

MULTIPLICITY MOD 2 AS A METRIC INVARIANT

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ABSTRACT. We study the multiplicity modulo 2 of real analytic hypersurfaces. We prove that, under some assumptions on the singularity, the multiplicity modulo 2 is preserved by subanalytic bi-Lipschitz homeomorphisms of \mathbb{R}^n . In the first part of the paper, we find a subset of the tangent cone which determines the multiplicity *mod* 2 and prove that this subset of S^n is preserved by the antipodal map. The study of such subsets of S^n enables us to deduce the subanalytic metric invariance of the multiplicity modulo 2 under some extra assumptions on the tangent cone. We also prove a real version of a theorem of Comte, and yield that the multiplicity modulo 2 is preserved by arc-analytic bi-Lipschitz homeomorphisms.

0. INTRODUCTION

During the three last decades, various criteria for topological stability of singular algebraic or analytic subsets were given. Numerous topological invariants were also found, especially in algebraic geometry. For classification problems, invariants are very useful to distinguish different classes. More recently, attention started to focus on the metric type of singularities. But if we have lots of invariants describing the topological type of singularities, we dispose of rather few ones devoted to the metric type of a singularity and very few of them are explicitly computable from the equation of an analytic set.

This paper deals with analytic hypersurfaces of \mathbb{R}^n and focuses on their multiplicity. By multiplicity of a real analytic germ of hypersurface defined by a reduced analytic equation $P = 0$, we mean the lowest degree in the expansion of P . In complex algebraic geometry the multiplicity has a geometric interpretation as the cardinality of a generic fiber of a generic projection onto a hyperplane. If we consider the multiplicity modulo 2, this geometric interpretation also holds for real analytic hypersurfaces.

O. Zariski asked whether two germs analytic hypersurfaces of \mathbb{C}^n which are homeomorphic have the same multiplicity. The Zariski conjecture is still open more than thirty years later. It is easy to find examples of real analytic sets which are homeomorphic and which have different multiplicities even modulo 2. For instance a cusp defined by $\{(x; y) \in \mathbb{R}^2 / y^2 = x^3\}$ is homeomorphic to a straight line. This raises a very natural question: is the multiplicity *mod* 2 a metric invariant for real analytic hypersurfaces ?

Metric classification refers to classification up **bi-Lipschitz maps** $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$, which are the maps satisfying:

$$\frac{1}{C}|x - x'| \leq |h(x) - h(x')| \leq C|x - x'|,$$

for some constant $C \geq 1$. The smallest constant C for which this inequality holds is then called the **Lipschitz constant** of h and is denoted by $Lip(h)$.

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One will observe that in the above example the two given curves are homeomorphic but *not* bi-Lipschitz homeomorphic.

In this paper we investigate this problem and give a partial answer. We prove that the multiplicity modulo 2 of real analytic hypersurfaces is preserved by subanalytic bi-Lipschitz maps under an extra assumption on the singularity which, roughly speaking, is somewhat stronger than requiring that the singularities of the link are isolated. Namely we introduce what we call the odd part of the tangent cone and are able to prove invariance of the multiplicity modulo 2 in the case where this subset has only isolated singularities (Theorem 4.2). We prove this result for all the hypersurfaces of \mathbb{R}^3 as well (Theorem 4.1). We also prove a real version of a theorem of George Comte: the multiplicity modulo 2 of a real analytic hypersurface is preserved by a subanalytic bi-Lipschitz homeomorphism whose Lipschitz constant is sufficiently close to one (Theorem 4.4). For complex analytic sets, a similar theorem has been proved by Comte in [C].

It is proved in [FFW] that any semialgebraic subset of S^n is the tangent cone of an algebraic singular set. It would be interesting to investigate the question whether every semialgebraic subset (or subanalytic subset) of S^n which is preserved by the antipodal map may be the odd part of the tangent cone of a subanalytic singular set.

Content of the paper. We start by showing (Proposition 1.4) how the multiplicity *mod* 2 can be computed from a part of the tangent cone (*called the odd part of the tangent cone*). In section 2 we prove that the odd part of the tangent cone is equivariant (preserved by the antipodal map). Section 3 is devoted to the topology of equivariant submanifolds of S^n . This enables us to show that the multiplicity modulo 2 is preserved by arc-analytic bi-Lipschitz homeomorphisms. Relating the multiplicity to the Euler characteristic of the odd part of the tangent cone when it has only isolated singularities, we prove that the multiplicity modulo 2 is a topological invariant of the odd part of the tangent cone. The last section concludes about our equimultiplicity results which are the main results of this article.

Notations. Throughout this paper we write $a \equiv b$ if a and b are two integers which are equal *mod* 2. We write $a \equiv b[4]$ if they are equal modulo 4. Given a subset X of \mathbb{R}^n we denote by $cl(X)$ its closure and $|\cdot|$ stands for the Euclidian norm of \mathbb{R}^n . We write $\mathbb{R}_{>0}$ for the positive real numbers.

All the **stratifications** are assumed topologically trivial. Given a subanalytic set C we write $Sing C$ for the singular locus of C i. e. the set of all the points of C at which the set C is not a topological manifold.

Given $\lambda \in S^n$ we write $\pi_\lambda : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ for the orthogonal projection onto λ^\perp .

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1. EULER CYCLES

In this section we relate the multiplicity of analytic hypersurfaces to their tangent cones (Proposition 1.4).

Notations. The projection π_λ defined above induces a mapping $\pi'_\lambda : S^n \setminus \pm\lambda \rightarrow S^{n-1}$ defined by $\pi'_\lambda(u) := \frac{\pi_\lambda(u)}{|\pi_\lambda(u)|}$. Let C be a subanalytic subset of S^n . We set:

$$\varphi_\lambda(x) := \text{card } \pi'^{-1}_\lambda(x) \cap C,$$

(which is finite for generic x and λ).

We start by characterizing the sets for which φ_λ is constant generically modulo 2.

Definition 1.1. An $(n-1)$ dimensional subanalytic set C is said to be an **Euler cycle** if it is a closed set and if, for a stratification of C (and hence for any), the number of $(n-1)$ dimensional strata containing a given $(n-2)$ dimensional stratum in their closure is even.

This is a well known property of real algebraic sets (see for instance [BCR], Theorem 11.1.1).

Proposition 1.2. *Let C be a subanalytic subset of S^n and let $\lambda \in S^n$ be generic. Then, $\varphi_\lambda(x)$ is constant modulo 2 for x generic in S^{n-1} if and only if C is an Euler cycle.*

Proof. Considering a cylindrical decomposition we may extract a finite partition of S^{n-1} such that φ_λ is constant on each element of this partition. Fix some stratifications of C and S^{n-1} respectively in such a way that π'_λ maps submersively strata onto strata. Choose a $(n-2)$ dimensional stratum $\beta \subset S^{n-1}$. Then there exist exactly two strata of S^{n-1} , say α and α' , containing β in their closure.

Let $\pi'^{-1}_\lambda(\beta) \cap C := \{\beta_1, \dots, \beta_k\}$, $\pi'^{-1}_\lambda(\alpha) \cap C := \{\alpha_1, \dots, \alpha_m\}$ and $\pi'^{-1}_\lambda(\alpha') \cap C := \{\alpha'_1, \dots, \alpha'_{m'}\}$.

For $i \leq k$ we set

$$(1.1) \quad j_i := \text{card}\{\alpha_l/\beta_i \subset \text{cl}(\alpha_l)\}$$

(resp. $j'_i = \text{card}\{\alpha'_l/\beta_i \subset \text{cl}(\alpha'_l)\}$).

If C is an Euler cycle then $j_i \equiv j'_i$ for each i . So $\sum_{i=1}^k j_i \equiv \sum_{i=1}^k j'_i$ and hence $m \equiv m'$.

To prove the converse, assume that C is not an Euler cycle. Thus, the condition of the definition fails, say at β_1 . Let $\beta = \pi'_\lambda(\beta_1)$, then β lies in the closure of two $(n-1)$ dimensional strata of S^{n-1} , say α and α' . But, as we assumed that the condition of Definition 1.1 fails along the stratum β_1 , this implies (with the notation introduced in (1.1)): $j_1 \equiv 1 + j'_1$.

For a sufficiently generic λ we may assume that $\pi'^{-1}_\lambda(\pi'_\lambda(\beta_1) \cap C)$ is the union of β_1 and of some strata lying in the closure of exactly two strata (this holds in particular if they lie in the regular locus of C). This implies $j_i \equiv j'_i$ for $i > 1$ and hence $m \equiv 1 + m'$. \square

Hence, for an Euler cycle C , we denote by $m_\lambda(C)$ the generic value modulo 2 of $\varphi_\lambda(x)$. A priori this depends on λ .

1.1. Multiplicity mod 2 and tangent cone. If A is the germ of an analytic hypersurface of \mathbb{R}^{n+1} it is well known that $\text{card } \pi_\lambda^{-1}(x) \cap A$ is constant modulo 2 for λ generic in S^n and x generic in \mathbb{R}^n sufficiently close to zero. This number coincides with the **multiplicity mod 2** of A at the origin, we will denote it by m_A .

We are going to relate the notion of multiplicity *mod 2* of a real analytic set to the tangent cone. We recall that the tangent cone of a subanalytic set A of \mathbb{R}^{n+1} is defined by:

$$C_A = \{\lambda \in S^n / \exists \gamma : [0; \varepsilon[\rightarrow A, \gamma(t) = \lambda t + \dots \}.$$

Usually, the tangent cone is rather the cone over this subset, but it will be more convenient to work with subsets of S^n .

Every subanalytic map germ $h : (\mathbb{R}^{n+1}; 0) \rightarrow (\mathbb{R}^{n+1}; 0)$ is Lipschitz differentiable in the sense that $\lim_{r \rightarrow 0} \frac{h(rx)}{r}$ exists, for any $x \in \mathbb{R}^n$. Indeed, the arc $r \mapsto h(rx)$ is mapped onto a subanalytic arc which must have an endpoint. Denote by $d_0 h(x)$ the latter limit. Given a subanalytic bi-Lipschitz homeomorphism $h : (\mathbb{R}^{n+1}; 0) \rightarrow (\mathbb{R}^{n+1}; 0)$ we may now define a mapping $h' : S^n \rightarrow S^n$ by setting:

$$(1.2) \quad h'(x) := \frac{d_0 h(x)}{|d_0 h(x)|}.$$

The bi-Lipschitz condition implies that $d_0 h(x)$ is nonzero so that h' is well defined.

We will deal with arc-analytic functions. They have been introduced by Kurdyka in [K]. We recall their definition.

Definition 1.3. Let U be an open subset of the origin in \mathbb{R}^{n+1} . A function $f : U \rightarrow \mathbb{R}$ is **arc-analytic** if for every analytic arc $\gamma : [0; \varepsilon) \rightarrow U$ the function $f \circ \gamma$ is analytic.

It is possible to compute the multiplicity mod 2 of an Euler cycle from its tangent cone.

Proposition 1.4. *Let A be a germ (at 0) of analytic hypersurface of \mathbb{R}^{n+1} . Then, there exists an Euler cycle $C'_A \subset C_A$ of pure dimension $(n-1)$, if nonempty, such that:*

$$m_\lambda(C'_A) \equiv m_A$$

for λ generic.

Proof. First we are going to associate, to a given hypersurface A , a subanalytic function $s_A : C_A \setminus N \rightarrow \mathbb{N}$, where N is a subanalytic subset of S^n with $\dim N \leq n-2$, such that for any generic $\lambda \in S^n \setminus C_A$ we have

$$(1.3) \quad \sum_{y \in \pi_\lambda^{-1}(rx)} s_A(y) = \text{card } \pi_\lambda^{-1}(rx) \cap A$$

for any generic $x \in S^{n-1}$ and any positive real number r small enough.

A subanalytic function which takes only integral values, is constant on the strata of a subanalytic stratification. Thus, it will be enough to define C'_A as the closure of the union of the strata on which s_A is odd, in order to get a subset satisfying (1).

To define the function s_A let us consider the spherical blowing-up at 0:

$$e : S^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^{n+1}$$

defined by $e(x; r) = rx$. Then note that:

$$C_A = \text{cl}(e^{-1}(A)) \cap S^n \times \{0\},$$

and take a stratification S of $\text{cl}(e^{-1}(A))$ compatible with C_A . We may construct simultaneously a stratification S' of \mathbb{R}^n in such a way that $\pi_\lambda \circ e$ maps submersively strata onto strata. Now, for y in an $(n-1)$ -dimensional stratum Y of S included in C_A , we can define $s_A(y)$ as the number of strata Y' satisfying $\text{cl}(Y') \setminus Y' \supseteq Y$.

Let $y \in S^n$ be generic. For r small enough we set:

$$u := \text{card } \pi_\lambda^{-1}(ry) \cap A.$$

Let also:

$$\pi_\lambda^{-1}(e(y; r)) \cap A := \{\gamma_1(r), \dots, \gamma_u(r)\}$$

and $\pi_\lambda'^{-1}(y) \cap C_A := \{y_1, \dots, y_p\}$.

Note that each path $\tilde{\gamma}_i := e^{-1}(\gamma_i)$ ends at some y_i . Hence, to check (1.3) it is enough to check that the number of $\tilde{\gamma}_j$'s ending at y_i is precisely $s_A(y_i)$. Fix $j \leq p$ and let $Y_1, \dots, Y_{s_A(y_j)}$ be the strata such that $y_j \in cl(Y_i)$, $1 \leq i \leq s_A(y_j)$. As the map $\pi_\lambda \circ e|_{Y_i}$ is proper and submersive, it is a homeomorphism (we may assume the strata homeomorphic to open balls of \mathbb{R}^n). Therefore the γ_k 's may only belong to distinct Y_i . It means that $\sum s_A(y_j) \leq u$. Hence, it is enough to check that each Y_i contains one of the γ_k 's.

Fix $i \leq m$. By the curve selection lemma we can find a curve $\tilde{\gamma}$ in Y_i ending at y_j . Hence we see that $\gamma = e(\tilde{\gamma})$ is a subanalytic curve ending at the origin and tangent to y_j . Let us remark that the vector y is tangent to $\pi_\lambda(\gamma)$. Let M be the stratum containing $e(y; r)$. Note that $\pi_\lambda \circ e : Y_i \rightarrow M$ is a homeomorphism. As y is generic we can assume that $\dim M = n$ and that y is not in the tangent cone of the boundary of M . But as the vector y is tangent to $\pi_\lambda(\gamma(r))$ this implies that $e(y; r)$ is in the image of $e \circ \pi_\lambda|_{Y_i}$ for r small enough, as required.

By definition, for x generic, $s_A(x)$ is nothing but the number of connected components of

$$e^{-1}(A \setminus \{0\}) \cap B(x; \varepsilon),$$

for $\varepsilon > 0$ small enough. It is thus independent of the considered stratification. As C'_A is defined as the closure of the set of points in C'_A at which s_A is odd, we see that C'_A is independent of the stratification.

Note that the fact that C'_A is an Euler cycle follows from the preceding proposition. \square

Proposition 1.5. *(metric invariance of C'_A) Given two germs of analytic hypersurfaces A and B and a germ of subanalytic or arc-analytic bi-Lipschitz homeomorphism $h : (\mathbb{R}^{n+1}; 0) \rightarrow (\mathbb{R}^{n+1}; 0)$ such that $h(A) = B$, we have*

$$h'(C'_A) = C'_B,$$

(see (1.2) for h').

Proof. The map h' sends C_A into C_B . Two rays rx and rx' are mapped onto subanalytic arcs which may not be tangent since h is bi-Lipschitz. Hence h' is a homeomorphism. Let s_A be the function defined in the proof of the previous proposition for A and denote by s_B the analogous function for B .

Recall that, for x generic $s_A(x)$ is the number of connected components of

$$(cl(e^{-1}(A)) \setminus C_A) \cap B(x; \epsilon)$$

(for ϵ small enough), where $e : S^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^{n+1}$ is the spherical blowing-up $e(x; r) = rx$. As h is a homeomorphism, we clearly have $s_B(h'(x)) = s_A(x)$, for any x generic in C_A , and hence h' sends C'_A onto C'_B . \square

The function s_A in the proof corresponds to the multiplicities introduced by K. Kurdyka and G. Raby in [KR]. In the latter paper, the authors point out that those multiplicities are analytic invariants. The above argument proves that they are actually subanalytic metric invariants.

As the set C'_A corresponds to the points having an odd multiplicity we will call it **the odd part of the tangent cone**. This section shows that only the odd part is relevant to determine the multiplicity *mod* 2 of a given analytic hypersurface.

We shall need the following lemma.

Lemma 1.6. *Let $h : (\mathbb{R}^{n+1}; 0) \rightarrow (\mathbb{R}^{n+1}; 0)$ be a germ of subanalytic bi-Lipschitz homeomorphism with Lipschitz constant L . Then h' satisfies for any x and x' in S^n :*

$$(1.4) \quad \frac{1}{L^2}|x - x'| - \delta \leq |h'(x) - h'(x')| \leq L^2|x - x'| + L\delta$$

where $\delta := L - 1$.

Proof. Define:

$$\tilde{h}_r(x) := (|h(rx)|; \frac{h(rx)}{|h(rx)|}).$$

Now observe that, due to the bi-Lipschitz character of h , we must have:

$$||h(rx)| - r| \leq \delta r.$$

Therefore, from the definition of \tilde{h}_r , an easy computation shows that:

$$|\tilde{h}_r(x) - \tilde{h}_r(x')| \leq L^2|x - x'| + L\delta.$$

Arguing on \tilde{h}_r^{-1} we get the other inequality. \square

2. HYPERSURFACES AND PATHS

In this section the letter C will stand for a subanalytic closed Euler cycle of S^n , $n > 1$.

Definition 2.1. A subanalytic C^1 arc $\gamma : [0; 1] \rightarrow S^n$ is said to be an **allowed path for C** if for every t in the set $I_\gamma = \{t \in [0; 1] / \gamma(t) \in C\}$ the point $\gamma(t)$ is a regular point of C at which the mapping γ is transverse to C .

Note that the set I_γ is always finite, so that we may define the **length** of an allowed path γ by:

$$\text{lg}_C \gamma = \text{card } I_\gamma.$$

It is easy to see that two points of the complement of C in S^n may be joint by an allowed path. Therefore we may define for λ and μ in $S^n \setminus C$:

$$d_C(\lambda; \mu) := \min\{\text{lg}_C \gamma / \gamma \text{ allowed path joining } \lambda \text{ and } \mu\}.$$

This is not a distance on $S^n \setminus C$ since it can vanish at two distinct points (if they lie in the same connected component). Actually this is a distance on the (finite) set of the connected components of $S^n \setminus C$. It is well known that the homology class of the intersection of two cycles only depends on the homology class of the cycles. We use this fact in the next proposition to derive a property of the distance d_C which will be useful for our purpose.

Proposition 2.2. *The value of $d_C(\lambda; \mu) \bmod 2$ is independent of the allowed path joining λ and μ .*

Proof. Let γ_1 and γ_2 be two allowed paths joining λ and μ . The union γ of these two allowed paths defines a 1-geometric cycle γ . Therefore, as C is an Euler cycle, the homology class (in $H_0(S^n; \mathbb{Z}_2)$) of $C \cap \gamma$ only depends on the homology classes of C and γ .

But, since $H^1(S^n; \mathbb{Z}_2)$ is zero, the Poincaré dual of the fundamental class of C is zero and thus its cup product with the dual of the fundamental class of γ vanishes as well. On the other hand, by definition, if $[\cdot]$ denotes the dual of the fundamental class we have:

$$\text{lg}_C \gamma \equiv [\gamma([0; 1]) \cap C]$$

(identifying $H^n(S^n; \mathbb{Z}_2)$ with \mathbb{Z}_2). Hence $\text{lg}_C \gamma_1 + \text{lg}_C \gamma_2 \equiv 0$, as required. \square

Note that an immediate consequence of Proposition 2.2 is:

$$(2.5) \quad d_C(\lambda; \mu) \equiv d_C(\lambda; \tau) + d_C(\tau; \mu)$$

for any λ, τ and μ in $S^n \setminus C$.

Remark 1. We may associate a graph G_C to C , whose vertices are the connected components of $S^n \setminus C$ and the edges are connecting two vertices V and V' satisfying $\dim cl(V) \cap cl(V') \geq n - 1$. The distance d_C is now the geodesic distance in the graph. Proposition 2.2 says that the graph G_C is bi-colorable.

2.1. Multiplicity mod 2 and d_C . In this section A is the germ of an analytic hypersurface of \mathbb{R}^{n+1} . The relation between d_C and the multiplicity is given by the following proposition.

Proposition 2.3. *For any $\lambda \in S^n \setminus C_A$, we have:*

$$m_A \equiv d_{C'_A}(\lambda; -\lambda).$$

Proof. Choose a generic fiber of π'_λ . This provides an allowed path from λ to $-\lambda$. Its length modulo 2 is $m_\lambda(C'_A)$ if the fiber is sufficiently generic. But, by Proposition 1.4, $m_\lambda(C'_A) \equiv m_A$. \square

As a consequence we see that $d_{C'_A}(\lambda; -\lambda)$ only depends on the connected component of $S^n \setminus C_A$ where λ is chosen. The number m_λ merely says whether λ and $-\lambda$ are of the same color in the coloring of $S^n \setminus C'_A$.

Definition 2.4. We will say that a set $C \subseteq S^n$ is **a -invariant** if it is preserved globally by the antipodal map (i. e. $a(C) = C$, with $a(x) = -x$). In this case $d_C(-\lambda; -\mu) = d_C(\lambda; \mu)$ for any λ and μ in $S^n \setminus C$.

Proposition 2.5. *Modulo 2, the number $d_C(\lambda; -\lambda)$ is independent of $\lambda \in S^n \setminus C$ if and only if C is a -invariant.*

Proof. To prove the "only if" remark that by (2.5) we have

$$d_C(-\lambda; -\lambda) \equiv d_C(\lambda; \mu) + d_C(\mu; -\mu) + d_C(-\mu; -\lambda).$$

Then the result follows from $d_C(\lambda; \mu) = d_C(-\mu; -\lambda)$. For the "if" part, as C is closed, it is enough to show $a(x) \in C$, for x in a dense subset of C . Take x generic in C , with $a(x)$ not in C . If x is sufficiently generic, then C is a topological manifold at C . The set C separates a little neighborhood of x in S^n in two connected components. Thus, we may pick λ and μ close to x such that $d_C(\lambda; \mu) = 1$. Using the above identity, we see that, if $d_C(\lambda; -\lambda) \equiv d_C(\mu; -\mu)$ then

$$d_C(-\lambda; -\mu) \equiv 1.$$

But as $a(x)$ does not belong to C , $d_C(\lambda; \mu) = 0$, if λ and μ are chosen sufficiently close to x . A contradiction. \square

Remark 2. As a consequence of Propositions 2.3 and 2.5, the odd part of the tangent cone is a -invariant. In view of the two above propositions it is natural to set $m_C := d_C(\lambda; -\lambda)$, for a generic λ , so that now

$$m_A = m_{C'_A}.$$

It is already enough to establish the invariance of the multiplicity modulo 2 under arc-analytic bi-Lipschitz mappings. Arc-analytic bi-Lipschitz mappings are studied in [FKP] where it is shown that the inverse of a bilipschitz arc-analytic map is arc-analytic.

Theorem 2.6. *Multiplicity mod 2 is an arc-analytic bi-Lipschitz invariant for real analytic hypersurfaces of \mathbb{R}^n .*

Proof. Let A and B be two germs of analytic sets and let h be an arc-analytic bi-Lipschitz homeomorphism sending A onto B . By Proposition 1.5, h' is a homeomorphism sending C'_A onto C'_B . Observe that, as h is arc-analytic, a ray $r \mapsto r\lambda$ (with $\lambda \in S^n$) is sent onto an analytic arc which may not be a cusp since h is bi-Lipschitz. This clearly means that the tangent half-lines at 0 of the image of this analytic arc have opposite directions, that is to say $h'(-\lambda) = -h'(\lambda)$. Therefore

$$m_A = d_{C'_A}(\lambda; -\lambda) = d_{C'_B}(h'(\lambda); h'(-\lambda)) = m_B.$$

□

It is proved in [KK] that the lines $\{x^2 = y^3\}$ and $\{x^3 = y^7\}$ are arc-analytically homeomorphic. Hence the hypothesis "bi-Lipschitz" may not be omitted.

Remark 3. As the Yang's index (see [Y]) is preserved by equivariant mappings, the Yang's index of C'_A is an arc-analytic bi-Lipschitz invariant.

3. ON a -INVARIANT HYPERSURFACES OF S^n

3.1. The case of S^2 . In this section we focus on hypersurfaces of S^2 . This will enable us to prove the invariance through a subanalytic bi-Lipschitz homeomorphism of the multiplicity mod 2 for hypersurfaces of \mathbb{R}^3 in section 4. Actually in this setting we have a better result. The multiplicity mod 2 may be characterized just by means of the distance d_C introduced above. This fact is no longer true in higher dimensions.

More precisely, let us define the diameter of an Euler cycle $C \subset S^2$ as the integer:

$$\delta_C = \sup\{d_C(\lambda; \mu)/\lambda, \mu \in S^2 \setminus C\}.$$

Theorem 3.1. *If C is an a -invariant Euler cycle of S^2 of dimension one then*

$$\delta_C \equiv m_C.$$

Proof. We need to check that $\delta_C \equiv d_C(-\lambda; \lambda)$, for some $\lambda \in S^2 \setminus C$. Let λ and μ be such that $\delta_C = d_C(\lambda; \mu)$. We actually are going to see that

$$d_C(\lambda; \mu) \leq \max(d_C(\lambda; -\lambda); d_C(\mu; -\mu)).$$

Let γ be an allowed path from λ to $-\lambda$ realizing the distance. Observe that $-\gamma$ is also such a path. The union $\gamma \cup -\gamma$ separates the sphere into at least two connected components. Remark that, as this path is a -invariant, the points μ and $-\mu$ may not lie in the same connected component of $S^2 \setminus \gamma \cup -\gamma$ (see for instance [J] Corollary 3). It means that there is a geodesic σ from μ to $-\mu$ which cuts a geodesic σ' from λ to $-\lambda$ (either γ or $-\gamma$). Let θ be a connected component crossed by both σ and σ' .

Note that, as θ is crossed by a geodesic from μ to $-\mu$, we have

$$d_C(\mu; \theta) + d_C(\theta; -\mu) = d_C(\mu; -\mu).$$

So that if $d_C(\theta; -\mu) \geq d_C(\lambda; \theta)$ then we get

$$d_C(\lambda; \mu) \leq d_C(-\mu; \mu).$$

Hence we may suppose $d_C(\theta; -\mu) \leq d_C(\lambda; \theta)$. But then as we also have:

$$d_C(\lambda; \theta) + d_C(\theta; -\lambda) = d_C(\lambda; -\lambda)$$

we get:

$$d_C(-\lambda; -\mu) \leq d_C(\lambda; -\lambda).$$

As $d_C(\lambda; \mu) = d_C(-\lambda; -\mu)$, we are done. \square

Even for a hypersurface of S^3 , the above proposition is no longer true. For instance, the torus

$$\{x \in \mathbb{R}^4 : x_1^2 + x_2^2 = \frac{1}{2}, x_3^2 + x_4^2 = \frac{1}{2}\}$$

is an a -invariant subset of S^3 and has a diameter equal to 1 while the distance from a connected component to its opposite is zero.

3.2. On a -invariant submanifolds of S^n . We will say for short that an a -invariant Euler cycle C of S^n is **even** if $m_C = 0$ and **odd** if $m_C = 1$ (see Remark 2). The question raised by section 2 is the following: can an even hypersurface of S^n be homeomorphic to an odd one?

We answer negatively in this section in some particular cases. We start by considering the case of manifolds that are submanifolds of an odd dimensional sphere. In this case, it is known that we can have a characterization of the multiplicity in terms of the Euler characteristic (see for instance [Vi], 1.3.B).

Proposition 3.2. *Let C be an a -invariant subanalytic submanifold of S^{2k+1} of dimension $2k$. Then*

$$m_C \equiv \frac{1}{2}\chi(C).$$

Proof. Let $\pi : S^{2k+1} \rightarrow \mathbb{R}P^{2k+1}$ be the quotient map and let $C' := \pi(C)$. Observe that $m_C \equiv 0$ if and only if the number of intersection points of C' with a generic projective line is even, which means that the number of intersection points with any closed subanalytic generic closed path is even.

Therefore, if $m_C = 0$ then we may associate a color (say either white or black) to each connected component of $\mathbb{R}P^{2k+1} \setminus C'$ such that two adjacent connected components have opposite colors. But, thanks to Poincaré duality, this implies that

$$\chi(C') = \sum_{F \in \mathcal{C}} \chi(\partial F) = \sum_{F \in \mathcal{C}} 2 \cdot \chi(F)$$

where \mathcal{C} is the family constituted by the closure of the white connected components. As $\chi(C) = 2\chi(C')$ this proves the result in the case where m_C is even.

Now, if m_C is nonzero we may stratify C' and choose a generic projective hyperplane P cutting transversely the strata so that $P \cap C'$ is a topological manifold of dimension $2k - 1$. This implies that

$$\chi(P \cap C') = 0.$$

Observe that $P \cup C'$ is even and so, like in the even case, we may choose a coloring of the complement. The same computation yields

$$\chi(P \cup C') \equiv 0,$$

which means that $\chi(C')$ is odd from which we conclude that $\frac{1}{2}\chi(C)$ is odd. \square

Proposition 3.2 does generalize to singular sets as it is shown by the following example:

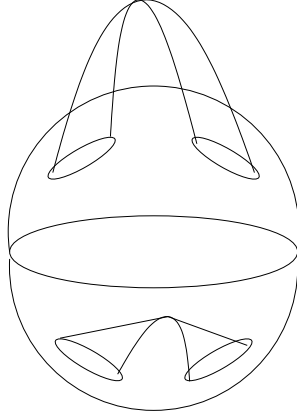


FIGURE 1.

The above picture shows a singular a -invariant subset of $\mathbb{R}^3 \cup \{pt\} \approx S^3$. It is invariant with respect to the involution of S^3 (and not that of \mathbb{R}^3). Thus, the two pinched handles are interchanged by the antipodal map. This set is odd but the Euler characteristic is zero. However, we are going to show that in the case where we have only isolated singularities, an even subset may not be homeomorphic to an odd one (Proposition 3.3).

In the case where we only have an isolated singularities we may deduce the result invariance of the multiplicity by deleting the singularities.

Proposition 3.3. *Let C_1 and C_2 be two a -invariant subanalytic hypersurfaces of S^{2k+1} which have only isolated singularities. Assume that there exists a homeomorphism $h : S^{2k+1} \rightarrow S^{2k+1}$ which maps C_1 onto C_2 . Then C_1 is even iff C_2 is.*

Proof. Assume that C_1 is odd and C_2 even. We first choose a coloring of the family of the connected components of the complement of C_1 and C_2 , as explained in Remark 1. We may assume that h preserves the colors. Pulling back the antipodal map by h (i. e. setting $*x := h^{-1}(-h(x))$) we get an involution $*$ on S^n preserving C_1 and preserving the colors.

We denote by W the union of all the white connected components of the complement of C_1 and by B the union of all the black ones. Let us define:

$$\Lambda_{=} := \{x \in \text{Sing } C_1 : \chi(W \cap S(x; \varepsilon)) = \chi(B \cap S(x; \varepsilon))\},$$

where $\text{Sing } C_1$ denotes the points at which C_1 is not a topological manifold. It is well known that $\Lambda_{=}$ is independent of ε if it is chosen small enough. Let also $\Lambda_{\neq} := \text{Sing } C_1 \setminus \Lambda_{=}$.

Observe that $\Lambda_{=}$ and Λ_{\neq} are a -invariant subsets. Moreover, since we have assumed that the singularities are isolated, these sets are finite. We shall define a map $c : \Lambda_{\neq} \rightarrow \{B, W\}$, invariant for both involutions $-$ and $*$. Actually, it is enough to set:

$$c(\lambda) = W, \text{ if } \chi(B \cap S(x; \varepsilon)) < \chi(W \cap S(x; \varepsilon))$$

and $c(\lambda) = B$, otherwise.

Choose also a mapping $d : \Lambda_{=} \rightarrow \{W, B\}$ such that $d(-x) = -d(x)$ (it is clearly possible since $-$ has no fixed point). Now, let us set

$$C_- := (C_1 \setminus \bigcup_{x \in \text{Sing} C_1} B(x; \varepsilon)) \cup \left(\bigcup_{x \in \Lambda_{=}} S(x; \varepsilon) \cap d(x) \right) \cup \left(\bigcup_{x \in \Lambda_{\neq}} S(x; \varepsilon) \cap c(x) \right).$$

It defines an a -invariant subset of S^{2k+1} . Define now an equivariant subset of S^{2k+1} invariant under the action of $*$ by setting:

$$C_* := (C_2 \setminus \bigcup_{x \in \text{Sing} C_2} B(h(x); \varepsilon)) \cup \left(\bigcup_{x \in \Lambda_{=}} S(h(x); \varepsilon) \cap h(W) \right) \cup \left(\bigcup_{x \in \Lambda_{\neq}} S(h(x); \varepsilon) \cap c'(x) \right),$$

where $c'(x) = h(c(x))$.

Note that, as C_1 has been modified only locally at the singularities, the values of d_C have been modified only for the points lying close to $\text{Sing} C_1$ and therefore C_- is still odd and C_* is still even.

Remark that as we have cut off the singular locus, the set $C' := C_1 \setminus \bigcup_{x \in \text{Sing} C_1} B(x; \varepsilon)$ is a manifold with boundary. Near $x \in \text{Sing} C_1$, this boundary is also the boundary of the closure of the manifolds $S(x; \varepsilon) \cup W$ and $S(x; \varepsilon) \cup B$. Therefore, near every $x \in \text{Sing} C_1$, the sets $C' \cup (S(x; \varepsilon) \cap W)$ and $C' \cup (S(x; \varepsilon) \cap B)$ are topological manifolds and hence so are C_- and C_* . We claim that:

$$\chi(C_-) = \chi(C_*),$$

which will clearly contradict Proposition 3.2 since we have already observed that C_- is even and C_* odd.

To prove this claim, first remark that the set C_- and C_* are the respective disjoint union of three sets. The first and the third set involved in each union are germs and links at singular points of homeomorphic sets. As the Euler characteristic of the link is a topological invariant of the singularity, they have the same Euler characteristic. Thus, it is enough to prove that the Euler characteristics of the second set coincide, i. e.:

$$\sum_{x \in \Lambda_{=}} \chi(S(h(x); \varepsilon) \cap h(W)) = \sum_{x \in \Lambda_{=}} \chi(S(x; \varepsilon) \cap d(x)).$$

But, by definition of $\Lambda_{=}$ we have

$$\chi(S(x; \varepsilon) \cap W) = \chi(S(x; \varepsilon) \cap B),$$

for any $x \in \Lambda_{=}$. This implies the preceding equality (no matter $d(x)$ is W or B) and yields the claim. \square

Remark 4. Actually the odd hypersurfaces of S^n are those having Yang's index equal to $(n-1)$ (see [J, Y]). Hence, we have proved in this section that an a -invariant subanalytic subset of S^n with only isolated singularities cannot be homeomorphic to one having a Yang index strictly inferior.

4. MULTIPLICITY MOD 2 AND BI-LIPSCHITZ MAPS

4.1. The main results. We end this article by proving four equimultiplicity theorems that are consequences of the main results of the previous section. We start by the case of hypersurfaces of \mathbb{R}^3 .

Theorem 4.1. *Multiplicity mod 2 is a subanalytic bi-Lipschitz invariant for real analytic hypersurfaces of \mathbb{R}^3 .*

Proof. Let A be the germ of a real analytic hypersurface of \mathbb{R}^3 . By proposition 2.3 and Proposition 1.4 we have

$$m_A = d_{C'_A}(\lambda; -\lambda)$$

for $\lambda \notin C'_A$. By Theorem 3.1 we know that the number $d_{C'_A}(\lambda; -\lambda)$ is a topological invariant of C'_A . But by (2) of Proposition 1.5 if an analytic germ A is subanalytically bi-Lipschitz homeomorphic to an analytic germ B then C'_A is homeomorphic to C'_B . \square

We also can state the following criteria of equimultiplicity *mod* 2:

Theorem 4.2. *Let A be the germ of an analytic hypersurface of \mathbb{R}^n with n even. Assume that C_A has only isolated singularities. Then the multiplicity *mod* 2 of A is preserved by subanalytic bi-Lipschitz homeomorphisms.*

Let us make a little more precise the statement of the above Theorems. By "the multiplicity *mod* 2 of A is preserved by subanalytic bi-Lipschitz homeomorphisms" we mean that another germ of analytic hypersurface B of \mathbb{R}^n , for which we have a germ of subanalytic bi-Lipschitz homeomorphism $h : (\mathbb{R}^n; 0) \rightarrow (\mathbb{R}^n; 0)$ carrying A onto B , has the same multiplicity *mod* 2 as A . Actually the theorem holds not only for analytic hypersurfaces but for all Euler cycles having a "well defined multiplicity" (that is to say an a -invariant odd part of the tangent cone, see Proposition 2.5).

Proof. This can be checked in the same way as Theorem 4.1, replacing Theorem 3.1 by Proposition 3.3. Note that if C_A has only isolated singularities, C'_A has isolated singularities as well. \square

Remark that Proposition 3.2 shows that when n is even it is not necessary to have a homeomorphism of the ambient space.

In the case where n is odd we may prove the metric invariance of the multiplicity if C'_A is a manifold. This is due to the fact that we may take the suspension of the odd part of the tangent cone.

Theorem 4.3. *If C'_A is a topological manifold, the conclusion of the previous theorem holds for any n .*

Proof. We may assume that n is odd. Observe that the suspension of C'_A provides a a -equivariant subset of S^n , which has only isolated singularities. The respective suspensions of two homeomorphic sets are clearly homeomorphic. Thus, we may apply exactly the same argument as in the previous theorem. \square

Of course, if C_A is a topological manifold then so is C'_A .

In order to state the last equimultiplicity theorem, we introduce a notation. Given a germ of analytic hypersurface $A \subset \mathbb{R}^{n+1}$, let:

$$\rho_A := \sup_{x \in S^n} d(x; C'_A),$$

(where C'_A is defined by Proposition 1.4).

We are going to see that, two germs of real analytic sets A and B which are subanalytically bi-Lipschitz homeomorphic have the same multiplicity when the Lipschitz constant is close enough to one. In the complex case, a similar theorem has been proved by Comte in [C].

In the theorem below, the bound for the Lipschitz constant of h could be slightly improved but this complicates the computations. Actually, the most important fact is that we give a bound which only depends on A .

Theorem 4.4. *Let A and B be germs at 0 of real analytic hypersurfaces of \mathbb{R}^{n+1} . Let $h : (\mathbb{R}^{n+1}; 0) \rightarrow (\mathbb{R}^{n+1}; 0)$ be the germ of a subanalytic bi-Lipschitz homeomorphism with $|Lip(h) - 1| \leq \frac{\rho_A^2}{48}$ and $h(A) = B$. Then $m_A \equiv m_B$.*

Proof. By Proposition 1.5, the mapping $h' : S^n \rightarrow S^n$ satisfies $h'(C'_A) = C'_B$ and, by Lemma 1.6, (1.4) holds. Set for simplicity $L := Lip(h)$ and $\delta := L - 1$.

Take $x \in S^n$ such that $d(x; C'_A) = \rho_A$. First we observe that, by definition of ρ_A , we have:

$$d(h'(x); C'_B) \geq \frac{\rho_A}{C^2} - \frac{\rho_A^2}{48} \geq \rho_A - 2\frac{\rho_A^2}{48}\rho_A - \frac{\rho_A^2}{48} \geq \frac{3}{4}\rho_A,$$

(since $\frac{1}{L^2} \geq 1 - 2\delta$ and $\rho_A \leq 2$).

Now remark that if

$$d(-h'(-x); h'(x)) < d(h'(x); C'_B),$$

then $-h'(-x)$ and $h'(x)$ belong to the same connected component of $S^n \setminus C'_B$, and hence, by Proposition 2.3 and Proposition 2.2, the conclusion of the theorem ensues. Therefore, thanks to the previous inequality, it is enough to show that

$$d(-h'(-x); h'(x)) < \frac{3}{4}\rho_A.$$

But by (1.4):

$$d(h'(x); h'(-x)) \geq \frac{2}{L^2} - \frac{\rho_A^2}{48} \geq 2 - 5\frac{\rho_A^2}{48}.$$

By Pythagoras this implies that:

$$d(h'(x); -h'(-x)) \leq \sqrt{\frac{20\rho_A^2}{48}} < \frac{3}{4}\rho_A,$$

as required. □

4.2. A conjecture. The general case still remains undecided, even in \mathbb{R}^4 . As the results of section 1 and 2 are general, it is enough to generalize Proposition 3.3. In other words, the subanalytic bi-Lipschitz invariance of the multiplicity mod 2 for real analytic hypersurfaces would be established if the following conjecture were proved:

Conjecture. Let C and C' be two a -invariant Euler cycles of S^n of dimension $(n - 1)$. Assume that there exists a subanalytic homeomorphism $h : S^n \rightarrow S^n$ such that $h(C) = C'$. Then C is even iff so is C' (see section 3 for the definition).

It seems to be true at least in S^3 (on pictures). Note that it is enough to solve the problem for odd dimensional spheres (or even dimensional) since we may take the suspension of the subsets.

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