**Definition**: An $r \times n$ matrix consisting of the numbers 1, \ldots, $n$ is called a $r \times n$ latin rectangle if the same number never appears twice in any row or any column. If $r = n$, it is a latin square.

For example,

$$
\begin{pmatrix}
1 & 3 & 4 & 2 & 5 \\
4 & 1 & 5 & 3 & 2 \\
2 & 4 & 1 & 5 & 3
\end{pmatrix}
$$

is a $3 \times 5$ latin rectangle.

**Proposition 0.1.** Let $r < n$. Any $r \times n$ latin rectangle can be extended to a $(r + 1) \times n$ latin square.

**Proof.** Consider the bipartite graph where one of the partite sets of vertices corresponds to columns $c_1, \ldots, c_n$ of the Latin rectangle, and the other partite set of vertices $N_1, \ldots, N_n$ corresponds to the numbers 1, \ldots, $n$. Put an edge between vertex $c_i$ and vertex $N_j$ if the column corresponding to $c_i$ does not contain the number corresponding to the vertex $N_j$. For the example above, we have

A perfect matching of this graph corresponds to a way of assigning to each column a number which is not already present in that column. This gives a recipe for extending the latin square by one row.

$$
\begin{pmatrix}
1 & 3 & 4 & 2 & 5 \\
4 & 1 & 5 & 3 & 2 \\
2 & 4 & 1 & 5 & 3
\end{pmatrix}
$$

To complete the proof, it suffices to verify that this graph is $(n - r)$ regular. The edges incident to the $c_i$ vertices correspond to numbers that don’t appear in column $c_i$. There are $(n - r)$ such numbers, so $d(c_i) = n - r$. Next, notice that each number appears $r$ times in the $r \times n$ latin rectangle because it appears once in each row. Therefore, for each number, there are exactly $n - r$ columns which don’t have that number, so $d(N_i) = n - r$.

By $(n - r)$ repeated applications of this proposition to an $r \times n$ latin rectangle, we arrive at a latin square, proving the following corollary.

**Corollary 0.2.** Every Latin rectangle can be extended to a Latin square