| \leq C \frac{|q - q'|}{d(q; X_{s_1-1})}.$$

We recall that a stratification is said to be **compatible with a subset** if this subset is a union of strata.

As we have said, in the case $\Lambda = \Lambda(\mathcal{S})$ (the union of the tangent bundles of the strata), we have by the definitions that $\mathcal{S}$ is (L) regular (in the sense of Mostowski) if and only if $(\mathcal{S}; \Lambda(\mathcal{S}))$ is.

Assume that $i \geq 1$ (and recall that $j \geq i$). We define inductively the following sequences of numbers:

$$n_s := \text{card} \{m < t \leq s : X_{t-1} \subsetneq X_t\}$$

and

$$(2.10) \quad \alpha_s := n_s(2 - i - j) + j - i - 1.$$

This enables us to define the following coverings of the X_s 's, $s \geq m$.

First let

$$U_s^0 := \{q \in X_s : d(q; X_{s-1}) \geq e_0 d(q; Y)^{j-1-\alpha_{s-1}}\}.$$

Let $U_m^0 = \emptyset$ and for $1 \leq l \leq s - m$ let

$$U_s^l := \{q \in X_s : d(q; U_{s-1}^{l-1}) \leq e d(q; Y)^{j-1-\alpha_{s-1}}\}.$$

if $X_s \neq X_{s-1}$ (where e_0, e are some strictly positive constants) and $U_s^l := U_{s-1}^{l-1}$ whenever $X_s = X_{s-1}$. Note that, given e , we can choose e_0 in such a way that $\{U_s^0, \dots, U_s^{s-m}\}$ covers X_s . If it is needed we will write $U_s^l(e)$ instead of U_s^l to specify e .

Lemma 2.2.3. *Given $c > 1$, there exists $\nu > 0$ such that, for every $q \in U_s^l(e)$, if $q_{s_1}, \dots, q_{s_r}$ is a c -chain for q then $q_{s_2}, \dots, q_{s_t}$ belong to $U_s^l(\nu e)$, where t is the greatest integer for which $s_t \geq s - l$.*

Proof. We do the proof for q_{s_2} , the argument may be iterated to yield the result for the other q_{s_p} 's using (2.6). For simplicity set $l_2 = l - (s_1 - s_2)$. First, as $-\alpha_s$ is increasing we have:

$$(2.11) \quad d(q; U_{s_2-1}^{l_2-1}) \leq (s_1 - s_2) e d(q; Y)^{j-1-\alpha_{s_2-1}}.$$

Remark also that by definition of U_S^l :

$$d_{s_2}(q) \leq d(q; U_{s_2}^{l_2}) \leq e(s_1 - s_2) d(q; Y)^{j-1-\alpha_{s_2}}.$$

This implies that:

$$d(q; q_{s_2}) \leq c d_{s_2}(q) \leq c e(s_1 - s_2) d(q; Y)^{j-1-\alpha_{s_2}}.$$

Then, together with (2.11) we have:

$$d(q_{s_2}; U_{s_2-1}^{l_2-1}) \leq d(q_{s_2}; q) + d(q; U_{s_2-1}^{l_2-1}) \leq (s_1 - s_2)(c e d(q; Y)^{j-1-\alpha_{s_2}} + e d(q; Y)^{j-1-\alpha_{s_2-1}}).$$

Thus, it is enough to choose $\nu \geq (c + 1)(s_1 - s_2)$. $\square$

We now prove a proposition which will be useful at the end of the next section.

Proposition 2.2.4. *Assume that $i \geq 1$ and that $(\Lambda; \mathcal{S})$ is (L) regular. Then, there exists a horn neighborhood of Γ_v of order i , say H , such that for each $s \geq m$ and $l \leq s - m$ there exists an m -distribution $\mathcal{D}_s^l$ on $U_s^l \cap H$, Lipschitz with exponent at least α_s (see (2.10)), with $\mathcal{D}_s^l(q) \subset \Lambda_q$ for all q , and such that $(v; \mathcal{D}_s^l)$ is $(t_1^{i,j})$ if $(v; \Lambda)$ is.*

Proof. We are going to construct by induction on $s > m$ the m -distributions $\mathcal{D}_s^l$ on X_s , $0 \leq l \leq s - m$. We start by checking that we may find a horn neighborhood H of Γ_v of order i such that:

$$(2.12) \quad H \cap X_m = \{0\}.$$

Replacing v by its Taylor expansion we may assume that it is polynomial. Thus, if (2.12) failed for any H , there would exist an analytic arc $\gamma(t) \in X_m$ satisfying:

$$(2.13) \quad |v(\gamma_1) - \gamma_2| \ll |\gamma_1|^i,$$

where $\gamma = (\gamma_1; \gamma_2)$ in $N \oplus Y$. Let $\tilde{\gamma} := (\gamma_1; v(\gamma_1))$. Then $|\gamma(r) - \tilde{\gamma}(r)| \ll r^i$ (we parameterize these paths by their distance to the origin). This means that

$$(2.14) \quad \delta(T_{\gamma(r)}\gamma; T_{\tilde{\gamma}(r)}\tilde{\gamma}) \ll r^{i-1}.$$

Let Q_r be the normal space to $\tilde{\gamma}'(r)$ in $T_{\tilde{\gamma}(r)}\Gamma_v(r)$ and let $P_r := \gamma'(r) \oplus Q_r$. Then, by (2.14):

$$(2.15) \quad \delta(P_r; T_{\tilde{\gamma}(r)}\Gamma_v) \ll r^{i-1}.$$

Note that $\gamma'(r) \in T_{\gamma(r)}X_m \cap P_r$ so that $\dim T_{\gamma(r)}X_m \cap P_r \geq 1$. Hence, the linear mapping $\lambda : N \rightarrow Y$ whose graph is P_r belongs to $\Sigma(T_{\gamma(r)}X_m)$. By (2.15) we deduce that

$$d(d_x v; \Sigma(T_{\gamma(r)}X_m)) \ll r^{i-1}.$$

As r is equivalent to $d(q; Y)$ along γ and Λ_q is a subset of $T_q X_m$ for any $q \in X_m$, this contradicts the definition of the (t^i) condition.

Hence we must have on some horn neighborhood H of Γ_v of order i :

$$\frac{1}{d(q; X_m)} \lesssim d(q; Y)^{-i}.$$

But the (L) condition implies that on $H \cap X_{s_0}$ we must have whenever $(q; q')$ satisfies $|q - q'| \leq \frac{1}{2c} d(q; Y)$:

$$|P_q - P_{q'}| \lesssim |q - q'| d(q; Y)^{-i}.$$

Let s_1 be the first integer greater than m such that $n_{s_1} = 1$.

We can define $\mathcal{D}_{s_1}^0 : U_{s_1}^0 \rightarrow \mathbb{G}_d^m$ by $\mathcal{D}_{s_1}^0(q) := P_q(\mathcal{D}_v(q))$ (if $l > 0$ then $U_{n_{s_1}}^l$ is empty). By the above inequality and (2.3) we get for $w \in \mathcal{D}_v(q)$ and $q \in H \cap X_{s_1}$:

$$\frac{|P_q(w) - P_{q'}(w)|}{|P_q(w)|} \lesssim |q - q'| d(q; Y)^{1-2i}.$$

This implies that the m -distribution $\mathcal{D}_{s_1}^0$ is Lipschitz with exponent $(1 - 2i)$. As $n_{s_1} = 1$ we see that $\alpha_{s_1} = 1 - 2i$ which completes the first step of the induction.

Assume the result true until $s - 1 > m$. If $X_s = X_{s-1}$ then $n_s = n_{s-1}$ and there is nothing to prove. Hence we may assume $X_{s-1} \neq X_s$ and $n_s = n_{s-1} + 1$.

Fix $0 < l \leq s - m$ (we will construct $\mathcal{D}_s^0$ later on). Using Proposition 1.0.5 we can extend $\mathcal{D}_{s-1}^{l-1}$ (given by the induction hypothesis) to the ambient space to a distribution $\mathcal{D}'$ which is still Lipschitz with exponent α_{s-1} . We set for $q \in H \cap U_s^l$: $\mathcal{D}_s^l := P_q(\mathcal{D}')$.

We first check that the condition $(t_1^{i,j})$ is fulfilled by $\mathcal{D}_s^l$. We have to check that the inequalities (2.4) and (2.5) (which hold for $\mathcal{D}_{s-1}^{l-1}$) still hold for $\mathcal{D}_{s|H \cap U_s^l}^l$. Let $q \in H \cap U_s^l$ and let $q_{j_1}, \dots, q_{j_p}$ be a chain of q . As in Lemma 2.2.3, let t be the greatest integer for which $s_t \geq s - l$. Then, by the definition of chains and by the definition of U_s^l :

$$(2.16) \quad d(q_{s_t}; X_{s-l}) \geq \varepsilon d(q; Y)^{j-1-\alpha_{s-1}}$$

for some strictly positive constant ε .

By Lemma 2.2.3 we know that $q_{s_2}, \dots, q_{s_t}$ belong to $U_s^l(\nu.e)$ (we may apply the induction hypothesis to $U_{s-1}^{l-1}(\nu.e)$ and construct $\mathcal{D}_s^l$ only on $U_s^l(e)$). We may also assume that $q_{s_2}, \dots, q_{s_t}$ belong to a given horn neighborhood of order i by taking H small enough.

As $\mathcal{D}'$ is Lipschitz with exponent at least α_{s-1} (see Remark 1 (1)), given a unit vector w in $\mathcal{D}'(q)$, we may find $w(q_{s_2}), \dots, w(q_{s_t})$ in $\mathcal{D}'(q_{s_2}), \dots, \mathcal{D}'(q_{s_t})$ respectively such that for each u :

$$(2.17) \quad |w(q_{s_{u+1}}) - w(q_{s_u})| \leq C d(q_{s_{u+1}}; X_{s_u}) d(q_{s_u}; Y)^{\alpha_{s-1}}.$$

Now thanks to (2.7), (2.16), and (2.17) we have:

$$\begin{aligned} |P_q(w(q)) - w(q)| &\leq \sum_{u=1}^t |P_{q_{s_1}}^\perp P_{q_{s_2}} \cdots P_{q_{s_u}}(w(q_{s_{u+1}}) - w(q_{s_u}))| + \\ &\quad |P_{q_{s_1}}^\perp P_{q_{s_2}} \cdots P_{q_{s_t}}(w(q_{s_t}))| \\ &\leq C e |q - q_{s_2}| d(q; Y)^{\alpha_{s-1}}. \end{aligned}$$

Using again (2.17) and the definition of U_s^l we may derive that (for some constant C) we have:

$$(2.18) \quad |P_q(w(q)) - w(q_{s_2})| \leq C e |q - q_{s_2}| d(q; Y)^{\alpha_{s-1}} \leq c C e d(q; Y)^{j-1}.$$

Since $\mathcal{D}_{s-1}^{l-1}$ is $(t_1^{i,j})$ this implies that $\mathcal{D}_s^l$ is $(t_1^{i,j})$ as well, provided e is chosen small enough (recall that $i \leq j$).

Let us now check that the constructed distribution $\mathcal{D}_s^l$, $l = 1, \dots, s$ is Lipschitz with exponent α_s . Fix q and q' in $H \cap U_s^l$ satisfying $|q - q'| \leq \frac{1}{2}d(q; Y)$. Note that (2.18) implies that:

$$\delta(\mathcal{D}_s^l(q); \mathcal{D}_s^l(q_{s_2})) \leq C |q - q_{s_2}| d(q; Y)^{\alpha_{s-1}}.$$

Therefore, if $|q - q'| > \frac{1}{2}d(q; X_{s_2})$ we are done. So we may assume that:

$$|q - q'| \leq \frac{1}{2}d(q; X_{s_2}).$$

As $\mathcal{D}'$ is Lipschitz with exponent at least α_{s-1} , for w unit vector of $\mathcal{D}'(q)$, we may find a unit vector $w' \in \mathcal{D}'(q')$ for which:

$$(2.19) \quad |w - w'| \leq |q - q'| d(q; Y)^{\alpha_{s-1}}.$$

But thanks to the definition of the (L) condition we may write:

$$\begin{aligned} |P_q(w) - P_{q'}(w)| &\leq \left| \sum_{u=1}^t (P_q - P_{q'}) P_{q_{s_2}} \cdots P_{q_{s_u}} (w(q_{s_{u+1}}) - w(q_{s_u})) \right| + \\ &\quad |(P_q - P_{q'}) P_{q_{s_2}} \cdots P_{q_{s_t}} (w(q_{s_t}))| \\ &\leq C |q - q'| d(q; Y)^{\alpha_{s-1}}, \end{aligned}$$

using (2.8), (2.16), (2.17) and the definition of chains. Together with (2.19) this implies that

$$|P_q(w) - P_{q'}(w')| \leq C |q - q'| d(q; Y)^{\alpha_{s-1}}.$$

By (2.17) (for $u = 0$) and (2.18), as w is a unit vector, we have

$$|P_q(w)| \geq \varepsilon,$$

for a strictly positive number ε independent of q and w . Together with the preceding inequality this implies that:

$$(2.20) \quad |\mu_q - \mu_{q'}| \leq C |q - q'| d(q; Y)^{\alpha_{s-1}} \leq C |q - q'| d(q; Y)^{\alpha_s},$$

(since $n_{s-1} < n_s$) where μ_q is the orthogonal projection on $\mathcal{D}_s^l(q)$.

If $l = 0$ it is enough to set $\mathcal{D}_s^0(q) := P_q(\mathcal{D}_v(q))$. By definition, we see that the $(t_1^{i,j})$ condition is clearly satisfied by this distribution when it is satisfied by $(v; \Lambda)$. Furthermore, as $\mathcal{D}_v$ is a Lipschitz for w unit vector of $\mathcal{D}_v(q)$, we may find a unit vector $w' \in \mathcal{D}_v(q')$ for which $|w - w'| \leq C |q - q'|$. But thanks to (2.9) this implies that (for any e_0) we have on $H \cap U_s^0$ for a constant C

$$|P_q(w) - P_{q'}(w')| \leq C |q - q'| d(q; Y)^{1-j+\alpha_{s-1}}.$$

Now, as $(v; \Lambda)$ is (t^i) we have:

$$|P_q(w)| \geq \varepsilon d(q; Y)^{i-1},$$

for some strictly positive constant ε independent of the unit vector w in $\mathcal{D}_v(q)$. This implies that:

$$|\mu_q - \mu_{q'}| \leq C|q - q'| d(q; Y)^{2-i-j+\alpha_{s-1}},$$

where μ denotes the orthogonal projection on $\mathcal{D}_s^0$. But $\alpha_{s-1} - i - j + 2 = \alpha_s$. $\square$

In the above proof e has to be small enough but e_0 is arbitrary so that we have a family of distributions which are defined on a covering of X_s for every s .

3. PULLBACK AND REGULARITY CONDITIONS

In this section m' will be an integer and Y' will be $\{0_{\mathbb{R}^n}\} \times \mathbb{R}^{m'}$. We will write $\mathbb{R}^{d'}$ for $N \oplus Y'$. We fix a direct transversal v and a h -distribution Λ , with h positive integer. As in the previous section P_q will be the orthogonal projection on Λ_q .

Given a mapping $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ (resp. $f : \mathbb{R}^{d'} \rightarrow \mathbb{R}^{d'}$) we will denote by f_N its component along N , f_Y (resp. $f_{Y'}$) its component along Y (resp. Y'), and f_{Y_s} (resp. $f_{Y'_s}$) the s^{th} component f_Y (resp. $f_{Y'}$) with respect to the canonical basis of Y (resp. Y').

We define as in [TW] the notion of deformation of transversals.

Let $\rho \in \mathbb{N}$ and let U be a neighborhood of the origin in $\mathbb{R}^{m'}$. Let $h_1, \dots, h_{m'} : N \rightarrow Y$ be smooth functions on N satisfying $|h_r(x)| = o(|x|^\rho)$ (resp. $|h_r(x)| = O(|x|^\rho)$) and $|d_x h_r| = o(|x|^{\rho-1})$ (resp. $|d_x h_r| = O(|x|^{\rho-1})$) for any r . Then

$$\begin{aligned} F : N \times U &\rightarrow N \times Y, \\ F(x; u) &= (x, f(x, u)) = (x, f_u(x)) \\ &= \left(x, v(x) + \sum_{r=1}^{m'} u_r h_r(x) \right) \end{aligned}$$

is called a *deformation of order ρ_+* (resp. *of order ρ*).

Deformations of transversals induce a pull-back transformation over the graphs of applications from $N \times Y$ into $\mathbb{G}_d$. We define it as in [TW].

Let $\mathcal{S} = \{S_0, \dots, S_d\}$ be a stratification of a closed set $X \subset \mathbb{R}^d$. If for each point $p \in U$ the mapping F is transverse to the strata we may define $F^*\mathcal{S}$ as the stratification of $F^{-1}(X)$ given by the manifolds $F^{-1}(S_k)$.

Similarly, if at each point $p \in \mathbb{R}^{d'}$, the linear mapping $d_p F$ is transverse to $\Lambda_{F(p)}$ we may define $F^*\Lambda$ by setting $F^*\Lambda_p := d_p F^{-1}(\Lambda_{F(p)})$. In particular if Λ is an m -distribution, we thus define an m' -distribution.

We may also push forward a transversal u by F by setting $F_*u(x) := F(x; u(x))$.

Important. Throughout this section ρ will be an element of $\mathbb{Z} \cup \mathbb{Z}_+$, and F a deformation of order ρ . We will assume that $\rho \geq j$ and $\max(\rho - 2; 0) \geq k$.

We recall a result proved by D. Trotman and L. Wilson in [TW]. The terminology is a bit different, it is adapted to the one of the present paper which is more convenient to generalize these results for Lipschitz conditions. The reader is referred to [TW] for the proof of this theorem. The idea is that the pull-back by a deformation of transversals increases the regularity.

Theorem 3.0.5. [TW] *If $(v; \Lambda)$ satisfies the condition (t^i) and $\rho \geq i$ then $F^*\Lambda$ is rugose with exponent $\rho - i + 1$.*

3.1. Lipschitz distributions. We denote by $r_0(v)$ the greatest integer satisfying $|v(x)| \lesssim |x|^{r_0(v)}$. We set $\phi_v(x; u) = (x; u - v(x))$.

The following proposition will be useful since it will reduce the problem to the case where $v = 0$.

Proposition 3.1.1. *Assume that $k \leq r_0(v) - 2$. Then Λ is Lipschitz with exponent k if and only if $\phi_{-v}^*\Lambda$ is.*

Proof. Note that v for couples $(x; x') \in N \times N$ satisfying $|x - x'| \leq \frac{1}{2}|x|$ we have:

$$(3.1) \quad |d_x v - d_{x'} v| \lesssim |x - x'| |x|^{r_0(v)-2}.$$

This implies that $d_x \phi_v$ also satisfies this property. Now a straightforward computation shows that $\phi_{-v}^*\Lambda$ is still Lipschitz with exponent k if $k \leq r_0(v) - 2$. $\square$

3.2. Pullback of distributions. We recall that F is a deformation of order ρ with $\rho \geq \max(j; k + 2)$.

Theorem 3.2.1. *Assume that Λ is Lipschitz with exponent $k \leq r_0(v) - 2$. If $(v; \Lambda)$ is $(t_1^{i,j})$ then there exists an m -sub-distribution of $F^*\Lambda$ which is Lipschitz with exponent $k + \rho + \min(2 - (i + j); 0)$.*

We postpone the proof of this theorem and shall prove some preliminary results.

Notations. We denote by $\mathcal{D}$ the distribution generated by the vectors $P_q(\sum_{l=1}^{m'} t_l \partial_x h_{l,s} - \partial_s)$, $s = 1, \dots, m$, by $\mathcal{D}'_r(p)$ the subspace generated by the vectors $(\sum_{l=1}^{m'} t_l \partial_x h_{l,s} - \partial_s)$, with $s \neq r$, where $q = F(x; t)$, and by $\mathcal{D}_r(p)$ the vector subspace $P_q(\mathcal{D}'_r(p))$. Let $\pi_{r,p}$ be the orthogonal projection onto the orthogonal complement of $\mathcal{D}_r(p)$ in $\mathcal{D}(q)$.

Proposition 3.2.2. *We assume the same hypotheses as in Theorem 3.2.1 for the transversal $v = 0$. We also assume $i \geq 1$. Then there exists a neighborhood U of Y' such that for any $r = 1, \dots, m$:*

- (i) $\mu(Y; \Lambda_q) \lesssim |\pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)|$
- (ii) *For any couple $(p; p') \in U \cap X \times U \cap X$ satisfying $|p - p'| \leq \frac{1}{2c} d(p; Y')$:*

$$|\pi_{r,p} - \pi_{r,p'}| \lesssim |p - p'| d(q; Y)^{k-i-j+2} |\pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)|^2.$$

where $q = F(p) = (x; t_1; \dots; t_m)$.

Proof. It is clear from the definition of the pull-back that, as $\rho \geq j$, the inverse image of a horn neighborhood of Γ_v of order j is a neighborhood of Y' . Hence, if U is a sufficiently small neighborhood of Y' the inequalities of the condition $(t_1^{i,j})$ will hold on its image by F .

We show (i) by way of contradiction. Suppose:

$$(3.2) \quad |\pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)| \ll \mu(Y; \Lambda_q)$$

for a sequence in $U \cap X$.

Let $p \in U \cap X$, $q = F(p)$ and let $\hat{\pi}_{r,p}$ be the orthogonal projection onto $\mathcal{D}_r(p)$. We have $\hat{\pi}_{r,p} + \pi_{r,p} = P_q$. Let $w(p)$ be the vector of $\mathcal{D}'_r(p)$ which projects onto $\hat{\pi}_{r,p}(\partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r})$ via P_q . Then

$$P_q(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r) = P_q(w) + \pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)$$

and so by (3.2):

$$|P_q(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r - w(p))| = |\pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)| \ll \mu(Y; \Lambda_q).$$

By (2.3) we have $\frac{1}{\mu(Y; \Lambda_q)} \lesssim d(q; Y)^{1-i}$. Moreover since $p \in H$, we have

$$|\partial_x h_{l,r}| \lesssim |x|^{\rho-1},$$

for each l between 1 and m' .

As $\rho \geq i$ we get $|\sum_{l=1}^{m'} t_l \partial_x h_{l,r}| \ll \mu(Y; \Lambda_q)$. Thus:

$$|P_q(\partial_r - w(p))| \ll \mu(Y; \Lambda_q).$$

As $w \in \mathcal{D}'_r(p)$ the norm of $(\partial_r - w(p))$ is bounded below away from zero. This contradicts the definition of $\mu(Y; \Lambda_q)$ and proves the point (i).

Now let us show (ii). Note that, as Λ is Lipschitz with exponent k we have, thanks to (i) (and the condition (t^i)):

$$|P_q - P_{q'}| \lesssim |q - q'| d(q; Y)^{k-2i+2} |\pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)|^2.$$

This already means that (ii) holds if, in the left-hand side, we replace $\pi_{r,p}$ by P_q . Since $\hat{\pi}_{r,q} + \pi_{r,q} = P_q$, it is enough to show (ii) with the projection $\hat{\pi}_{r,p}$ instead of $\pi_{r,p}$ in the left-hand side.

Let $(p; p') \in U \cap F^*X \times U \cap F^*X$ such that $|p - p'| \leq \frac{1}{2c} d(p; Y')$. We set $q = F(p)$, $q' = F(p')$ so that, as F is Lipschitz $|q - q'| \leq C|p - p'|$ and therefore since $(p; p')$ satisfies $|p - p'| \leq \frac{1}{2c} d(p; Y')$, $(q; q')$ fulfills an inequality of the same type (with respect to Y) provided c is chosen sufficiently big. Now, as Λ is Lipschitz with exponent k , we have:

$$(3.3) \quad |(P_q - P_{q'}) (\partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r})| \lesssim |q - q'| d(q; Y)^k.$$

Moreover for all $l = 1, \dots, m'$, we have:

$$\begin{aligned}
 |t_l \partial_x h_{l,r} - t'_l \partial_{x'} h_{l,r}| &\leq |t_l - t'_l| |\partial_x h_{l,r}| + |\partial_{x'} h_{l,r} - \partial_x h_{l,r}| |t'_l r| \\
 &\lesssim |p - p'| (d(q; Y)^{\rho-1} + d(q; Y)^{\max(\rho-2; 0)}) \\
 (3.4) \quad &\lesssim |p - p'| d(q; Y)^{\max(\rho-2; 0)}.
 \end{aligned}$$

As $\max(\rho - 2; 0) \geq k$, putting together (3.3) and (3.4) we get the following inequality:

$$|P_q(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r) - P_{q'}(\sum_{l=1}^{m'} t'_l \partial_{x'} h_{l,r} - \partial_r)| \lesssim |p - p'| d(q; Y)^k.$$

Moreover it is easy to derive from the definition of μ_1 that:

$$\mu_1(Y; \Lambda_q) \leq d(P_q(\partial_r); V_r) \cdot \min\{|P_q(w)| / |w \perp \partial_r, |w| = 1, w \in Y\},$$

where V_r denotes the subspace of Λ_q generated by the $P_q(\partial_s)$, $s \neq r$. And as for any integer $s \leq m$, $|\sum_{l=1}^{m'} t_l \partial_x h_{l,s}| \ll \mu_1(Y; \Lambda_q)$ (due to the condition (t_1^j) and $\rho - j \geq 0$), in particular we have

$$|\sum_{l=1}^{m'} t_l \partial_x h_{l,s}| \ll \min_{\substack{w \perp \partial_r \\ |w|=1}} (|P_q(w)|) \quad \text{and} \quad |\sum_{l=1}^{m'} t_l \partial_x h_{l,r}| \ll d(P_q(\partial_r); V_r),$$

in a sufficiently small neighborhood of the origin. Hence:

$$(3.5) \quad d(P_q(\partial_r); V_r) \sim d(P_q(\partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r}); V_r).$$

($\sim$ means that the ratio is bounded above and below.)

Moreover if $w \in Y$ is a unit vector such that $w \perp \partial_r$, we have:

$$d(w, \mathcal{D}'_r(q)) \leq \max_s (|\sum_{l=1}^{m'} t_l \partial_x h_{l,s}|),$$

so that:

$$(3.6) \quad \min_{\substack{w \in \mathcal{D}'_r(q) \\ |w|=1}} (|P_q(w)|) \sim \min_{\substack{w \in Y, w \perp \partial_r \\ |w|=1}} (|P_q(w)|).$$

By (3.5) et (3.6) we deduce that:

$$(3.7) \quad \mu_1(Y; \Lambda_q) \lesssim d(P_q(\partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r}); V_r) \min_{\substack{w \in \mathcal{D}'_r(q) \\ |w|=1}} |P_q(w)|.$$

But by definition of $\mathcal{D}_r$ any element w of $\mathcal{D}_r(p)$ satisfies $P_q(w) = \widehat{\pi}_{r,p}(w)$. For simplicity set $V'_{r,p} = \mathcal{D}_r(p)^\perp \cap P_q$. We have:

$$\begin{aligned} d(P_q(\partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r}); V_r) &\leq d(P_q(\partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r}); V'_r) + \delta(V_r; V'_r) \\ &\leq d(P_q(\partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r}); V'_r) + \max_s |\sum_{l=1}^{m'} t_l \partial_x h_{l,s}| \\ &= |\pi_{r,p}(\partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r})| + \max_s |\sum_{l=1}^{m'} t_l \partial_x h_{l,s}|. \end{aligned}$$

Moreover, by (i) : $\mu(Y; \Lambda_q) \lesssim |\pi_{r,p}(\partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r})|$ and as $\max_s |\sum_{l=1}^{m'} t_l \partial_x h_{l,s}| \lesssim d(q; Y)^\rho \ll \mu(Y; \Lambda_q)$ we may write:

$$d(P_q(\partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r}); V_r) \lesssim |\pi_{r,p}(\partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r})|.$$

And we deduce from (3.7) that:

$$(3.8) \quad \mu_1(Y; \Lambda_q) \leq |\pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)| \cdot \min_{\substack{w \in \mathcal{D}'_r(q) \\ |w|=1}} |P_q(w)|.$$

Thus for any $w \in \mathcal{D}'_r(p)$:

$$\frac{|P_q(w) - P_{q'}(w)|}{|P_q(w)|} \lesssim |p - p'| d(q; Y)^{k+1-j} |\pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)|,$$

therefore:

$$\delta(\mathcal{D}_r(p); \mathcal{D}_r(p')) \lesssim |p - p'| d(q; Y)^{k+1-j} |\pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)|.$$

This implies:

$$|\widehat{\pi}_{r,p} - \widehat{\pi}_{r,p'}| \lesssim |p - p'| |x|^{k+1-j} |\pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)|.$$

And hence, via (3.4), we get:

$$|\widehat{\pi}_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r) - \widehat{\pi}_{r,p'}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)| \lesssim |p - p'| |x|^{k+1-j} |\pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)|,$$

which, together with (i), proves (ii). $\square$

We shall need a similar result in the case where $i \geq 1$. This case is actually easier and is the purpose of the following lemma.

Lemma 3.2.3. *We assume the same hypotheses as in Proposition 3.2.2. We assume as well that $i \leq 1$. Then there exists a neighborhood U of Y' in $\mathbb{R}^{d'}$ such that at any point $p \in F^*X \cap U$:*

$$|\pi_{r,p}(\partial_r) - \partial_r| \lesssim d(p; Y')^{1-i}$$

for any integer $1 \leq r \leq m$.

Proof. Let $p \in \mathbb{R}^{d'}$ and $q = F(p)$. Recall that $\pi_{r,p}$ is the projection onto $\mathcal{D}_r(p)^\perp \cap \Lambda_q$. Moreover since $\partial_r \in Y \cap \mathcal{D}_r^\perp(p)$, it is enough to see that

$$\delta(Y \cap \mathcal{D}_r^\perp(p); \Lambda_q \cap \mathcal{D}_r^\perp(p)) \lesssim d(p; Y')^{1-i}.$$

As $i \leq 1$ and thanks to Remark 3 we have $\mu(Y; \Lambda_q) \geq \varepsilon > 0$. Since $|\sum_{l=1}^{m'} t_l \partial_x h_{l,s}| \lesssim |x|^{\rho-1}$ and $\rho \geq i$ we may deduce that:

$$\mu(\mathcal{D}(p); \Lambda_q) \geq \frac{\varepsilon}{2}$$

for q close the origin. But as $\mathcal{D}_r(p) \subseteq \mathcal{D}(p)$ we get:

$$(3.9) \quad \mu(\mathcal{D}_r(p); \Lambda_q) \geq \frac{\varepsilon}{2}$$

But:

$$\begin{aligned} \delta(Y \cap \mathcal{D}_r^\perp(p); \Lambda_q \cap \mathcal{D}_r^\perp(p)) &\leq \frac{\delta(Y \cap \mathcal{D}_r^\perp(p); \Lambda_q)}{\mu(\mathcal{D}_r(p); \Lambda_q)} \\ &\lesssim \delta(Y \cap A(p); \Lambda_q) \quad (\text{by (3.9)}) \\ &\lesssim d(q; Y)^{1-i}, \end{aligned}$$

as required. $\square$

Proof of Theorem 3.2.1. Remark first that we may assume, without loss of generality, that v is the zero map. Actually we may set $\widehat{F} = \phi_v \circ F$ so that $\widehat{F}_*0 = 0$. Remark that:

$$F^* \Lambda = (\phi_{-v} \circ \widehat{F})^* \Lambda = \widehat{F}^* (\phi_{-v}^* \Lambda),$$

and by Proposition 3.1.1 $\phi_{-v}^* \Lambda$ is Lipschitz with exponent k if and only if Λ is. Moreover (see Remark 4) the condition $(t_1^{i,j})$ is preserved by the pull-back by ϕ_{-v} . Hence it is enough to show the result for $\widehat{F}$ with $(0; \phi_{-v}^* \Lambda)$.

The theorem will be proved if we can find a neighborhood of Y' such that Λ admits a family of sections $\zeta_1, \dots, \zeta_{m'}$ which are Lipschitz with exponent $k + \rho + i + j - 2$ satisfying $\pi(\zeta_l) = \partial_l$, $l \in \{1, \dots, m'\}$.

Given $p \in U$ and $r \in \{1, \dots, m\}$ let:

$$w_r(p) = \frac{\pi_{r,p}(\partial_r)}{\langle \partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r} ; \pi_{r,p}(\partial_r) \rangle}.$$

It is easy to derive from Proposition 3.2.2 (ii):

$$(3.10) \quad |w_r(p) - w_r(p')| \lesssim |p - p'| d(q; Y)^{k-i-j+2}.$$

On the other hand from the construction of w_r we know that, for $r \neq s$,

$$\langle w_r(p); \sum_{l=1}^{m'} t_l \partial_x h_{l,s} - \partial_s \rangle = 0$$

and hence for $s \in \{1, \dots, m\}$ different from r :

$$(3.11) \quad \langle w_r(p); \sum_{l=1}^{m'} t_l \partial_x h_{l,s} \rangle = \langle w_r(p); \partial_s \rangle = w_{r,s}(p).$$

And as $\langle w_r(p); \partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r} \rangle = 1$, we get :

$$(3.12) \quad \langle w_{r,N}(p); \sum_{l=1}^{m'} t_l \partial_x h_{l,r} \rangle = \langle w_r(p); \partial_r \rangle - 1 = w_{r,Y_r}(p) - 1.$$

Then we set for $l \in 1, \dots, m'$:

$$(3.13) \quad \zeta_l(p) = \left(\sum_{r=1}^m h_{l,r}(x) w_{r,N}(p); 0; \dots; 1; \dots; 0 \right)$$

where the number 1 lies in l^{th} position so that:

$$\pi(\zeta_l(p)) = \partial_l.$$

Moreover as:

$$\begin{aligned} \frac{\partial F}{\partial x}(p) &= (Id; \sum_{l=1}^{m'} t_l d_x h_{l,1}; \dots; \sum_{l=1}^{m'} t_l d_x h_{l,m}) \\ \frac{\partial F}{\partial t_s}(p) &= (0; \dots; 0; h_{s,1}(x); \dots; h_{s,m}(x)). \end{aligned}$$

We may deduce using (3.11) et (3.12) :

$$\begin{aligned} dF(p)(\zeta_l(p)) &= \left(\sum_{r=1}^m h_{l,r}(x) w_{r,N}(p); \sum_{r=1}^m h_{l,r}(x) w_{r,Y}(p) \right) \\ &= \sum_{r=1}^m h_{l,r}(x) w_r(p). \end{aligned}$$

In consequence ζ_l is a section of $F^* \Lambda$.

Moreover $(\zeta_l(p) - \partial_l) = (\sum_{r=1}^m h_{l,r}(x) w_{r,N}(q); 0)$. By Proposition 3.2.2 (i), we have:

$$|w_{r,N}(p)| = \frac{1}{|\pi_{r,p}(\sum_{l=1}^{m'} t_l \partial_x h_{l,r} - \partial_r)|} \lesssim \frac{1}{\mu(Y; \Lambda_q)}.$$

Now, if $i \geq 1$, the inequality (2.3) shows that:

$$\frac{1}{\mu(Y; \Lambda_q)} \lesssim d(q; Y)^{1-i}.$$

If $i \leq 1$ we apply Lemma 3.2.3. This proves that $|w_{r,N}(p)| \lesssim d(p; Y)^{1-i}$.

In consequence, in any case we have $|w_{r,N}(p)| \lesssim d(q; Y')^{1-i}$ and so since $|h_l(x)| \lesssim d(q; Y)^\rho$ we get for $l = 1, \dots, m'$:

$$(3.14) \quad |\zeta_l(p) - \partial_l| \lesssim d(p; Y)^{\rho-i+1}.$$

Hence ζ_l is a section rugose with exponent $\rho - i + 1$ which lifts ∂_r . As $j \geq k$, ζ_l is a fortiori a section rugose with exponent $k + \rho - i - j + 2$. Thus, it remains to prove that for $(p; p') \in U$ satisfying $|p - p'| \leq \frac{1}{2c}d(p; Y')$ we have:

$$|\zeta_l(p) - \zeta_l(p')| \lesssim d(p; Y')^{k+\rho-i-j+2}|p - p'|.$$

But:

$$\begin{aligned} |\zeta_l(p) - \zeta_l(p')| &= \left| \sum_{r=1}^m h_{l,r}(x) w_{l,N}(p) - \sum_{r=1}^m h_{l,r}(x') w_{r,N}(p') \right| \\ &\leq \left| \sum_{r=1}^m (h_{l,r}(x) - h_{l,r}(x')) |w_{r,N}(p)| \right| + \left| \sum_{r=1}^m h_{l,r}(x') |w_{r,N}(p) - w_{r,N}(p')| \right| \end{aligned}$$

and as the ρ -jet of h_l vanishes at the origin, we may apply (3.14) and (3.10) to complete the proof. $\square$

Remark 5. Distributions that are Lipschitz with exponent 0_+ and which are C^1 outside Y are actually C^1 distributions. This will enable us to provide determinacy theorems for the C^1 equivalence in the case where the stratification is composed by only two strata (see section 5.1).

3.3. Pull-back and (L) stratification. The proof of the following theorem is actually a variation of that of Theorem 3.2.1. The setting is actually the one of section 2.2. Let $\mathcal{S} = \{S_0, \dots, S_d\}$ be a stratification compatible with Y (with $\dim S_l = l$ or $S_l = \emptyset$) and let $\Lambda : \mathbb{R}^d \rightarrow \mathbb{G}_d$ be such that Λ_q is a subspace of tangent space to the stratum containing q .

Assume $j \geq i \geq 1$.

Theorem 3.3.1. *Let F be a deformation of order ρ . Assume that $(\Lambda; \mathcal{S})$ is (L) regular and that $(v; \Lambda)$ is $(t_1^{i,j})$. Then every smooth vector field on Y' may be extended to a vector field in a neighborhood of Y' in $\mathbb{R}^{d'}$ tangent to the strata of $F^*\mathcal{S}$ and Lipschitz with exponent $\rho + \alpha_d + 2 - i - j$.*

Proof. By Proposition 2.2.4 we may find m -distributions $\mathcal{D}_d^0, \dots, \mathcal{D}_d^{d-m}$, respectively defined on the intersection of $U_d^0, \dots, U_d^{d-m}$ with some horn neighborhood of order i , Lipschitz with exponent at least α_d and such that $(v; \mathcal{D}_d^l)$ is $(t_1^{i,j})$ for $l = 0, \dots, d-m$. Thanks to Proposition 3.2.2 (with $\mathcal{D}_d^l$ instead of Λ) we know that the vector field on $H \cap U_d^l$ defined by

$$w_r(p) = \frac{\pi_{r,p}(\partial_r)}{\langle \partial_r - \sum_{l=1}^{m'} t_l \partial_x h_{l,r} ; \pi_{r,p}(\partial_r) \rangle}$$

is Lipschitz with exponent $\alpha_d - (i + j) + 2$ (see (3.10)). By Proposition 1.0.4 and the definition of the U_l^s 's we may find a partition of unity adapted to the covering $U_d^0, \dots, U_d^{d-m}$ Lipschitz with exponent at least α_d . Thus a straightforward computation shows that we may paste these vector fields into one still Lipschitz with the same exponent. Now we can

end the proof by defining the sections ζ_l as in the proof of Theorem 3.2.1 (see (3.13)). The same computation clearly yields that they are Lipschitz with exponent $\rho + \alpha_d + 2 - i - j$. $\square$

4. BI-LIPSCHITZ SUFFICIENCY OF JETS

Given $r \in \mathbb{N}$, the r -jet of a mapping $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$, is given by $(u(0); d_0 u; \dots; d_0^r u)$ where $d_0 u, \dots, d_0^r u$ are the successive derivatives of the mapping u at the origin.

All the results of this section are consequences of those of the previous one. We are going to give some explicit criteria for sufficiency of jets. For simplicity we set for this section:

$$\rho := i + j - 2 - \alpha_d.$$

We assume $j \geq i \geq 1$.

4.1. Determinacy of transversals. For this section we fix a stratification of $\mathbb{R}^d$, $\mathcal{S} = \{S_0, \dots, S_d\}$ compatible with Y and satisfying the (L) condition of Mostowski. Let $\Lambda = \Lambda(\mathcal{S})$ the union of the tangent bundles of the strata of $\mathcal{S}$.

Definition 4.1.1. We will say that two transversals v and u are bi-Lipschitz $\mathcal{S}$ -equivalent if there exists a bi-Lipschitz map $H : \Gamma_v \times [0; 1] \rightarrow \mathbb{R}^d$ such that for any t , H_t is a bi-Lipschitz map preserving the strata of $\mathcal{S}$, $H_0 = Id$ and H_1 sends Γ_u onto Γ_v .

A jet is said to be **bi-Lipschitz $\mathcal{S}$ -sufficient** in C^ρ whenever all the C^ρ mappings u having this jet at the origin are bi-Lipschitz $\mathcal{S}$ -equivalent.

We will write $j^\alpha v(0)$ for the α -jet of v at 0.

Theorem 4.1.2. (1) If $(v; \Lambda)$ satisfies the condition $(t_1^{i,j})$ then $z = j^\rho v(0)$ is bi-Lipschitz $\mathcal{S}$ -sufficient in C^ρ .

(2) If $(u; \Lambda)$ satisfies the condition $(t_1^{i,j})$, for any polynomial representative u of $j^\rho v(0)$ jet then $z = j^{\rho-1} v(0)$ is bi-Lipschitz $\mathcal{S}$ -sufficient in C^ρ .

Proof. Let h be a C^r mapping with a ρ -jet vanishing at the origin and let $F(x; t) = (x; v(x) + th(x))$. Then F is a deformation of order ρ_+ . By Theorem 3.3.1, as $\rho = i + j - 2 - \alpha_d$, $F^* \Lambda$ is a Lipschitz m -distribution which implies, by Lemma 1.1.2, that $F^{-1}(X)$ is bi-Lipschitz trivial along Y' . In other words there exists a bi-Lipschitz isotopy carrying the graph of v onto the graph of $v + th$, for $t \in [0; 1]$. For (2) we apply the same argument, the only difference is that the deformation is of order ρ . $\square$

In [V1] some theorems of determinacy with respect to other relations are also provided. For instance it is given some explicit criteria for a transversal to be finitely determined up to a homeomorphism which is bi-Lipschitz with exponent k , for $k \geq 0$ (with an explicit order of determinacy) or which is a quasi-isometry. Some criteria using the condition (t^{i-}) are also provided.

Remark 6. In order to make more explicit our criterion, let us recall that in our setting the (t^i) condition is fulfilled by the couple $(v; \Lambda)$ if and only if for any q in a horn neighborhood of Γ_v of order i we have for $l > m$:

$$(4.1) \quad \mu(\mathcal{D}_v(q); T_q S_l) \geq \varepsilon |x|^{i-1}$$

where $q = (x; t)$.

Similarly, the (t_1^j) condition is fulfilled by the couple $(v; \mathcal{S})$ if and only if for any q in a horn neighborhood of Γ_v of order j we have for $l > m$:

$$(4.2) \quad \mu_1(\mathcal{D}_v(q); T_q S_l) \geq \varepsilon |x|^{j-1}$$

where $q = (x; t)$.

4.2. SV-Determinacy. We are going to give a determinacy theorem for smooth mappings with respect to the SV -equivalence. This is roughly speaking the determinacy of the zero locus (see below for an exact definition). We fix a subset X of N and a stratification $\mathcal{S} := \{\{0\}, S_1, \dots, S_d\}$ of X . In [TW] are given some analogous theorems but in the topological point of view. Here we will focus on such an equivalence involving bi-Lipschitz maps. This equivalence preserve the strata the stratification $\mathcal{S}$.

Two given germs of C^ρ mappings, $f, g : N \rightarrow Y$ are said **bi-Lipschitz SV -equivalent** if there exists a bi-Lipschitz homeomorphism of N , preserving the strata, bi-Lipschitz and sending $f^{-1}(0)$ onto $g^{-1}(0)$.

We set $\mathcal{S}_0 = \{Y \setminus \{0\}, S_0 \times \{0\}, \dots, S_d \times \{0\}\}$ so that f and g are bi-Lipschitz SV -equivalent if and only if they are bi-Lipschitz $\mathcal{S}_0$ -equivalent. We recall that we have set $\rho = i + j - 2 - \alpha_d$.

Theorem 4.2.1. *If $(v; \mathcal{S}_0)$ is $(t_1^{i,j})$ then $z = j^\rho v(0)$ is bi-Lipschitz SV -sufficient in C^ρ .*

Proof. This theorem is a direct consequence of Theorem 4.1.2 and the above observation that f and g are bi-Lipschitz SV -equivalent if and only if f and g are bi-Lipschitz $\mathcal{S}_0$ -equivalent. $\square$

Remark 7. Therefore, equations (4.1) and (4.2) provide explicit criteria for SV -bi-Lipschitz determinacy. It suffices to replace $\mathcal{S}$ by $\mathcal{S}_0$. Note that in this case the functions $\mu(\mathcal{D}_v(x); T_q S_l)$ and $\mu_1(\mathcal{D}_v(x); T_q S_l)$ may be explicitly expressed as follows:

$$\mu(\mathcal{D}_v(x); T_q S_l) = \inf\{|d_x v(w)| : |w| = 1, w \in T_q S_l\}$$

and

$$\mu_1(\mathcal{D}_v(x); T_q S_l) = \inf\{|d_x v(w)| \cdot |d_x v(w')| : |w'| = |w| = 1, w \in T_q S_l, w' \in T_q S_l\}.$$

Also, in this case, the horn neighborhood on which equation (4.1) (resp. (4.2)) has to hold is of type:

$$(4.3) \quad \{x \in \mathbb{R}^n : |v(x)| \leq C|x|^i\},$$

for some constant C (resp. $\{x \in \mathbb{R}^n : |v(x)| \leq C|x|^j\}$).

4.3. $\mathcal{R}$ -sufficiency of jets. We say that two mappings f and g from $\mathbb{R}^n$ to $\mathbb{R}^m$ are **bi-Lipschitz $\mathcal{R}$ -equivalent** whenever there exists a bi-Lipschitz mapping φ from N to N , preserving the strata and such that $f = g \circ \varphi$. We say that a jet is **bi-Lipschitz $\mathcal{R}$ -sufficient** if all the mappings having this jet at the origin are bi-Lipschitz $\mathcal{R}$ -equivalent.

In this section X will be a subanalytic subset of N , stratified by a subanalytic stratification of $\mathbb{R}^d$, $\mathcal{S} = \{\mathcal{S}_1, \dots, \mathcal{S}_d\}$ compatible with X and Y , and satisfying the condition (L). Let $\Lambda_q := T_x S_{p_x} \times \{0_{\mathbb{R}^m}\}$ where S_{p_x} is the stratum containing x if $q = (x; t)$. Recall that $\rho = i + j - 2 - \alpha_d$.

Theorem 4.3.1. *If $(v; \Lambda)$ is $(t_1^{i,j})$ then $z = j^\rho v(0)$ is bi-Lipschitz $\mathcal{R}$ -sufficient in C^ρ .*

Proof. We use a similar argument as in the proof of Theorem 4.1.2. Consider a mapping h having a zero ρ -jet at the origin and let $F(x; t) := (x; v(x) + th(x))$. By Theorem 3.3.1 then $F^* \Lambda$ admits a section which is bi-Lipschitz. By Lemma 1.1.2 this implies that $F^* \mathcal{S}$ is bi-Lipschitz trivial along Y . $\square$

Remark 8. To get an explicit criterion for $(v; \Lambda)$ to be $(t_1^{i,j})$ regular just take the one given in Remark 7 but omitting the restriction to the horn neighborhood given by (4.3).

5. THE TWO STRATA CASE.

We are going to give some determinacy theorems in the case where the stratification is composed by only two strata. In this case, vector fields smooth on X which are Lipschitz with exponent 0_+ actually extend to C^1 vector field to a neighborhood of Y . Hence we get the C^1 - $\mathcal{S}$ -determinacy. Again we will study successively two types of determinacy: The $\mathcal{S}$ -determinacy and the $\mathcal{R}$ -determinacy. For this section we set:

$$\rho := \max(i + j - 2; 2).$$

5.1. $\mathcal{S}$ -determinacy in the two strata case. In this section we fix a stratification of X composed by two strata $\mathcal{S} := \{Y, X \setminus Y\}$. We assume that the distribution $q \mapsto T_q X$, is Lipschitz with exponent $k \in \mathbb{Z}$ (with k possibly positive). In other words we assume that we have for any q and q' in X satisfying $|q - q'| \leq \frac{1}{2c} d(q; Y)$:

$$(5.1) \quad \delta(T_q X; T_{q'} X) \lesssim |q - q'| \cdot d(q; Y)^k.$$

with k integer.

Note that, thanks to the Łojasiewicz inequality, such an inequality may always be obtained by taking k sufficiently large.

Now the C^1 - $\mathcal{S}$ -equivalence is defined in the same way as the bi-Lipschitz equivalence, just replacing bi-Lipschitz by C^1 . The notion of C^1 - $\mathcal{S}$ -equivalence immediately gives rise to the notion of C^1 - $\mathcal{S}$ -sufficiency, derived in the same way.

Theorem 5.1.1. *Assume $\max(\rho - 2; 0) \geq k$. If $(v; \mathcal{S})$ satisfies the condition $(t_1^{i,j})$ then $z = j^{\rho-k} v(0)$ is C^1 - $\mathcal{S}$ -sufficient in $C^{\rho-k}$.*

Proof. The proof is readily the same as that of Theorem 4.1.2. Just use Theorem 3.2.1 instead of Theorem 3.3.1. The C^1 determinacy is a consequence of the above observation

that, in the two strata case, vector fields smooth on X which are Lipschitz with exponent 0_+ actually extend to C^1 vector field to a neighborhood of Y . $\square$

When $i = 1$ the transversality is good and the required conditions for determinacy are simpler. In particular:

Corollary 5.1.2. *The C^1 type of the germ of the intersection of a (L) stratified set with a C^2 transversal to a given stratum at a given point does not depend on the considered transversal.*

Proof. Note that the (L) condition of Mostowski implies the (w) condition of Kuo-Verdier which is the condition (t^0) . If (t^0) holds then a fortiori (t^1) holds then by definitions $(t_1^{1,1})$ also holds; so, by the preceding theorem (with $\rho = k = 0$), we get that the 0-jet of v is C^1 - $\mathcal{S}$ -sufficient. $\square$

Of course in the above theorem the hypothesis $\max(\rho-2; 0) \geq k$ may always be obtained when we have $(t_1^{i,j})$ since in this case we may choose i, j as large as we please.

5.2. Germs of sets with an isolated singularity. In this section we will be interested in the case of subsets X such that $X \setminus \{0\}$ is a smooth manifolds. We assume as well that the tangent distribution to X is Lipschitz with exponent k .

We assume now $i \geq 2$ (for simplicity, some results with $i < 2$ could be stated but have no interest) so that now $\rho = i + j - 2$.

We denote by $\mathcal{S}$ the stratification constituted by the pair $(X \setminus \{0\}; Y)$. We define C^1 - $\mathcal{R}$ -equivalence exactly like bi-Lipschitz $\mathcal{R}$ equivalence, just replacing bi-Lipschitz by C^1 (see section 4.3). Immediately, we also define in an analogous way C^1 sufficiency for jets.

Theorem 5.2.1. *Assume $\rho - 2 \geq k$ and $k \leq r_0(v) - 2$. If $(v; \mathcal{S})$ satisfies $(t_1^{i,j})$ then $z = j^{\rho-k}v(0)$ is C^1 - $\mathcal{R}$ -sufficient in $C^{\rho-k}$.*

Proof. Let $\Lambda_q = T_x X$ where $q = (x; t) \in N \oplus Y$. If we take a deformation of transversals $F(x; t) = (x; v(x) + th(x))$ where h has a vanishing ρ -jet we get by Theorem 3.2.1 $F^* \Lambda$ is a C^1 distribution. The theorem follows from Lemma 1.1.2. $\square$

By Remarks 7 and 8 we get an explicit criterion. Note that a consequence of this theorem is that (t^i) is enough to have the sufficiency of the $3i$ -jet in C^{3i} . Note also that, as in the case of bi-Lipschitz determinacy we could also provide some criteria for C^1 -equivalence devoted to the SV -equivalence.

Again, in the preceding theorem, the hypothesis $\rho - 2 \geq k$ and $k \leq r_0(v) - 2$ may always be obtained when we have $(t_1^{i,j})$ since we may choose i, j , and k as large as we want. In [V1], we also give some criteria for which k may be chosen in $\mathbb{Z}_-$.

In the case where $X = N$ the inequality (5.1) is true for any $k \in \mathbb{Z}$. Thus the conditions $(t_1^{i,j})$ give the determinacy and we get the immediate following corollary.

Corollary 5.2.2. *If $(v; N)$ satisfies $(t_1^{i,j})$ then $z = j^{\rho-r_0(v)}v(0)$ is $\mathcal{R}$ -sufficient in $C^{\rho-r_0(v)}$.*

Proof. As we have said the inequality (5.1) is true for any $k \in \mathbb{Z}$. Hence, just put $k = r_0(v) - 2$. Note that by the definition of (t^i) , we have

$$\inf_{|u|=1} |d_x v(u)| \geq \varepsilon |x|^{i-1},$$

for some $\varepsilon > 0$, on some horn neighborhood of Γ_v of order i . By the definition of $r_0(v)$ we necessarily have $i \geq r_0(v)$ and so $\rho = i + j - 2 \geq r_0(v) + j - 2 \geq k + 2$ (since $j \geq 2$). $\square$

It can be seen that this criterion always improves the one given by C. T. C. Wall in [Wa]. Let us illustrate it by an example.

Example 1. Let $v : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be defined by $v(x; y; z; t) = (2x^2 - y^2 + 6zt^2 - z^3; xy - 3z^2t + 2t^3)$. Denote by $N(d_x v)$ the sum of the squares of all minors of order 2. The criterion provided in [Wa] requires $N(d_x v) \geq \varepsilon |x|^{i-2}$ for the $i - r_0(v) + 1$ determinacy.

If $x = y = 0$ there is only one minor of order 2 which does not vanish. This minor is $\Delta(z; t) = (6t^2 - 3z^2)^2 + 72z^2t^2$. The best lower bound that we can get is $|(z; t)|^4 \lesssim \Delta(z; t)$. The best exponent to bound N below is not less than 8.

Moreover $\Delta_{1,1}(x; y) = 4x^2 + 2y^2$ is greater or equal to $|(x; y)|^2$. As the norm of $|q|$ is always equivalent either to $|(x; y)|$ or to $|(z; t)|$ we deduce that $|q|^8 \lesssim N(d_q v)$. Then, according to the theorem given in [Wa] the 9-jet is thus sufficient.

In dimension 2, the functions μ_1^2 and N coincide. The condition (t_1^5) thus holds for the couple $(v; \mathcal{G}_{\mathcal{R}}(N))$. Moreover by the same arguments as for the lower bound of N , we see that (t^3) holds for this couple. As $r_0(v) = 2$ we get by the above corollary that actually the 6-jet is sufficient.

6. TWO OTHER CONSEQUENCES

In this section we give further results which are also consequences of the results of section 3. We start by proving a finiteness result. Finiteness of topological types of intersection of C^∞ transversals is known since the work by D. Trotman and L. Wilson. We first give a result of finiteness of metric types. We then prove give another theorem about the so-called Kuo's blowing up.

6.1. A finiteness result.

Theorem 6.1.1. *Let S be a subanalytic stratification of X of $\mathbb{R}^d$ satisfying the (a) condition of Whitney and let S be the a stratum. Then, for $x_0 \in S$, there exists an integer r such that the number of Lipschitz types of germs at x_0 of intersection of X with a C^r direct transversal is finite.*

Proof. As it is local we may assume that S is Y . As the stratification is (a) regular we may find a C^0 m -distribution Λ which is a subanalytic sub-distribution of the tangent to the tangent bundle of the strata and satisfying $\Lambda|_Y \equiv Y$. By the Łojasiewicz inequality we may find an integer k such that this distribution is Lipschitz with exponent k . Moreover $(t_1^{1,1})$ is fulfilled since Λ is continuous. By, theorem 3.2.1, this implies that the pull-back $F^* \Lambda$ by a deformation of transversals F of contact at least $|k|$ is Lipschitz with exponent

0. Therefore by Lemma 1.1.2 we then may trivialize F^*X and thus the $|k|$ -jet of v is bi-Lipschitz $\mathcal{S}$ -sufficient in $C^{|k|}$ for any direct transversal v .

On the other hand, it is well known that the number of Lipschitz types of a global subanalytic family is finite [P] [V2]. Therefore, the number of Lipschitz types of germs of intersections of a polynomial direct transversals to Y of bounded degree with a X is finite. $\square$

Remark 9. Actually we do not need the (a) regularity with respect to Y . The (t^i) regularity (for some i) for the couples $(Y; S_s)$ would be enough.

6.2. Kuo and Trotman's Blowing up and (L) conditions. The blowing-up of T. C. Kuo and D. Trotman [KuT] is a transformation of stratified spaces. Actually, up to a chart of the Grassmanian it is a particular case of pull-back by a deformation of transversals.

We fix a stratification $\{S, Y\}$ of X composed by two strata, that is to say, we assume $Y \subseteq \overline{S} \setminus S$ and S is smooth. Let us define:

$$\begin{aligned}\tilde{Y} &= \{(0; P) \in \{0_{\mathbb{R}^d}\} \times \mathbb{G}_d^n / P \pitchfork Y\} \\ \tilde{S} &= \{(q; P) \in \mathbb{R}^d \times \mathbb{G}_d^n / q \in P \cap S, P \in \tilde{Y}\}.\end{aligned}$$

In [KuT] it is proved that if $\mathcal{S}$ satisfies the (t^i) condition then $\{\tilde{S}, \tilde{Y}\}$ is a stratification (t^{i-1}) regular. In particular, if a couple of strata $\mathcal{S}$ satisfies the (t) condition then $(\tilde{S}; \tilde{Y})$ is a (w) -regular stratification. Here we prove that:

Theorem 6.2.1. *If $\mathcal{S}$ satisfies the (L) condition of Mostowski then the tangent distribution to $\tilde{Y}$ may be extended to a distribution tangent to $\tilde{S}$, C^1 in the sense of Whitney. In particular $\tilde{X}$ is C^1 trivial along $\tilde{Y}$.*

Proof. Up to a chart of the grassmannian manifold we may write $\tilde{S}$ and $\tilde{Y}$ as follows:

- (1) $\tilde{Y} = \{0_{\mathbb{R}^d}\} \times U$ where U is an open subset of $\mathbb{R}^{nm}$.
- (2) $\tilde{S} = \{(x; t; u) \in S_i \times U / \sum_{i=1}^n x_i u_{ji} = t_j \quad \forall j \in \{1, \dots, m\}\}$
where $x = (x_1; \dots; x_n)$, $t = (t_1; \dots; t_m)$, $u = (u_{ji})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$.

Let $p(x; t; u) = (x; u) \in \mathbb{R}^n \times \mathbb{R}^{nm}$ and $\hat{S} = p(\tilde{S})$. Now let us set

$$F(x; u) = (x; \sum_{i=1}^n u_{1i} x_i; \dots; \sum_{i=1}^n u_{mi} x_i).$$

Then F is a deformation of order 1. Let Λ be the m -distribution on S defined by $\Lambda_q = P_q(Y)$ where P_q denotes the orthogonal projection onto $T_q S$. As $\mathcal{S}$ satisfies the condition (L) , the couple $(\Lambda; \mathcal{S})$ fulfills it as well and by Theorem 3.2.1 we get that $F^* \Lambda$ is Lipschitz with exponent 1 which means that it is C^1 distribution. By definition of the pull-back $F^* S = \hat{S}$ and in consequence $F^* \Lambda$ induces over $\hat{S}$ a sub-distribution of the tangent distribution $\hat{S}$. As p restricted to $\tilde{S}$ is a diffeomorphism we are done. $\square$

Example 2. Let us define the set X by: $X = \{(x; y; z;)/y^3 = z^4x^5 + x^7\}$. Then the singular locus of X is the axis O_z . It is possible to see [Y] that the stratification $(X \setminus O_z; O_z)$ is a Lipschitz stratification in the sense of Mostowski. Then by the above theorem the Grassmann blow-up $\tilde{X}$ is C^1 trivial along the set of vector spaces directly transverse to O_z .

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