A VARIANT OF LEHMER'S CONJECTURE IN THE CM CASE

by

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Abstract

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Lehmer's conjecture asserts that $\tau(p) \neq 0$, where τ is the Ramanujan τ-function. This is equivalent to the assertion that $\tau(n) \neq 0$ for any n. A related problem is to find the distribution of primes p for which $\tau(p) \equiv 0 \pmod{p}$. These are open problems. However, the variant of estimating the number of integers n for which n and $\tau(n)$ do not have a non-trivial common factor is more amenable to study. More generally, let f be a normalized eigenform for the Hecke operators of weight $k \geq 2$ and having rational integer Fourier coefficients $\{a(n)\}$. It is interesting to study the quantity $(n, a(n))$. It was proved by S. Gun and V. K. Murty (2009) that for Hecke eigenforms f of weight 2 with CM and integer coefficients $a(n)$

$$\{n \leq x \mid (n, a(n)) = 1\} = \frac{(1+o(1))U_f x}{\sqrt{L_1(x)L_3(x)}} \tag{1}$$

for some constant U_f. We extend this result to higher weight forms.

We also show that

$$\{n \leq x \mid (n, a(n)) \text{is a prime}\} = \frac{(1+o(1))U_f x L_4^2(x)}{\sqrt{L_1(x)L_3(x)}} \tag{2}$$

the number of $n \leq x$ such that $\gcd(n, a(n))$ is a prime is.

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Chapter 1

Introduction

1.1 The τ function

Consider the cusp form of Ramanujan:

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z} = q \left\{ \prod_{k=1}^{\infty} \left(1 - q^k\right) \right\}^{24},$$

where $q = e^{2\pi i z}$. The coefficients $\tau(n)$ have received extensive arithmetic scrutiny following the ground-breaking investigations of Ramanujan himself.

Of the many problems that are open, there is Lehmer's conjecture that asserts that for any prime p,

$$\tau(p) \neq 0.$$

Equivalently, for any $n \geq 1$,

$$\tau(n) \neq 0.$$

More generally, let

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

be the Fourier expansion of a normalized eigenform of weight $k \geq 2$ and level N, and suppose $a(n) \in \mathbb{Z}$. We can ask a question about the vanishing of the coefficients $a(p)$ or $a(n)$. If the weight is 2, it is known that there are infinitely many primes p for which $a(p) = 0$. If the weight is ≥ 4, it is expected (but not known) that the set of primes p for which $a(p) = 0$ is finite.

A problem closely related to the vanishing of $a(p)$ is to ask whether we can have

$$a(p) \equiv 0 (\text{mod } p).$$

We might expect this to be rare.

In the case of the Ramanujan τ-function, it is known that

$$\tau(p) \equiv 0 (\text{mod } p)$$

holds for primes $p = 2, 3, 5, 7, 2411, 7758337633$ and these are the only primes up to 10^{10} that satisfy this congruence [6], but it is not known if there are infinitely many such primes. Nor do we know any good upper bounds of the number of such primes. In particular, is it true that

$$\#\{p \leq x : \tau(p) \equiv 0 \pmod{p}\} = o(\pi(x))?$$

Or in general, is it true that

$$\#\{p \leq x : a(p) \equiv 0 \pmod{p}\} = o(\pi(x))?$$

Since we have the Ramanujan-Petersson estimate, proved by Deligne [1], [2],

$$|a(p)| \leq 2p^{(k-1)/2},$$

we see that in the weight $k = 2$ case, for $p > 3$ the condition $p|a(p)$ is equivalent to $a(p) = 0$. Heuristically, if the weight is > 2, then the number of primes p up to x for which $p|a(p)$ may grow like $\log\log x$, though we do not even know if these are of density zero. If we assume that the values of $a(p)$ are equidistributed in the interval $[-2p^{\frac{k-1}{2}}, 2p^{\frac{k-1}{2}}]$, then we can roughly evaluate the number of such primes less than x in the following way. Given an $a \in [-2p^{\frac{k-1}{2}}, 2p^{\frac{k-1}{2}}]$, we might expect that the probability that $a(p) = a$ is

$$\frac{1}{4p^{\frac{k-1}{2}} + 1},$$

and therefore the number of $p \leq x$ for which $a(p) = a$ is asymptotically

$$\frac{cx}{p^{\frac{k-1}{2}}}.$$

Since approximately $\frac{1}{p}$ integers in the interval $[-2p^{\frac{k-1}{2}}, 2p^{\frac{k-1}{2}}]$ are divisible by p, we might expect that the probability that $a(p) \equiv 0 (\mathrm{mod}\ p)$ is proportional to $\frac{1}{p}$.

$$\sum_{\substack{p \leq x \\ a(p) \equiv 0 \pmod{p}}} 1 \sim c \sum_{p \leq x} \frac{p^{\frac{k-1}{2}-1}}{p^{\frac{k-1}{2}}} \sim c \sum_{p \leq x} \frac{1}{p} \sim \log\log x.$$

Besides the fact that this is a very rough heuristic, we note that the $a(p)$ are not expected to be equidistributed. Indeed, there is a skewing due to the Sato-Tate measure. However, this will only affect the above heuristic by a constant.

Analogous to the vanishing of $a(p)$ is the vanishing of $a(n)$. Analogous to the question whether $p|a(p)$ is whether $(n, a(n)) = 1$. In particular, we might ask whether

$$\#\{n \leq x : (n, a(n)) \neq 1\} = o(x).$$

In fact, as explained in [7], this is not true, and the correct question to ask is the opposite, namely whether it is true that

$$\#\{n \leq x : (n, a(n)) = 1\} = o(x). \tag{1.1}$$

This is the variant of Lehmer's conjecture that we discuss. We remark that the motivation for the question arises from the possibility of using modular forms for new factoring algorithms.

Let $L_1(x) = \log x$ and $L_i(x) = \log L_{i-1}$.
In a recent work [7] by V. K. Murty (1.1) was considered and the following theorem proved:

Theorem 1.1.1. *(V. K. Murty): For a normalized Hecke eigenform f with rational integer coefficients $a(n)$, one has*

$$\#\{n \leq x|\ (n, a(n)) = 1\} \ll \frac{x}{\log\log\log x}.$$

In [7] it was also anticipated that if f has complex multiplication (CM), a stronger result should hold. Such a result was obtained in [3] for the case that f has CM and is of weight 2:

Theorem 1.1.2. *(S. Gun, V. K. Murty):*
Let f be a normalized eigenform of weight 2 with rational integer Fourier coefficients $\{a(n)\}$. If f is of CM-type, then there is a constant $U_f > 0$ so that

$$\#\{n \leq x|\ (n, a(n)) = 1\} = (1 + o(1))\frac{U_f x}{(\log x \cdot \log\log\log x)^{\frac{1}{2}}}.$$

The constant is given explicitly.

In this thesis we study the analogue of Theorem 1.1.2 for CM forms of weight > 2. We obtain an asymptotic formula in this case as well. In order to do this, we need to surmount some technical obstacles. In [3], essential use is made of a result of Schaal [12] in order to obtain an estimate for the sum

$$\sum_{\substack{q \leq x \\ a(q) \equiv 0 \ (\bmod p)}} \frac{1}{q}$$

of the form $\frac{1}{p}\log\log x$. For weight > 2, the argument given in [3] breaks down and we need to find a replacement. In particular, we are not able to use Schaal's estimate. Rather, we rely on a clever use of the Chebotarev density theorem. This allows us to establish the key Lemma 4.4.1. It will be of interest to see whether our technique can actually be used to strengthen Schaals' theorem. We do not pursue this theme here, but hope to return to it in future work.

A new product that emerges in our estimates is:

$$\prod_{\substack{p \leq x \\ a(p) \equiv 0 \ (\bmod p)}} \left(1 - \frac{1}{p}\right). \tag{1.2}$$

We need to assume that this product converges. Equivalently, we need to have that the sum

$$\sum_{a(p) \equiv 0 \ (\bmod p)} \frac{1}{p}$$

converges.
Denote by Z_f the set

$$Z_f = \{p \text{ prime} \mid p|a(p)\},$$

and by $z(x)$ the number of primes in Z_f such that $p \leq x$:

$$z(x) = \{p \leq x \mid p \in Z_f\}.$$

Hypothesis Z. $z(x) \sim \dfrac{x}{(\log x)^{1+\epsilon}}$.

Hypothesis Z implies that (1.2) converges to a constant in the asymptotic formula.
This product did not emerge in the weight $k = 2$ case, because for $p > 3$ the condition $p|a(p)$ is equivalent

to $a(p) = 0$ and we know that

$$\sum_{a(p)=0} \frac{1}{p}$$

converges. Unfortunately, we do not know the analogue of this for the set Z_f.
However, we show cases when there can be at most finitely many primes in Z_f.

1.2 CM case for higher weight forms

Let

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Let f be a normalized Hecke eigenform of weight $k \geq 2$ for $\Gamma_0(N)$ with complex multiplication and rational integer Fourier coefficients, and let K be the imaginary quadratic field associated to f. I.e. there is a Hecke character Ψ of K with conductor $\mathfrak{m}$ such that

$$f(z) = \sum_{\substack{\mathfrak{a} \in \mathcal{O}_K, \\ (\mathfrak{a}, \mathfrak{m})=1}} \Psi(\mathfrak{a}) e^{2\pi i N(\mathfrak{a}) z}.$$

Here the sum is over integral ideals $\mathfrak{a}$ of the ring of integers $\mathcal{O}_K$ of K which are coprime to $\mathfrak{m}$, and $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$. Thus,

$$a(n) = \sum_{\substack{N(\mathfrak{a})=n, \\ (\mathfrak{a}, \mathfrak{m})=1}} \Psi(\mathfrak{a}),$$

where $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$ is the Fourier expansion of f at infinity. In particular, if a prime p is inert in K, then $a(p) = 0$.
Set

$$M_{f,1}(x) = \#\{n \leq x \mid a(n) \neq 0, \ \ p|n \ \Rightarrow \ p \notin Z_f\}.$$

Then we show that there is a constant u_f so that

$$M_{f,1}(x) = (1 + o(1)) \frac{u_f x}{(\log x)^{\frac{1}{2}}}.$$

Denote by

$$M_{f,1} = \{n \mid a(n) \neq 0, \ \ p|n \ \Rightarrow \ p \notin Z_f\}.$$

Later on, whenever we are dealing with a set of natural numbers E, $E(x)$ will denote the cardinality of the set $\{n \leq x, n \in E\}$.

We also show that there is a constant $\omega_f > 0$ so that

$$\prod_{\substack{p<x \\ a(p) \neq 0}} \left(1 - \frac{1}{p}\right) \sim \frac{\mu_f}{(\log x)^{\frac{1}{2}}},$$

where μ_f is given in Proposition 2.1.1.
The main results of this thesis are the following theorems.

Theorem 1.2.1. *Let f be a normalized eigenform of weight $k \geq 2$ with rational integer Fourier coefficients $\{a(n)\}$ and of CM-type. Assume Hypothesis Z. Then there is a constant $U_f > 0$ so that*

$$\#\{n \leq x \mid (n, a(n)) = 1\} = (1 + o(1)) \frac{U_f x}{(\log x \cdot \log\log\log x)^{\frac{1}{2}}}.$$

The constant is given explicitly in terms of f during the course of the proof.

We also prove the following theorem

Theorem 1.2.2.

$$\#\{n \leq x \mid (n, a(n)) \text{ is prime}\} = (1 + o(1)) \frac{U_f x L_4^2(x)}{\sqrt{L_1(x) L_3(x)}}.$$

Chapter 2

Vanishing of Fourier coefficients

2.1 Vanishing of $a(p)$

We denote by d_K the discriminant of the field K. Recall that if p is inert in K, then $a(p) = 0$, since in this case there exists no ideal of norm p, and so the corresponding coefficient in the definition of a CM-form is equal to zero. Also, p ramifies in K if and only if $p|d_K$.
The following result will be useful in establishing the main result.

Proposition 2.1.1. *There is a constant $\mu_f > 0$ so that*

$$\prod_{\substack{p<z \\ a(p)\neq 0}} \left(1 - \frac{1}{p}\right) = \frac{\mu_f}{\sqrt{\log z}} + O_f\left(\frac{1}{(\log z)^{3/2}}\right).$$

Proof. Using Rosen [11], Theorem 2, we have

$$\prod_{N\mathfrak{p}\leq z} \left(1 - \frac{1}{N\mathfrak{p}}\right)^{-1} = e^{\gamma}\alpha_K \log z + O_K(1).$$

Here the product is over primes $\mathfrak{p}$ of K and α_K is the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$. Note that $\alpha_K = L(1, \chi_K)$, where χ_K is the quadratic character defining K and $L(s, \chi_K)$ is the associated L-function, which is the extension to the whole complex plane of the Dirichlet L-series:

$$L(s, \chi_K) = \sum_{n=1}^{\infty} \frac{\chi_K(n)}{n^s}.$$

It follows that

$$\prod_{N\mathfrak{p}\leq z} \left(1 - \frac{1}{N\mathfrak{p}}\right) = \frac{e^{-\gamma}L(1, \chi_K)^{-1}}{\log z} + O_K\left(\frac{1}{(\log z)^2}\right),$$

and

$$\prod_{N\mathfrak{p}\leq z} \left(1 - \frac{1}{N\mathfrak{p}}\right)^{\frac{1}{2}} = \frac{e^{-\frac{\gamma}{2}}L(1, \chi_K)^{-\frac{1}{2}}}{\sqrt{\log z}} + O_K\left(\frac{1}{(\log z)^{\frac{3}{2}}}\right).$$

Now,

$$\prod_{N\mathfrak{p}\leq z}\left(1-\frac{1}{N\mathfrak{p}}\right)=\prod_{\substack{p\text{ splits in }K\\ p\leq z}}\left(1-\frac{1}{p}\right)^2\prod_{\substack{p\text{ inert in }K\\ p\leq\sqrt{z}}}\left(1-\frac{1}{p^2}\right)\prod_{\substack{p\text{ ramifies in }K\\ p\leq z}}\left(1-\frac{1}{p}\right)$$

So,

$$\prod_{\substack{p\leq z\\ a(p)\neq 0}}\left(1-\frac{1}{p}\right)=\prod_{\substack{p\leq z\\ p\text{ splits in }K\\ a(p)\neq 0}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\leq z\\ p|d_K\\ a(p)\neq 0}}\left(1-\frac{1}{p}\right)=$$

$$=\prod_{\substack{p\leq z\\ p\text{ splits in }K}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\leq z\\ p|d_K\\ a(p)\neq 0}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\leq z\\ p\text{ splits in }K\\ a(p)=0}}\left(1-\frac{1}{p}\right)^{-1}=$$

$$=\prod_{\substack{p\leq z\\ p\text{ splits in }K}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\leq\sqrt{z}\\ p\text{ inert in }K}}\left(1-\frac{1}{p^2}\right)^{\frac{1}{2}}\prod_{\substack{p\leq z\\ p|d_K}}\left(1-\frac{1}{p}\right)^{\frac{1}{2}}\prod_{\substack{p\leq\sqrt{z}\\ p\text{ inert in }K}}\left(1-\frac{1}{p^2}\right)^{-\frac{1}{2}}\prod_{\substack{p\leq z\\ p|d_K}}\left(1-\frac{1}{p}\right)^{-\frac{1}{2}}\cdot$$

$$\cdot\prod_{\substack{p\leq z\\ p|d_K\\ a(p)\neq 0}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\leq z\\ p\text{ splits in }K\\ a(p)=0}}\left(1-\frac{1}{p}\right)^{-1}=$$

$$=\prod_{N\mathfrak{p}\leq z}\left(1-\frac{1}{N\mathfrak{p}}\right)^{\frac{1}{2}}\prod_{\substack{p\leq z\\ p|d_K\\ a(p)\neq 0}}\left(1-\frac{1}{p}\right)^{\frac{1}{2}}\prod_{\substack{p\leq\sqrt{z}\\ p\text{ is inert in }K}}\left(1-\frac{1}{p^2}\right)^{-\frac{1}{2}}\prod_{\substack{p\leq z\\ p|d_K\\ a(p)=0}}\left(1-\frac{1}{p}\right)^{-\frac{1}{2}}\cdot$$

$$\cdot\prod_{\substack{p\leq z\\ p\text{ splits in }K\\ a(p)=0}}\left(1-\frac{1}{p}\right)^{-1}=$$

$$=\left(\frac{e^{-\frac{\gamma}{2}}L(1,\chi_K)^{-\frac{1}{2}}}{\sqrt{\log z}}+\ O_K\left(\frac{1}{(\log z)^{\frac{3}{2}}}\right)\right)\prod_{\substack{p\leq\sqrt{z}\\ p\text{ inert in }K}}\left(1-\frac{1}{p^2}\right)^{-\frac{1}{2}}\prod_{\substack{p\leq z\\ p\text{ splits in }K\\ a(p)=0}}\left(1-\frac{1}{p}\right)^{-1}\cdot$$

$$\cdot\prod_{\substack{p\leq z\\ p|d_K\\ a(p)\neq 0}}\left(1-\frac{1}{p}\right)^{\frac{1}{2}}\prod_{\substack{p\leq z\\ p|d_K\\ a(p)=0}}\left(1-\frac{1}{p}\right)^{-\frac{1}{2}},$$

where all the products are bounded by a constant, because $\prod_{\substack{p|d_K\\ a(p)\neq 0}}\left(1-\frac{1}{p}\right)^{\frac{1}{2}}$, $\prod_{\substack{p|d_K\\ a(p)=0}}\left(1-\frac{1}{p}\right)^{-\frac{1}{2}}$ are finite products; $\prod_{p\text{ inert in }K}\left(1-\frac{1}{p^2}\right)^{-\frac{1}{2}}$ is convergent,

and $\prod_{\substack{p \text{ splits in } K \\ a(p)=0}} \left(1-\frac{1}{p}\right)^{-1}$ is convergent, because $\sum_{\substack{p \text{ splits in } K \\ a(p)=0}} \frac{1}{p} = \sum_{p\in S_2} \frac{1}{p}$ converges.
Thus,

$$\prod_{\substack{p\leq z \\ a(p)\neq 0}} \left(1-\frac{1}{p}\right) = \frac{\mu_f}{\sqrt{\log z}} + O_f\left(\frac{1}{(\log z)^{3/2}}\right),$$

where

$$\mu_f = e^{-\gamma/2} L(1,\chi_K)^{-\frac{1}{2}} \prod_{p \text{ inert}} \left(1-\frac{1}{p^2}\right)^{-\frac{1}{2}} \prod_{\substack{p \text{ splits in } K \\ a(p)=0}} \left(1-\frac{1}{p}\right)^{-1} \prod_{\substack{p\leq z \\ p|d_K \\ a(p)\neq 0}} \left(1-\frac{1}{p}\right)^{\frac{1}{2}} \prod_{\substack{p\leq z \\ p|d_K \\ a(p)=0}} \left(1-\frac{1}{p}\right)^{-\frac{1}{2}}.$$

Here we split the primes for which $a(p)=0$ into three sets: 1) p that split in K and $a(p)=0$, 2) p that ramify in K and $a(p)=0$ and 3) p that are inert in K.

□

2.2 The number of non-zero Fourier coefficients

We begin by considering a slightly more general setting as in Serre ([14], §6), which parts of this section follow closely. Let $\{a(n)\}$ be the Fourier coefficients of f. Then $a(n)$ is a multiplicative function, which means that $a(nm) = a(n)a(m)$ whenever $(n,m)=1$.
As before, denote by Z_f the set of all primes p that divide their coefficient $a(p)$.

$$Z_f = \{p- \text{ prime } : p|a(p)\}.$$

Define the multiplicative function

$$a^0(n) := \begin{cases} 1, & \text{if } a(n)\neq 0, p|n \Rightarrow p\notin Z_f \\ 0, & \text{otherwise.} \end{cases}$$

Note that for p - prime

$$a^0(p) := \begin{cases} 1, & \text{if } a(p)\neq 0, p\notin Z_f \\ 0, & \text{otherwise.} \end{cases}$$

We want the asymptotic behaviour of

$$M_{f,d}(x) := \#\{n\leq x \mid a(n)\neq 0,\ p|n \Rightarrow p\notin Z_f,\ d|n\} = \sum_{dn\leq x} a^0(dn),$$

for any positive integer d.
4.1. **The case** $d=1$.

$$M_{f,1}(x) = \#\{n\leq x \mid a(n)\neq 0,\ p|n \Rightarrow p\notin Z_f\} = \sum_{n\leq x} a^0(n).$$

Definition 2.2.1. A set of primes P is called *frobenien* of density α if there exists a finite Galois extension $K/\mathbb{Q}$ and a subset H of the group $G = \mathrm{Gal}(K/(Q))$ such that

- H is stable under conjugation,
- $|H|/|G| = \alpha$,

- for p sufficiently large, $p \in P \Leftrightarrow \sigma_p(K/\mathbb{Q}) \in H$, where $\sigma_p(K/\mathbb{Q})$ denotes the class of a Frobenius automorphism associated to p.

Definition 2.2.2. A set of primes P is called *regular* of density α if

$$\sum_{p\in P} \frac{1}{p^s} = \alpha \log \frac{1}{s-1} + \theta_P(s),$$

where $\theta_P(s)$ extends to a holomorphic function in the region $Re(s) \geq 1$.

Lemma 2.2.3. *If P is frobenien of density α, then P is regular of density α.*

Proof. (Serre [13]) Let P be frobenien of density α, with Galois group G and the subset H that satisfies the properties of the definition. Then

$$\sum_{p\in P} \frac{1}{p^s} = \frac{1}{|G|} \sum_{\chi} \overline{\chi}(H) \log L(s,\chi) + g(s),$$

where χ runs through irreducible characters of G, $L(s,\chi)$ is the Artin L-function of the extension $K/\mathbb{Q}$ and character χ, g is a Dirichlet series that converges absolutely for $\Re(s) > 1/2$ (thus, it is holomorphic for $\Re(s) \geq 1$), and $\overline{\chi}(H) = \sum_{h\in H} \overline{\chi}(h)$.
From the elementary properties of the functions $L(s,\chi)$:

$$\log L(s,\chi) = \delta_\chi \log \frac{1}{s-1} + \theta_\chi(s),$$

where $\delta_\chi = \begin{cases} 0, & \text{if } \chi \neq 1, \\ 1, & \text{if } \chi = 1 \end{cases}$, and $\theta_\chi(s)$ is holomorphic for $\Re(s) \geq 1$. Thus,

$$\sum_{p\in P} \frac{1}{p^s} = \frac{|H|}{|G|} \log \frac{1}{s-1} + \theta_P(s) = \alpha \log \frac{1}{s-1} + \theta_P(s).$$

□

Consider the Dirichlet series

$$\varphi(s) = \sum_n \frac{a^0(n)}{n^s} = \prod_p \varphi_p(s)$$

where

$$\varphi_p(s) = \sum_{m=0}^{\infty} a^0(p^m) p^{-ms}.$$

This holds because $a(n) = a(p_1^{\gamma_1}) a(p_2^{\gamma_2}) \cdots a(p_l^{\gamma_l})$, where $n = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_l^{\gamma_l}$ is the prime decomposition of n. Let

$$Z_f(x) = \#\{p \leq x \mid a(p) = 0 \text{ or } p \in Z_f\} = \#\{p \leq x \mid p \in Z_f\},$$

and

$$Z_f^{'}(x) = \#\{p \leq x \mid a(p) \neq 0,\ p \notin Z_f\} = \#\{p \leq x \mid p \notin Z_f\} = \sum_{p\leq x} a^0(p).$$

Lemma 2.2.4.

$$\log(\varphi(s)) = \sum_p \log(\varphi_p(s)) = \sum_p \frac{a^0(p)}{p^s} + \epsilon_1(s),$$

where $\epsilon_1(s)$ is also analytic in the neighbourhood of $s = 1$.

Proof.

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{a^0(n)}{n^s} = \prod_p \varphi_p(s).$$

Note that

$$a^0(p) = \begin{cases} 1, & \text{if } p \notin Z_f, \\ 0, & \text{if } p \in Z_f. \end{cases}$$

Thus, $\varphi_p(s)$ will start with $1 + \frac{1}{p^s}$ if and only if $p \notin Z_f$. So,

$$\varphi(s) = \prod_p \varphi_p(s) = \prod_{p \notin Z_f} \left(1 + \frac{1}{p^s} + \frac{a^0(p^2)}{p^{2s}} + \dots\right) \cdot \prod_{p \in Z_f} \left(1 + \frac{a^0(p^2)}{p^{2s}} + \dots\right) =$$

$$= \prod_{p \notin Z_f} \left(1 + \frac{1}{p^s}\right) \left(1 + \frac{a^0(p^2)}{p^{2s}} + \dots\right) \cdot \prod_{p \in Z_f} \left(1 + \frac{a^0(p^2)}{p^{2s}} + \dots\right) = \prod_{p \notin Z_f} \left(1 + \frac{1}{p^s}\right) \cdot \epsilon_2(s),$$

where

$$\epsilon_2(s) = \prod_{p \notin Z_f} \left(1 + \frac{a^0(p^2)}{p^{2s}} + \dots\right) \cdot \prod_{p \in Z_f} \left(1 + \frac{a^0(p^2)}{p^{2s}} + \dots\right)$$

is analytic for $\Re(s) > 1/2$, $\epsilon_2(s) \neq 0$. So, $\log \epsilon_2(s)$ is analytic for $\Re(s) > \frac{1}{2}$. Thus,

$$\log \varphi(s) = \sum_{p \notin Z_f} \log \left(1 + \frac{1}{p^s}\right) + \log \epsilon_2(s) = \sum_{p \notin Z_f} \frac{1}{p^s} + \epsilon_3(s) + \log \epsilon_2(s) = \sum_{p \notin Z_f} \frac{1}{p^s} + \epsilon_1(s),$$

where $\epsilon_1(s) = \log \epsilon_2(s) + \epsilon_3(s)$ is holomorphic for $\Re(s) > 1/2$.
Thus,

$$\log \varphi(s) = \sum_{p \notin Z_f} \frac{1}{p^s} + \epsilon_1(s) = \sum_p \frac{a^0(p)}{p^s} + \epsilon_1(s)$$

□

We record another useful equality:

$$\log \varphi(s) = \sum_{p \notin Z_f} \frac{1}{p^s} + \epsilon_1(s) = \sum_{p \notin S_1} \frac{1}{p^s} - \sum_{p \in S_2} \frac{1}{p^s} - \sum_{p \in S_3} \frac{1}{p^s} + \epsilon_1(s) = \sum_{p \notin S_1} \frac{1}{p^s} + \epsilon_4(s), \tag{2.1}$$

where Z_f is written as a disjoint union of three sets $Z_f = S_1 \cup S_2 \cup S_3$, with

$$\begin{aligned} S_1 &= \{p \text{ - prime} \mid p \text{ does not split in } K\}, \\ S_2 &= \{p \text{ - prime} \mid p \text{ splits in } K \text{ and } a(p) = 0\}, \\ S_3 &= \{p \text{ - prime} \mid p \in Z_f,\ a(p) \neq 0\}. \end{aligned}$$

Note that S_1 and S_2 together constitute the primes for which $a(p) = 0$.
The series $\sum_{p \in S_2} \frac{1}{p^s}$ converges at $s = 1$:

$$\left.\sum_{p \in S_2} \frac{1}{p^s}\right|_{\text{at } s=1} = \sum_{p \in S_2} \frac{1}{p} = O\left(\int_2^x \frac{1}{t}\, d\left(\frac{t}{(\log t)^2}\right)\right) =$$

$$= O\left(\frac{1}{(\log x)^2}\bigg|_2^\infty + \int_2^x \frac{t}{(\log t)^2}\cdot\frac{1}{t^2}\,dt\right) = O\left(\frac{1}{(\log 2)^2} + \frac{1}{\log 2}\right).$$

Thus, the function $\displaystyle\sum_{p\in S_2}\frac{1}{p^s}$ is holomorphic at $s=1$.

Assume that $\displaystyle\sum_{p\in S_3}\frac{1}{p^s} = \sum_{\substack{p\in Z_f,\\ a(p)\neq 0}}\frac{1}{p^s}$ is holomorphic at $s=1$.

So, the function $\displaystyle\epsilon_4(s) = \epsilon_1(s) - \sum_{p\in S_2}\frac{1}{p^s} - \sum_{p\in S_3}\frac{1}{p^s}$ is holomorphic as well.

Lemma 2.2.5. *$S_1 = \{p$ - prime $\mid p$ does not split in $K\}$ is frobenien of density $1/2$.*

Proof. $K/\mathbb{Q}$ is a quadratic extension, so $G = \mathrm{Gal}(K/(Q))$ consists of 2 elements: $G = \pm 1$. We know that $1 \notin H$. Take $H = -1$. Then σ_p makes sense for p that do not ramify. For p sufficiently large $p \in S_1 \Leftrightarrow p$ remains prime in K. For those primes $\sigma_p = -1$.
Thus, the density is $\frac{|H|}{|G|} = \frac{1}{2}$. □

The orthogonality of characters gives us:

$$\sum_{\chi}\overline{\chi}(H)\chi(\sigma_p) = \begin{cases} 0, & \text{if } \sigma_p \neq H,\\ \frac{|G|}{|H|}, & \text{if } \sigma_p = H.\end{cases} = \begin{cases} 0, & \text{if } p \notin S_1,\\ \frac{|G|}{|H|}, & \text{if } p \in S_1.\end{cases}$$

Thus, S_1 is frobenien implies

$$\sum_{p\in S_1}\frac{1}{p^s} = \sum_{\text{all } p}\frac{1}{p^s}\left(\sum_{\chi}\overline{\chi}(H)\chi(\sigma_p)\right)\frac{|H|}{|G|} = \frac{|H|}{|G|}\sum_{\chi}\sum_{p}\frac{1}{p^s}\overline{\chi}(H)\chi(\sigma_p)$$

If $\chi \neq 1$ this is analytic. Only $\chi = 1$ contributes

$$\frac{|H|}{|G|}\sum_{p}\frac{1}{p^s}\overline{\chi}(H)\chi(\sigma_p) = \frac{|H|}{|G|}\sum_{p}\frac{1}{p^s}.$$

Thus,

$$\sum_{p\in S_1}\frac{1}{p_s} = \frac{|H|}{|G|}\sum_{\chi}\sum_{p}\frac{1}{p^s}\overline{\chi}(H)\chi(\sigma_p) = \frac{|H|}{|G|}\sum_{p}\frac{1}{p^s} + \epsilon_5(s) = \frac{1}{2}\sum_{p}\frac{1}{p^s} + \epsilon_5(s), \tag{2.2}$$

where $\epsilon_5(s)$ converges at $s=1$.

If $\displaystyle\sum_{\substack{p\in Z_f\\ a(p)\neq 0}}\frac{1}{p^s}$ is convergent at $s=1$, we have the following:

Lemma 2.2.6.

$$\sum_{p}\frac{a^0(p)}{p^s} = \frac{1}{2}\log\left(\frac{1}{s-1}\right) + \epsilon_7(s),$$

where $\epsilon_7(s)$ is analytic in the neighbourhood of $s=1$.

Proof. If S_1 is regular of density α, then $S_1^{'} = \{p$ - prime, $p \notin S_1\}$ is regular of density $1-\alpha$ ([13]). Since S_1 is frobenien of density $1/2$, we have S_1' is frobenien of density $(1-1/2)$, and so by Lemma 2.2.3, S_1' is regular of density $1/2$. Thus,

$$\sum_{p\notin S_1}\frac{1}{p^s} = \frac{1}{2}\log\left(\frac{1}{s-1}\right) + \theta_{S_1}(s), \tag{2.3}$$

where $\theta_{S_1}(s)$ extends to a holomorphic function in the region $\Re(s) \geq 1$.
Put (2.1) and (2.3) together to get:

$$\log \varphi(s) = \sum_{p \notin S_1} \frac{1}{p^s} + \epsilon_4(s) = \frac{1}{2} \log \left(\frac{1}{s-1} \right) + \epsilon_7(s),$$

where $\epsilon_7(s) = \epsilon_4(s) + \theta_{S_1}(s)$. □

Also,

$$\log \varphi(s) = \sum_p \frac{a^0(p)}{p^s} + \epsilon_1(s),$$

and so,

$$\sum_p \frac{a^0(p)}{p^s} = \frac{1}{2} \log \left(\frac{1}{s-1} \right) + \epsilon_8(s),$$

where $\epsilon_8(s) = \epsilon_7(s) - \epsilon_1(s)$. Thus, we get

$$\varphi(s) = \frac{e^{\epsilon_7(s)}}{(s-1)^{1/2}} = \frac{\epsilon_9(s)}{(s-1)^{1/2}},$$

where $\epsilon_9(s)$ is analytic near $s = 1$ and $\epsilon_9(s) \neq 0$.
Thus, we have the following decomposition:

$$\varphi(s) = \frac{1}{(s-1)^{1/2}} \cdot (e_0 + e_1(s-1) + e_2(s-1)^2 + \ldots).$$

From (2.2) we get:

$$\sum_{p \notin S_1} \frac{1}{p^s} = \frac{1}{2} \sum_p \frac{1}{p^s} + \epsilon_{10}(s),$$

where $\epsilon_{10}(s) = -\epsilon_5(s)$. Thus,

$$\log \varphi(s) = \frac{1}{2} \sum_p \frac{1}{p^s} + \epsilon_{11}(s),$$

where $\epsilon_{11}(s) = \epsilon_{10}(s) + \epsilon_4(s)$.
So

$$\varphi(s) = \exp \left\{ \frac{1}{2} \sum_p \frac{1}{p^s} \right\} \cdot e^{\epsilon_{11}(s)}.$$

Also,

$$\begin{aligned} (\zeta(s))^{\frac{1}{2}} &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-\frac{1}{2}} = \exp \left\{ -\frac{1}{2} \sum_p \log \left(1 - \frac{1}{p^s}\right) \right\} \\ &= \exp \left\{ -\tfrac{1}{2} \left(-\sum_p \frac{1}{p^s} + \sum_p \left(\frac{c_2}{p^{2s}} + \frac{c_3}{p^{3s}} + \ldots \right) \right) \right\} = \exp \left\{ \tfrac{1}{2} \sum_p \frac{1}{p^s} \right\} \cdot \epsilon_{12}(s), \end{aligned}$$

where $\epsilon_{12}(s)$ is analytic near $s = 1$ and $\epsilon_{12}(s) \neq 0$. Thus,

$$\varphi(s) = (\zeta(s))^{\frac{1}{2}} \cdot h(s), \tag{2.4}$$

where $h(s) = \dfrac{e^{\epsilon_{11}(s)}}{\epsilon_{12}(s)}$ is analytic near $s = 1$ and $h(s) \neq 0$. We will use (2.4) later.

Some preliminary lemmas.

Put

$$b^0(x) = \sum_{n \le x} a^0(n) \log \frac{x}{n}.$$

The next three lemmas show that

$$b^0(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \varphi(s) x^s \cdot s^{-2} ds.$$

Consider the following integral:

$$J(w) = \int_{2-i\infty}^{2+i\infty} \frac{e^{ws}}{s^2} ds,$$

where w is a real number. It converges, since $\left|\frac{e^{ws}}{s^2}\right| = \frac{e^{2w}}{4+t^2}$, where $s = 2 + it$.

Lemma 2.2.7.

$$J(w) = \begin{cases} 0, & \text{if } w \le 0 \\ 2\pi i w, & \text{if } w \ge 0 \end{cases}$$

Proof. (From [4]) Put

$$J(w,T) = \int_{2-iT}^{2+iT} \frac{e^{ws}}{s^2} ds.$$

Then

$$J(w) = \lim_{T \to \infty} J(w,T).$$

Consider the two cases:
1) $w \ge 0$. Let $T > 2$. By the Cauchy's integral formula

$$J(w,T) - 2\pi i \text{ Res}\left(\frac{e^{ws}}{s^2}, s = 0\right) = \int_\gamma \frac{e^{ws}}{s^2} ds,$$

where γ denotes the left semicircle connecting the points $2 + iT$ and $2 - iT$ and is traversed from $2 + iT$ to $2 - iT$.

$$\text{Res}\left(\frac{e^{ws}}{s^2}, s = 0\right) = \lim_{s \to 0} \frac{d}{ds}\left(s^2 \cdot \frac{e^{ws}}{s^2}\right) = \lim_{s \to 0} w \cdot e^{ws} = w$$

Thus,

$$J(w,T) - 2\pi i w = \int_\gamma \frac{e^{ws}}{s^2} ds.$$

The length of the semicircle is πT. Also $|s| \ge T - 2$ and $\Re(s) \le 2$, thus

$$\left|\frac{e^{ws}}{s^2}\right| \le \frac{e^{2w}}{(T-2)^2},$$

and so

$$|J(w,T) - 2\pi i w| = \int_\gamma \frac{e^{ws}}{s^2} ds \le \frac{\pi T e^{2w}}{(T-2)^2} \to 0, \text{ as } T \to \infty.$$

Thus,

$$J(w) = \lim_{T \to \infty} J(w,T) = 2\pi i w.$$

2) $w \le 0$. Again we use Cauchy's theorem, only we will integrate along the other semicircle, so that there are no poles inside the area bounded by the closed curve of integration. Then,

$$J(w,T) = \int_\gamma \frac{e^{ws}}{s^2} ds.$$

The length of the arc is πT; $|s| \ge T$ implies that $\left|\frac{e^{ws}}{s^2}\right| \le \frac{e^{2w}}{T^2}$. Thus

$$|J(w,T)| \le \frac{\pi e^{2w}}{T} \to 0, \text{ as } T \to \infty.$$

$$J(w) = 0 \text{ for } w \le 0.$$

□

Lemma 2.2.8.

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^s s^{-2} ds = \begin{cases} 0, & \text{if } 0 < y \le 1 \\ \log y, & \text{if } y \ge 1 \end{cases}$$

Proof. This is an immediate consequence of the previous Lemma with $w = \log y$. □

Lemma 2.2.9.

$$b^0(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \varphi(s) x^s \cdot s^{-2} ds$$

Proof. The statement follows from Lemma 2.2.8 with $y = \frac{x}{n}$:

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \varphi(s) x^s \cdot s^{-2} ds = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \sum_{n=1}^{\infty} a^0(n)\left(\frac{x}{n}\right)^s s^{-2} ds =$$

$$= \frac{1}{2\pi i}\sum_{n=1}^{\infty} a^0(n)\int_{2-i\infty}^{2+i\infty}\left(\frac{x}{n}\right)^s s^{-2} ds = \sum_{n=1}^{x} a^0(n)\log\frac{x}{n} = b^0(x).$$

□

In the next Lemma we shall state some properties of $\zeta(s)$ which we will need in order to determine the behaviour of the function $\varphi(s)$. Here $c_1, c_2, \ldots$ denote positive constants, $s = \sigma + ti$ is a complex variable.

Lemma 2.2.10. *1. $\zeta(s) - (s-1)^{-1}$ is holomorphic for $\sigma > 0$.*

2. There exists c_1 such that $\zeta(s) \ne 0$ for $\sigma \ge 1 - \frac{c_1}{(\log|t|)^9}$, $|t| \ge 3$, and for $\sigma \ge 1 - \frac{c_1}{(\log 3)^9}$, $|t| \le 3$.

3. There exists c_2 such that

$$|\zeta(s)| < c_2 \log|t|$$

for $\sigma \ge 1 - \frac{1}{\log|t|}$, $|t| \ge 3$, and c_3 such that

$$|\log\zeta(s)| < c_3(\log|t|)^9$$

for $\sigma \ge 1 - \frac{c_1}{(\log|t|)^9}$, $|t| \ge 3$.

4. There exist c_4, c_5 and c_6 such that

$$|\zeta(s)| < c_4 \quad \text{and} \quad |\log\zeta(s)| < c_5$$

for $1-\dfrac{c_1}{(\log 3)^9} \le \sigma \le 1-c_6 < 1,\ t \le 3.$

Proof. The properties given in parts 1, 2 and 3 of the lemma are contained in §42, §48 and §64 of Landau [4]. Part 4 is an immediate consequence of the rest of the lemma. □

The next lemma follows immediately from Lemma 2.2.10 and the definition of $\varphi(s)$. Recall that $\varphi(s) = (\zeta(s))^{\frac{1}{2}}h(s)$.

Lemma 2.2.11. *For suitable positive constants d_1, d_2 and d_3, we have*

1. *The function $\varphi(s)$ is holomorphic for $\sigma \ge 1-\dfrac{d_1}{(\log|t|)^9}$, $|t| \ge 3$ and for $\sigma \ge 1-\dfrac{d_1}{(\log 3)^9}$, $|t| \le 3$ except for a singularity at $s=1$.*
2. *$|\varphi(s)| < d_2(\log|t|)^{k_1}$ for $\sigma \ge 1-\dfrac{d_1}{(\log|t|)^9}$, $|t| \ge 3$, where $k_1 > 0$, and $|\varphi(s)| < d_3$ for $\sigma = 1-\dfrac{d_1}{(\log 3)^9}$, $|t| \le 3$.*

Proof. □

Lemma 2.2.12. *If $|s-1| \le d_1(\log 3)^{-9}$, then*

$$|x^s\varphi(s)s^{-2} - x^s h(1)(s-1)^{-\frac{1}{2}}| = O\left(x^s\right).$$

Proof. By Lemma 2.2.10: part 1), $(s-1)\zeta(s)$ is holomorphic for $\sigma > 0$, and

$$\lim_{s\to 1}(s-1)\zeta(s) = 1. \tag{2.5}$$

$$\varphi(s)\cdot s^{-2} = (\zeta(s))^{\frac{1}{2}}h(s)s^{-2} = (s-1)^{-\frac{1}{2}}((s-1)\zeta(s))^{\frac{1}{2}}h(s)s^{-2},$$

where $((s-1)\zeta(s))^{\frac{1}{2}}h(s)s^{-2}$ is holomorphic and bounded when $|s-1| \le d_1(\log 3)^{-9}$ and hence can be expanded as a convergent power series in the form:

$$((s-1)\zeta(s))^{\frac{1}{2}}h(s)s^{-2} = \sum_{k=0}^{\infty} w_k(s-1)^k.$$

From (2.5) we have that

$$w_0 = h(1).$$

Thus,

$$\varphi(s)s^{-2} = h(1)(s-1)^{-\frac{1}{2}} + (s-1)^{\frac{1}{2}}\sum_{k=1}^{\infty} w_k(s-1)^{k-1},$$

and so

$$|x^s\varphi(s)s^{-2} - x^s h(1)(s-1)^{-\frac{1}{2}}| = O\left((s-1)^{\frac{1}{2}}\sum_{k=1}^{\infty} w_k(s-1)^{k-1}\right) = O(x^s),$$

for $|s-1| \le d_1(\log 3)^{-9}$. □

Lemma 2.2.13. *There is a constant u_f such that*

$$M_{f,1}(x) = (1+o(1))\frac{u_f x}{(\log x)^{1/2}}.$$

Proof.

$$\varphi(s) = \sum_n \frac{a^0(n)}{n^s} = \sum_{n \in M_{f,1}} \frac{1}{n^s}$$

$\varphi(s)$ extends to a holomorphic function in the following region:

The branches C and D are defined by $\Re(s) = 1 - \frac{b}{\log^A T}$, with $T = 2 + |\Im(s)|$. And in this region $|\varphi(s)| = O(\log^A T)$ as $T \to \infty$.
Indeed, take $b = c_1$ and $A \geq 9(\log\log 3)(\log\log 2)^{-1}$ (so that the branches C, D lie in the desired region) and apply Lemma 2.2.10.

Recall, that we defined

$$b^0(x) = \sum_{n \leq x} a^0(n) \log \frac{x}{n}.$$

and established that

$$b^0(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \varphi(s) x^s \cdot s^{-2} ds.$$

Lemma 2.2.14.

$$b^0(x) = \frac{h(1)}{\sqrt{\pi}} x (\log x)^{-\frac{1}{2}} + O(x(\log x)^{-1}) \tag{2.6}$$

Proof. Cauchy's theorem shows that this integral is equal to the analogous integral along the left edge of the region in question (i.e. along branches D, E, F, E', C). The branches C and D contribute negligible amounts of the order $\dfrac{x}{\log^N x}$ for any N. The integral along the circle F tends to 0 as the radius of the circle tends to 0. Thus, the main term is provided by the two integrals along the horizontal segments E and E'. To evaluate these we need the development of $\dfrac{\varphi(s)}{s^2}$ in the neighbourhood of $s = 1$:

$$\frac{\varphi(s)}{s^2} = \frac{1}{\sqrt{s-1}} (e_0 + e_1(s-1) + \ldots + e_k(s-1)^k + \ldots)$$

From Lemma 2.2.12 we have:

$$\int_{1-d_1(\log 3)^{-9}}^{1} (x^s \varphi(s) s^{-2} - x^s h(1)(s-1)^{-\frac{1}{2}}) ds = O\left(\int_{1-d_1(\log 3)^{-9}}^{1} x^s ds \right) = O\left(\frac{x}{\log x} \right).$$

Hence,

$$\left\{ \int_{E'} + \int_E \right\} x^s \varphi(s) s^{-2} ds = \int_{1-d_1(\log 3)^{-9}}^{1} h(1) x^{s^-} (s^- - 1)^{-\frac{1}{2}} ds^- - \int_{1-d_1(\log 3)^{-9}}^{1} h(1) x^{s^+} (s^+ - 1)^{-\frac{1}{2}} ds^+,$$

where s^+ and s^- indicate the upper edge and the lower edge respectively on the cut. Since $(s^+ - 1) = (1 - s^+) e^{\pi i}$ and $(s^- - 1) = (1 - s^+) e^{-\pi i}$, it follows that

$$\left\{ \int_{E'} + \int_E \right\} x^s \varphi(s) s^{-2} ds =$$

$$= h(1) \int_{1-d_1(\log 3)^{-9}}^{1} x^{s^+} (1 - s^+)^{-\frac{1}{2}} (e^{\frac{\pi i}{2}} - e^{-\frac{\pi i}{2}}) ds^+ + O\left(\frac{x}{\log x} \right) =$$

$$= 2ih(1) \int_{1-d_1(\log 3)^{-9}}^{1} x^s (1-s)^{-\frac{1}{2}} ds + O\left(\frac{x}{\log x} \right)$$

$$\int_{1-d_1(\log 3)^{-9}}^{1} x^s(1-s)^{-\frac{1}{2}}ds = \Gamma(\frac{1}{2})x(\log x)^{-\frac{1}{2}} + O\left(\frac{x^{1-d_1(\log 3)^{-9}}}{\log x}\right),$$

Thus,

$$\left\{\int_{E'} + \int_{E}\right\} x^s\varphi(s)s^{-2}ds = \frac{2\pi i h(1)}{\Gamma(\frac{1}{2})}x(\log x)^{-\frac{1}{2}} + O\left(\frac{x}{\log x}\right),$$

and so

$$b^0(x) = \frac{h(1)}{\Gamma(\frac{1}{2})}x(\log x)^{-\frac{1}{2}} + O\left(\frac{x}{\log x}\right).$$

□

We apply this result with $x+\delta x$, with $\delta \sim \frac{1}{(\log x)^{K+1}}$ and find the needed estimate for $\sum_{n\le x} a^0(n)$. We have

Lemma 2.2.15. *Suppose that*

$$b^0(x) = B\frac{x}{(\log x)^{\frac{1}{2}}} + O\left(\frac{x}{\log x}\right). \tag{2.7}$$

Then

$$M_{f,1}(x) = B\frac{x}{(\log x)^{\frac{1}{2}}} + O\left(x(\log x)^{-\frac{3}{4}}\right).$$

Proof. Let $\delta = \delta(x) = o(1)$ be a positive function of x to be chosen later, and suppose that $x(1+\delta)$ is an integer. Then, since

$$\log(x(1+\delta)) = \log x + O(\delta),$$

$$\begin{aligned} b^0(x(1+\delta)) &= B(1+\delta)x(\log(1+\delta)x)^{-\frac{1}{2}} + O\left(x(1+\delta)(\log x(1+\delta))^{-1}\right) = \\ &= B(1+\delta)x\left\{(\log x)^{-\frac{1}{2}} + (-1/2)(\log x)^{-\frac{3}{2}}O(\delta) + O((\log x)^{-\frac{3}{2}})\right\} + \\ &+ O\left(x(1+\delta)(\log x)^{-1}\right) = \\ &= Bx(\log x)^{-\frac{1}{2}}\left(1+\delta+O(\delta(\log x)^{-1})\right) + O\left(x(\log x)^{-1}\right). \end{aligned} \tag{2.8}$$

By definition

$$\begin{aligned} b^0(x(1+\delta)) - b^0(x) &= \sum_{n=1}^{x(1+\delta)} a^0(n)\log x(1+\delta)/n - \sum_{n=1}^{x} a^0(n)\log x/n \\ &= \log(1+\delta)\sum_{n=1}^{x} a^0(n) + \sum_{n=x+1}^{x(1+\delta)} a^0(n)\log x(1+\delta)/n \\ &\ge \log(1+\delta)M_{f,1}(x), \end{aligned} \tag{2.9}$$

since the second sum is not negative. Similarly

$$\begin{aligned} b^0(x(1+\delta)) - b^0(x) &= \sum_{n=1}^{x(1+\delta)} a^0(n)\log x(1+\delta)/n - \sum_{n=1}^{x} a^0(n)\log x/n \\ &= \log(1+\delta)\sum_{n=1}^{x(1+\delta)} a^0(n) + \sum_{n=x+1}^{x(1+\delta)} a^0(n)\log x/n \\ &\le \log(1+\delta)M_{f,1}(x(1+\delta)), \end{aligned} \tag{2.10}$$

since the second sum is not positive. By (2.7), (2.8) and (2.9)

$$\begin{aligned} M_{f,1}(x) &\le \left\{b^0(x(1+\delta)) - b^0(x)\right\} / \log(1+\delta) \\ &= \left\{b^0(x(1+\delta)) - b^0(x)\right\} (1+O(\delta))\delta^{-1} \\ &= Bx(\log x)^{-\frac{1}{2}} \left\{1 + O(\delta) + O\left((\log x)^{-1}\right) + O\left(\delta^{-1}(\log x)^{-1}\right)\right\}. \end{aligned} \tag{2.11}$$

By (2.7), (2.8) and (2.10)

$$\begin{aligned} M_{f,1}(x(1+\delta)) &\ge \left\{b^0(x(1+\delta)) - b^0(x)\right\} / \log(1+\delta) \\ &= \left\{b^0(x(1+\delta)) - b^0(x)\right\} (1+O(\delta))\delta^{-1} \\ &= Bx(\log x)^{-\frac{1}{2}} \left\{1 + O(\delta) + O\left((\log x)^{-1}\right) + O\left(\delta^{-1}(\log x)^{-1}\right)\right\}. \end{aligned} \tag{2.12}$$

If we replace x by $x(1+\delta)$ in (2.12), we obtain

$$M_{f,1}(x) \quad \ge \quad Bx(\log x)^{-\frac{1}{2}} \left\{1 + O(\delta) + O((\log)^{-1}) + O(\delta^{-1}(\log x)^{-1})\right\}. \tag{2.13}$$

We now choose δ so that all the error terms of (2.11) and (2.13) are of a smaller order of magnitude than the first term. We can take

$$\delta = x^{-1}[x(\log x)^{-\frac{1}{4}}].$$

Then the error terms of (2.11) and (2.13) are

$$O\left(x(\log x)^{-\frac{1}{4}}(\log x)^{-1}\right) = O\left(x(\log x)^{-\frac{3}{4}}\right).$$

Hence, from (2.11) and (2.13) if follows that

$$M_{f,1}(x) = B\frac{x}{(\log x)^{\frac{1}{2}}} + O\left(x(\log x)^{-\frac{3}{4}}\right),$$

which is the result of the lemma. □

We use (2.6) and Lemma 2.2.15 with $B = u_f = \dfrac{h(1)}{\sqrt{\pi}}$ to obtain the result of Lemma 2.2.13. □

4.2. **Convolution with a secondary function.** Now consider another function $n \mapsto b(n)$ with the following properties:

1. There is an integer d so that $b(n) \neq 0$ implies that all prime divisors of n are prime divisors of d.

2. We have $|b(n)| \le 4^{\nu(n)}$, where $\nu(n)$ is the number of distinct prime divisors of n.

Let us set

$$\xi_d(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}.$$

We see that

$$\sum_{m \le x} |b(m)| \le \sum_{m \le x} |b(m)| \left(\frac{x}{m}\right)^{\frac{1}{4}} \le \sum_{m \in \mathbb{Z}_{\mho}} |b(m)| \left(\frac{x}{m}\right)^{\frac{1}{4}} =$$

since $b(m) = 0$ for the rest of m

$$= \sum_{p|m \to p|d} |b(m)| \left(\frac{x}{m}\right)^{\frac{1}{4}} \le \sum_{p|m \to p|d} 4^{\nu(m)} \left(\frac{x}{m}\right)^{\frac{1}{4}} = x^{\frac{1}{4}} \sum_{p|m \to p|d} \frac{4^{\nu(m)}}{m^{\frac{1}{4}}} =$$

$$= x^{\frac{1}{4}} \prod_{p|d} \left(1 + \frac{4}{p^{\frac{1}{4}}} + \frac{4}{(p^2)^{\frac{1}{4}}} + \ldots\right) = x^{\frac{1}{4}} \prod_{p|d} \left(1 + \frac{4}{p^{\frac{1}{4}} - 1}\right)$$

The number of factors in the product $\prod_{p|d} \left(1 + \frac{4}{p^{\frac{1}{4}} - 1}\right)$ is the number of distinct prime divisors of d, i.e. it is $\nu(d)$. For $p \geq 5^4$ we have $1 + \frac{4}{p^{\frac{1}{4}}-1} \leq 2$. Thus,

$$\prod_{p|d} \left(1 + \frac{4}{p^{\frac{1}{4}} - 1}\right) \ll 2^{\nu(d)}$$

and so

$$\sum_{m \leq x} |b(m)| \ll x^{1/4} 2^{\nu(d)}. \tag{2.14}$$

Moreover, using (2.14), we have

$$\sum_{z<m<2z} \frac{|b(m)|}{m} \ll z^{-3/4} 2^{\nu(d)}. \tag{2.15}$$

Indeed,

$$\sum_{z<m<2z} \frac{|b(m)|}{m} \leq \frac{1}{z} \sum_{z<m<2z} |b(m)| \leq \sum_{m<2z} |b(m)| \ll$$

we use (2.14)

$$\ll \frac{1}{z}(2z)^{1/4} 2^{\nu(d)} = \sqrt[4]{2} z^{-3/4} 2^{\nu(d)}.$$

Let $c = a^0 * b$ be the Dirichlet convolution (i.e., $c(n) = \sum_{d|n} a^0(d) b\left(\frac{n}{d}\right)$) and consider the function

$$\psi(s) = \sum_n \frac{c(n)}{n^s} = \sum_n \frac{a^0(n)}{n^s} \cdot \sum_n \frac{b(n)}{n^s} = \varphi(s)\xi_d(s)$$

by the property of Dirichlet series.
Then, we have

$$\sum_{n \leq x} c(n) = \sum_{n \leq x} \sum_{r|n} a^0(r) b\left(\frac{n}{r}\right) = \sum_{m \leq x} \sum_{mr \leq x} b(m) a^0(r) = \sum_{m \leq x} b(m) \sum_{r \leq x/m} a^0(r).$$

The contribution from the terms with $\sqrt{x} \leq m \leq x$ is

$$\sum_{\sqrt{x} \leq m \leq x} b(m) \sum_{r \leq x/m} a^0(r) \leq \sum_{\sqrt{x} \leq m \leq x} b(m) \cdot \frac{x}{m} \leq x \sum_{\sqrt{x} \leq m \leq x} \frac{|b(m)|}{m}.$$

Decomposing the sum into dyadic intervals $U < m \leq 2U$ and using (2.15) shows that the summation is $O(x^{-3/8} 2^{\nu(d)})$:

$$\sum_{\sqrt{x} \leq m \leq x} \frac{|b(m)|}{m} \leq \sum_{l=1}^{k} \sum_{2^{l-1}\sqrt{x} \leq m \leq 2^l \sqrt{x}} \frac{|b(m)|}{m} \leq \sum_{l=1}^{k} 2^{\nu(d)} \cdot 2^{-3(l-1)/4} \cdot x^{-3/8} =$$

$$= 2^{\nu(d)} \cdot x^{-3/8} \sum_{l=0}^{k-1} \left(\frac{1}{2^{3/4}}\right)^l = O(x^{-3/8} 2^{\nu(d)}),$$

hence the whole expression is $O(x^{5/8}2^{\nu(d)})$. Lemma 2.2.13 implies

$$\sum_{n\le x} c(n) = \sum_{m\le\sqrt{x}} b(m)\left\{\left(u_f + O\left(\frac{1}{\log x}\right)\right)\frac{x}{m(\log x/m)^{\frac{1}{2}}}\right\} + O(x^{5/8}2^{\nu(d)}). \tag{2.16}$$

Note that

$$\left(\log\frac{x}{m}\right)^{-\frac{1}{2}} = (\log x - \log m)^{-\frac{1}{2}} = (\log x)^{-\frac{1}{2}} + O((\log m)(\log x)^{-\frac{3}{2}}).$$

Using this and (2.15), the right hand side of (2.16) is equal to

$$\left(u_f + O\left(\frac{1}{\log x}\right)\right)\frac{x}{(\log x)^{\frac{1}{2}}}\left(\xi_d(1) + O(x^{-3/8}(\log x)^{-1}2^{\nu(d)})\right) + O(x^{5/8}2^{\nu(d)}).$$

Summarizing this discussion, we have proved the following

Proposition 2.2.16. *We have*

$$\sum_{n\le x} c(n) = u_f\xi_d(1)\frac{x}{(\log x)^{\frac{1}{2}}} + O\left(\frac{x2^{\nu(d)}}{(\log x)^{\frac{3}{2}}}\right)$$

uniformly in d.

4.3. **The case of general d.** Consider the Dirichlet series

$$\psi_d(s) = \sum_n \frac{a^0(dn)}{n^s}. \tag{2.17}$$

We may write it as

$$\begin{aligned}\psi_d(s) &= \left(\sum_{\substack{n_1=1\\ p|n_1\Rightarrow p|d}}^{\infty}\frac{a^0(dn_1)}{n_1^s}\right)\left(\sum_{\substack{n_2=1\\ (n_2,d)=1}}^{\infty}\frac{a^0(n_2)}{n_2^s}\right)\\ &= \left(\sum_{\substack{n_1=1\\ p|n_1\Rightarrow p|d}}^{\infty}\frac{a^0(dn_1)}{n_1^s}\right)\left(\sum_{\substack{n_2=1\\ (n_2,d)=1}}^{\infty}\frac{a^0(n_2)}{n_2^s}\right)\left(\sum_{\substack{n_2=1\\ p|n_2\Rightarrow p|d}}^{\infty}\frac{a^0(n_2)}{n_2^s}\right)\left(\sum_{\substack{n_2=1\\ p|n_2\Rightarrow p|d}}^{\infty}\frac{a^0(n_2)}{n_2^s}\right)^{-1}.\end{aligned}$$

Thus, we see that

$$\psi_d(s) = \varphi(s)\xi_d(s) = \psi(s),$$

where as before

$$\varphi(s) = \sum_{n_3=1}^{\infty}\frac{a^0(n_3)}{n_3^s}$$

and

$$\xi_d(s) = \left(\sum_{\substack{n_1=1\\ p|n_1\Rightarrow p|d}}^{\infty}\frac{a^0(dn_1)}{n_1^s}\right)\left(\sum_{\substack{n_2=1\\ p|n_2\Rightarrow p|d}}^{\infty}\frac{a^0(n_2)}{n_2^s}\right)^{-1}.$$

We have a factorization

$$\xi_d(s) = \prod_{p|d} \xi_{p,d}(s),$$

where

$$\xi_{p,d}(s) = \left(\sum_{m=0}^{\infty} \frac{a^0(p^{m+\mathrm{ord}_p d})}{p^{ms}} \right) \left(\sum_{m=0}^{\infty} \frac{a^0(p^m)}{p^{ms}} \right)^{-1},$$

and $\mathrm{ord}_p d$ is the power of p in the prime decomposition of d.

It makes sense to consider $\xi_{p,d}(s)$ for only those p that divide d.
Note that if $p \in Z_f$, then $a^0(p^m) = 0$ for all $m \geq 1$. Thus, for $p|d$, $p \in Z_f$ we have

$$\xi_{p,d}(s) = 0,$$

and so $\xi_d(s) = 0$ if at least one $p|d$ satisfies $p \in Z_f$.

We record the following estimate for later use:

Lemma 2.2.17. *If $p \notin Z_f$, then*

$$\xi_{p,d}(1) = a^0(p^{\mathrm{ord}_p d}) + O\left(\frac{1}{p}\right). \tag{2.18}$$

Proof. This follows from the formula for $\xi_{p,d}(s)$. □

We write

$$\xi_d(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

and suppose that $\xi_d(s)$ (that is, the coefficients $\{b(n)\}$) satisfies the conditions of Section 4.2. Recall that

$$M_{f,d}(x) = \#\{n \leq x \mid a(n) \neq 0,\ d|n,\ p|n \Rightarrow p \notin Z_f\}.$$

We have

$$M_{f,d}(x) = \sum_{dn \leq x} a^0(dn),$$

and by Proposition 2.2.16, we deduce the following.

Proposition 2.2.18. *If ξ_d satisfies the hypotheses of Section 4.2, then we have*

$$M_{f,d}(x) = \frac{u_f \xi_d(1) x/d}{(\log x/d)^{\frac{1}{2}}} + O\left(\frac{x 2^{\nu(d)}}{d(\log x/d)^{\frac{3}{2}}}\right)$$

uniformly in d.

We begin with some preliminary results. Let us set $i_f(p)$ to be the least integer $i \geq 1$ for which $a(p^i) = 0$. If for a given p, there is no such i, then let us set $i_f(p) = 0$. In particular, if p divides the level N, then $i_f(p) = 1$. Note that

$$i_f(p) = 1 \Rightarrow a(p) = 0 \Rightarrow p|a(p) \Rightarrow p \in Z_f.$$

So, for $p \in Z_f$ let us define $i_f(p) = 1$.

Lemma 2.2.19. *For $p \nmid N$, $p \notin Z_f$, we have*

1. *$i_f(p) \in \{0, 1, 2, 3, 5\}$.*

2. *If $i_f(p) > 0$, then $a_f(p^i) = 0$ for every $i > 0$ with*

$$i + 1 \equiv 0 \pmod{i_f(p) + 1}.$$

3. *If $a(p^i) = 0$ for some $i > 0$, then $i + 1 \equiv 0 \pmod{i_f(p) + 1}$.*

4. *For p sufficiently large (depending on f), we have $i_f(p) \in \{0, 1\}$.*

Note that if $p \in Z_f$, then $a^0(p^i) = 0$ for $i \geq 1$.

Proof. Let us suppose that $i_f(p) > 0$. Thus, $a(p^i) = 0$ for some $i \geq 1$. Let us write α_p and β_p for the roots of $X^2 - a(p)X + p$. Then, we have

$$a(p^i) = \frac{\alpha_p^{i+1} - \beta_p^{i+1}}{\alpha_p - \beta_p}. \tag{2.19}$$

Thus, $\alpha_p = \zeta\beta_p$, where $\zeta^{i+1} = 1$. Since $\zeta \in \mathbb{Q}(\alpha_p, \beta_p) = \mathbb{Q}(\alpha_p)$ and $\mathbb{Q}(\alpha_p) : \mathbb{Q} = 2$, we must have $\zeta^2 = 1$ or $\zeta^4 = 1$ or $\zeta^6 = 1$. This means that one of $\{\zeta - 1, \zeta^2 - 1, \zeta^3 - 1, \zeta^6 - 1\}$ is zero. This in turn means that one of $\{a(p), a(p^3), a(p^2), a(p^5)\}$ is zero.

$$\left[\begin{array}{l} \zeta^2 - 1 = 0 \\ \zeta^4 - 1 = 0 \\ \zeta^6 - 1 = 0 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \zeta^2 = 1 \\ \zeta^4 = 1 \\ \zeta^3 = 1 \\ \zeta^6 = 1 \end{array}\right.$$

1) $\zeta^2 = 1 \Rightarrow a(p) = \dfrac{\alpha_p^2 - \beta_p^2}{\alpha_p - \beta_p} = 0.$

2) $\zeta^4 = 1 \Rightarrow a(p^3) = \dfrac{\alpha_p^4 - \beta_p^4}{\alpha_p - \beta_p} = 0.$

3) $\zeta^3 = 1 \Rightarrow a(p^2) = \dfrac{\alpha_p^3 - \beta_p^3}{\alpha_p - \beta_p} = 0.$

4) $\zeta^6 = 1 \Rightarrow a(p^5) = \dfrac{\alpha_p^6 - \beta_p^6}{\alpha_p - \beta_p} = 0.$

This proves the first assertion.
The second follows from (2.19).
For the third assertion, we note that $\alpha_p = \zeta\beta_p$ where $\zeta^{i+1} = 1$. We also have $\zeta^{i_f(p)+1} = 1$. Hence, $\zeta^j = 1$, where $i + 1 \equiv j \pmod{i_f(p) + 1}$. If $j > 0$, then $a(p^{j-1}) = 0$. But $0 \leq j - 1 < i_f(p)$, a contradiction, unless $j = 1$. But then $a(1) = 0$, which is also a contradiction. Hence, we must have $j = 0$, proving the third assertion.
The fourth assertion follows from [9], Lemma 2.5. □

As before, let us set

$$\varphi_p(s) = \sum_{m=0}^{\infty} \frac{a^0(p^m)}{p^{ms}}.$$

From the above lemma we deduce the following.

Lemma 2.2.20. *We have for $p \nmid N$, $p \notin Z_f$,*

$$\varphi_p(s) = \begin{cases} \left(1 - \frac{1}{p^s}\right)^{-1} & \text{if } i_f(p) = 0, \\ p^s \left(\frac{1}{p^s - 1} - \frac{1}{p^{(i_f(p)+1)s} - 1}\right) & \text{if } i_f(p) > 0. \end{cases}$$

Note that $\varphi_p(s) = 1$ for $p|N$ or $p \in Z_f$, because in this case $a^0(p^i) = 0$ for all $i \geq 1$.

Next, we evaluate $\xi_d(1)$. We have the following.

Proposition 2.2.21. *Writing*

$$\xi_d(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

we have that

1. $b(n) = 0$ *if n is divisible by a prime that does not divide d, and*

2. *if $p|d$, we have $|b(p^m)| \leq 4$ for all m.*

In particular, the function $n \mapsto b(n)$ satisfies the conditions of Section 4.2. Moreover, we have for $p \nmid N$, $p \notin Z_f$,

$$\xi_{p,d}(1) = \begin{cases} 1 & \text{if } i_f(p) = 0, \\ 1 + p^{-1} - p^{v-2k_0+1} & \text{if } i_f(p) = 1, \\ \frac{1+p+\ldots+p^{i_f(p)} - p^{v-(k_0-1)(i_f(p)+1)}}{p+\ldots+p^{i_f(p)}} & \text{if } i_f(p) > 1. \end{cases}$$

$$\xi_{p,d}(1) = 0, \text{ if } p \in Z_f.$$

Here $v = \mathrm{ord}_p d$, where $\mathrm{ord}_p(d) := \max_{p^i|d}\{i\}$, and k_0 is the smallest integer $\geq \dfrac{v+1}{i_f(p)+1}$.

Proof. By a calculation similar to that of Lemma 2.2.20, we see that

$$\sum_{m=0}^{\infty} \frac{a^0(p^{m+v})}{p^{ms}} = \begin{cases} \left(1 - \frac{1}{p^s}\right)^{-1} & \text{if } i_f(p) = 0, \\ p^s \left(\frac{1}{p^s-1} - \frac{p^{\{v-(k_0-1)(i_f(p)+1)\}s}}{p^{(i_f(p)+1)s}-1}\right) & \text{if } i_f(p) > 1. \end{cases}$$

Hence, writing $i = i_f(p)$, we have

$$\xi_{p,d}(s) = \frac{p^{(i+1)s} - 1 - p^{\{v+1-(k_0-1)(i+1)\}s} + p^{\{v-(k_0-1)(i+1)\}s}}{p^{(i+1)s} - p^s},$$

which is equal to

$$\left(1 - \frac{1}{p^{\{k_0(i+1)-v-1\}s}} + \frac{1}{p^{\{k_0(i+1)-v\}s}} - \frac{1}{p^{(i+1)s}}\right)\left(1 - \frac{1}{p^{is}}\right)^{-1}$$

from which it follows that $|b(p^m)| \leq 4$. Moreover, as

$$\xi_d(s) = \prod_{p|d} \xi_{p,d}(s),$$

it follows also that $b(n) = 0$ unless every prime divisor of n also divides d. The last assertion of the proposition follows from the above formulas. □

Note that if $p \in Z_f$, $p|d$, then

$$\xi_{p,d}(s) = \left(\sum_{m=0}^{\infty} \frac{a^0(p^{m+\mathrm{ord}_p d})}{p^{ms}}\right)\left(\sum_{m=0}^{\infty} \frac{a^0(p^m)}{p^{ms}}\right)^{-1} = 0.$$

Thus, $\xi_d(s) = 0$ if there is a $p|d$ such that $p \in Z_f$, because $\xi_{p,d}(s) = 0$ for that p.

Remark 2.2.22. *Note that the dependence on d of $\xi_{p,d}$ is only through $ord_p d$. Thus, where the meaning is clear, for $p|d$ and d squarefree, we shall write ξ_p, since in the case of a squarefree d we have*

$$\mathrm{ord}_p d = \begin{cases} 1, & \textit{if } p|d \\ 0, & \textit{if } p\nmid d \end{cases}.$$

In the remainder of this section we will elaborate on the constant u_f, and in particular, relate it to L-function values. From Lemma 2.2.20, we have

$$\begin{aligned} \log\varphi(s) &= \sum_p \log\varphi_p(s) = \sum_{p\notin Z_f}\log\varphi_p(s) + \sum_{p\in Z_f}\log\varphi_p(s) \\ &= -\sum_{\substack{i_f(p)=0\\ p\notin Z_f}} \log\left(1-\frac{1}{p^s}\right) - \sum_{\substack{i_f(p)=1\\ p\notin Z_f}} \log\varphi_p(s) + \sum_{\substack{i_f(p)>1\\ p\notin Z_f}} \log\varphi_p(s) \\ &= -\sum_{\substack{i_f(p)=0\\ p\notin Z_f}} \log\left(1-\frac{1}{p^s}\right) + \sum_{\substack{i_f(p)>1\\ p\notin Z_f}} \log\varphi_p(s). \end{aligned}$$

Note that $\sum_{p\in Z_f} \log\varphi_p(s) = 0$ because $\varphi_p(s) = 0$ for $p\in Z_f$; and $i_f(p) = 1 \Leftrightarrow p \in Z_f$.
Also, note that by Lemma 2.2.19, (4), the third sum on the right ranges over a finite set of primes p.
Denote by χ_K the quadratic Dirichlet character that defines K. This means that if d_K is the discriminant of K, then $\chi_K(n) = \left(\frac{n}{d_K}\right)$ – Jacobi symbol. If $d_K = p_1^{\alpha_1}\dots p_l^{\alpha_l}$ is the prime decomposition of d_K, then $\left(\frac{n}{d_K}\right) = \left(\frac{n}{p_1}\right)^{\alpha_1}\cdots\left(\frac{n}{p_l}\right)^{\alpha_l}$, where $\left(\frac{n}{p_i}\right)^{\alpha_i}$ is the Legendre symbol:

$$\left(\frac{n}{p_i}\right) = \begin{cases} 0, & \text{if } n\equiv 0 (\text{mod } p_i), \\ 1, & \text{if } n\not\equiv 0(\text{mod } p_i) \text{ and } n\equiv x^2 (\text{mod } p) \text{ for some } x, \\ -1, & \text{if } n\not\equiv 0(\text{mod } p_i) \text{ and there is no such } x. \end{cases}$$

Let $L(s,\chi_K)$ be the associated Dirichlet series. Let us denote by S, I, R the set of primes that split, stay inert or ramify in K respectively. Then, we have

$$L(s,\chi_K) = \sum_{n=1}^{\infty}\frac{\chi_K(n)}{n^s} = \prod_p \left(1-\frac{\chi_K(p)}{p^s}\right)^{-1}$$

Since

$$\chi_K(p) = \left(\frac{d_k}{p}\right) = \begin{cases} -1, & \text{if } p \text{ is inert}, \\ 0, & \text{if } p \text{ ramifies}, \\ 1, & \text{if } p \text{ splits}. \end{cases},$$

we have

$$-\sum_{p\in S}\log\left(1-\frac{1}{p^s}\right) = \frac{1}{2}\log\zeta(s) + \frac{1}{2}\log L(s,\chi_K) + \frac{1}{2}\sum_{p\in I}\log\left(1-\frac{1}{p^{2s}}\right) + \frac{1}{2}\sum_{p\in R}\log\left(1-\frac{1}{p^s}\right),$$

and so

$$
\begin{aligned}
-\sum_{\substack{p\in S\\ p\notin Z_f}} \log\left(1-\frac{1}{p^s}\right) &= \frac{1}{2}\log\zeta(s)+\frac{1}{2}\log L(s,\chi_K)+\frac{1}{2}\sum_{p\in I}\log\left(1-\frac{1}{p^{2s}}\right)+\frac{1}{2}\sum_{p\in R}\log\left(1-\frac{1}{p^s}\right)\\
&\quad + \sum_{\substack{p\in S\\ p\in Z_f}}\log\left(1-\frac{1}{p^s}\right).
\end{aligned}
$$

Moreover, if $i_f(p)=0$, then $a(p)\neq 0$ and $a(p^i)\neq 0$ for any $i\geq 2$. Thus,

$$
\begin{aligned}
-\sum_{\substack{i_f(p)=0\\ p\notin Z_f}} \log\left(1-\frac{1}{p^s}\right) &= -\sum_{\substack{a(p)\neq 0\\ p\notin Z_f}}\log\left(1-\frac{1}{p^s}\right)+\sum_{\substack{i_f(p)>1\\ p\notin Z_f}}\log\left(1-\frac{1}{p^s}\right)\\
&= -\sum_{\substack{p\in S\\ p\notin Z_f}}\log\left(1-\frac{1}{p^s}\right)-\sum_{\substack{p\in R\\ p\notin Z_f}}\log\left(1-\frac{1}{p^s}\right)+\sum_{\substack{i_f(p)>1\\ p\notin Z_f}}\log\left(1-\frac{1}{p^s}\right),\\
&\quad \text{because } p\notin Z_f \Rightarrow a(p)\neq 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\log\varphi(s) &= -\sum_{\substack{i_f(p)=0\\ p\notin Z_f}}\log\left(1-\frac{1}{p^s}\right)+\sum_{\substack{i_f(p)>1\\ p\notin Z_f}}\log\varphi_p(s)\\
&= -\sum_{\substack{p\in S\\ p\notin Z_f}}\log\left(1-\frac{1}{p^s}\right)-\sum_{\substack{p\in R\\ p\notin Z_f}}\log\left(1-\frac{1}{p^s}\right)+\sum_{\substack{i_f(p)>1\\ p\notin Z_f}}\log\left(1-\frac{1}{p^s}\right)+\sum_{\substack{i_f(p)>1\\ p\notin Z_f}}\log\varphi_p(s)\\
&= \frac{1}{2}\log\zeta(s)+\frac{1}{2}\log L(s,\chi_K)+\frac{1}{2}\sum_{p\in I}\log\left(1-\frac{1}{p^{2s}}\right)+\frac{1}{2}\sum_{p\in R}\log\left(1-\frac{1}{p^s}\right)\\
&\quad +\sum_{\substack{p\in S\\ p\in Z_f}}\log\left(1-\frac{1}{p^s}\right)-\sum_{\substack{p\in R\\ p\notin Z_f}}\log\left(1-\frac{1}{p^s}\right)+\sum_{\substack{i_f(p)>1\\ p\notin Z_f}}\log\left(1-\frac{1}{p^s}\right)+\sum_{\substack{i_f(p)>1\\ p\notin Z_f}}\log\varphi_p(s)\\
&= \frac{1}{2}\log\frac{1}{s-1}+\frac{1}{2}\log\left(\zeta(s)(s-1)\right)+\frac{1}{2}\log L(s,\chi_K)+\log C(s),
\end{aligned}
$$

where

$$
\begin{aligned}
C(s) &= \prod_{p\in I}\left(1-\frac{1}{p^{2s}}\right)^{\frac{1}{2}}\prod_{p\in R}\left(1-\frac{1}{p^s}\right)^{\frac{1}{2}}\prod_{\substack{p\in S\\ p\in Z_f}}\left(1-\frac{1}{p^s}\right)\prod_{\substack{p\in R\\ p\notin Z_f}}\left(1-\frac{1}{p^s}\right)^{-1}\cdot\\
&\quad \cdot\prod_{\substack{i_f(p)>1\\ p\notin Z_f}}\left(\left(1-\frac{1}{p^s}\right)\varphi_p(s)\right).
\end{aligned}
$$

$C(s)$ is holomorphic because $\prod_{\substack{p\in S\\ p\in Z_f}} \left(1-\frac{1}{p^s}\right)$ is holomorphic:

$$\begin{aligned}
\prod_{\substack{p\in S\\ p\in Z_f}} \left(1-\frac{1}{p^s}\right) &= \prod_{\substack{p\in S\\ a(p)=0}} \left(1-\frac{1}{p^s}\right) \cdot \prod_{\substack{p\in S\\ a(p)\neq 0\\ p\in Z_f}} \left(1-\frac{1}{p^s}\right) = \\
&= \prod_{\substack{p\in S\\ a(p)=0}} \left(1-\frac{1}{p^s}\right) \cdot \prod_{\substack{p\in Z_f\\ a(p)\neq 0}} \left(1-\frac{1}{p^s}\right) \cdot \prod_{\substack{p\in R\\ a(p)\neq 0\\ p\in Z_f}} \left(1-\frac{1}{p^s}\right)^{-1},
\end{aligned}$$

because $p \in I \Rightarrow a(p) = 0$.

We have proven that $\prod_{\substack{p\in S\\ a(p)=0}} \left(1-\frac{1}{p^s}\right)$ converges, and assumed that $\prod_{\substack{p\in Z_f\\ a(p)\neq 0}} \left(1-\frac{1}{p^s}\right)$ converges. This gives us the convergence of $\prod_{\substack{p\in S\\ p\in Z_f}} \left(1-\frac{1}{p^s}\right)$, as the product over the $p \in R$ is finite. Thus, we have convergence of $C(s)$.

Putting the above discussion together, we see that

$$\varphi(s) = \frac{\epsilon(s)}{(s-1)^{1/2}},$$

where

$$\epsilon(s) = L(s, \chi_K)^{\frac{1}{2}} C(s),$$

and so

$$\begin{aligned}
u_f &= \epsilon(1) \\
&= L(1, \chi_K)^{\frac{1}{2}} C(1).
\end{aligned}$$

Chapter 3

CM Hecke eigenforms

3.1 Some hypotheses on the Hecke character

In this section, we give examples of CM Hecke eigenforms for which $p|a(p)$ is possible for only finitely many primes p.
Let Ψ be a Hecke character of K.

$$\Psi \ : \ I(\mathfrak{f}) \to \mathbb{Q}^{\times}$$

where $I(\mathfrak{f})$ denotes the set of fractional ideals of K, coprime to $\mathfrak{f}$, $\mathfrak{f}$ is the conductor.

$$\Psi((\alpha)) = \alpha^{k-1} \ \text{ for } \alpha \equiv 1 \pmod{\mathfrak{f}}.$$

$$a(p) = \begin{cases} 0, & \text{if } p \text{ does not split,} \\ \Psi(\mathfrak{p}) + \Psi(\overline{\mathfrak{p}}), & \text{if } p = \mathfrak{p}\overline{\mathfrak{p}}. \end{cases}$$

$$\Psi(\mathfrak{p})\Psi(\overline{\mathfrak{p}}) = (N\mathfrak{p})^{k-1} = p^{k-1}$$

Assume $p \mid a(p)$, i.e. $a(p) \equiv 0 \ (\bmod p)$. This means

$$\Psi(\mathfrak{p}) \equiv -\Psi(\overline{\mathfrak{p}}) \ (\bmod p).$$

Since $a(p) \in \mathbb{Z}$, we have $\Psi(\overline{\mathfrak{p}}) = \overline{\Psi(\mathfrak{p})}$, or $\Psi(\mathfrak{p})$, $\Psi(\overline{\mathfrak{p}})$ are both equal to $\frac{a(p)}{2}$.
Indeed, $\Psi(\mathfrak{p})$, $\Psi(\overline{\mathfrak{p}})$ are the roots of the equation

$$T^2 - a(p)T + p^{k-1} = 0.$$

So, depending on the discriminant $D = a(p)^2 - 4p^{k-1}$, they are either real or complex conjugates of each other.
But we also have $|a(p)| \leq 2p^{\frac{k-1}{2}}$, which gives us $D \leq 0$.
If $D = 0$, then $a(p) = \pm 2p^{\frac{k-1}{2}}$, then $\Psi(\mathfrak{p}) = \Psi(\overline{\mathfrak{p}}) = \pm p^{\frac{k-1}{2}}$.

Suppose Ψ takes values in K. Then $a(p) \equiv 0 (\bmod p)$ and $a(p) \neq 0$ implies that $\Psi(\mathfrak{p}) \equiv -\Psi(\overline{\mathfrak{p}}) (\bmod \mathfrak{p})$

We have

$$\Psi(\overline{\mathfrak{p}}) = \overline{\Psi(\mathfrak{p})}$$

3.2 Hecke characters of $\mathbb{Q}(\sqrt{-1})$

Let us consider some examples.

1. Suppose $K = \mathbb{Q}(\sqrt{-1})$. The class number in this case is $h_K = 1$.

$$\Psi \;:\; I(\mathfrak{f}) \to \mathbb{Q}^\times$$
$$\Psi((\alpha)) = \alpha^{k-1} \;\text{ for } \alpha \equiv 1 \pmod{\mathfrak{f}}.$$

Suppose p splits: $p = \mathfrak{p}\overline{\mathfrak{p}}$. Write $\mathfrak{p} = (\alpha)$, $\alpha = c + di$.

$$\mathfrak{p}\overline{\mathfrak{p}} = p = c^2 + d^2 = p^{k-1}$$

$$\Psi(\mathfrak{p}) = \Psi((c+di)) = (c+di)^{k-1}$$
$$\Psi(\overline{\mathfrak{p}}) = \Psi((c-di)) = (c-di)^{k-1}$$
$$p \mid a(p) = \Psi(\mathfrak{p}) + \Psi(\overline{\mathfrak{p}}) = (c+di)^{k-1} + (c-di)^{k-1} \equiv 0 \;(\bmod\; p)$$

Consider the cases of k being odd or even separately.
1) Let k be odd: $k = 2m+1$ for some $m \in \mathbb{N}$.

$$(c+di)^{k-1} + (c-di)^{k-1} = (c+di)^{2m} + (c-di)^{2m} \equiv 0 \;(\bmod\; p)$$

$$\sum_{j=0}^{2m} \binom{2m}{j} c^j (di)^{2m-j} + \sum_{j=0}^{2m} \binom{2m}{j} c^j (-di)^{2m-j} \equiv 0 \;(\bmod\; p)$$

$$\sum_{j=0}^{m} 2\binom{2m}{2j} c^{2j} d^{2m-2j} (-1)^{m-j} \equiv 0 \;(\bmod\; p)$$

Since $c^2 + d^2 = p$, we have $d^2 \equiv -c^2 (\bmod\; p)$.

$$2\sum_{j=0}^{m} \binom{2m}{2j} c^{2j} (-c^2)^{m-j} (-1)^{m-j} \equiv 0 \;(\bmod\; p)$$

$$2\sum_{j=0}^{m} \binom{2m}{2j} c^{2m} \equiv 0 \;(\bmod\; p).$$

Note that

$$2\sum_{j=0}^{m} \binom{2m}{2j} = \sum_{j=0}^{2m} \binom{2m}{j} 1^j 1^{2m-j} + \sum_{j=0}^{2m} \binom{2m}{j} 1^j (-1)^{2m-j} = (1+1)^{2m} + (1-1)^{2m} = 2^{2m}.$$

Thus,

$$2\sum_{j=0}^{m} \binom{2m}{2j} c^{2m} = c^{2m} \cdot 2\sum_{j=0}^{m} \binom{2m}{2j} = 2^{2m} c^{2m} \equiv 0 \;(\bmod\; p) \Rightarrow p|c.$$

This is a contradiction as $c^2 + d^2 = p$.

2) Let k be even: $k = 2m$ for some $m \in \mathbb{N}$.

$$(c+di)^{k-1} + (c-di)^{k-1} = (c+di)^{2m-1} + (c-di)^{2m-1} \equiv 0 \;(\bmod\; p)$$

$$\sum_{j=0}^{2m-1} \binom{2m-1}{j} c^j (di)^{2m-1-j} + \sum_{j=0}^{2m-1} \binom{2m-1}{j} c^j (-di)^{2m-1-j} \equiv 0 \pmod{p}$$

If j is even, then $2m-1-j$ is odd, so the two terms that correspond to each such number, will cancel each other. Thus, we need to sum by the odd js only.

$$\sum_{j=0}^{m-1} 2\binom{2m-1}{2j+1} c^{2j+1} d^{2m-2j-2} (-1)^{m-j-1} \equiv 0 \pmod{p}$$

Since $c^2+d^2=p$, we have $d^2 \equiv -c^2 \pmod{p}$.

$$2c \sum_{j=0}^{m-1} \binom{2m-1}{2j+1} c^{2j} (-c^2)^{m-j-1} (-1)^{m-j-1} \equiv 0 \pmod{p}$$

$$2c \sum_{j=0}^{m-1} \binom{2m-1}{2j+1} c^{2m-2} \equiv 0 \pmod{p}.$$

$$2c^{2m-1} \sum_{j=0}^{m-1} \binom{2m-1}{2j+1} \equiv 0 \pmod{p}.$$

Note that

$$2\sum_{j=0}^{m-1} \binom{2m-1}{2j+1} = \sum_{j=0}^{2m-1} \binom{2m-1}{j} 1^j 1^{2m-1-j} + \sum_{j=0}^{2m-1} \binom{2m-1}{j} 1^j (-1)^{2m-1-j} =$$

$$= (1+1)^{2m-1} + (1-1)^{2m-1} = 2^{2m-1}.$$

Thus,

$$2c^{2m-1} \sum_{j=0}^{m-1} \binom{2m-1}{2j+1} = c^{2m-1} \cdot 2^{2m-1} \equiv 0 \pmod{p} \Rightarrow p|c.$$

Again, this is a contradiction.

3.3 Characters of $\mathbb{Q}(\sqrt{-D})$

Consider the general case: $K = \mathbb{Q}(\sqrt{-D})$. Let us suppose that $\mathcal{O}_K = \mathbb{Z}[\sqrt{-D}]$.
We do not assume that the class number $h = h_K = 1$. Choose representatives $\mathfrak{C}_1, \ldots, \mathfrak{C}_h$ for the elements of the $\mathfrak{f}$-ideal class group.
Write $p = \mathfrak{p}\bar{\mathfrak{p}}$.
Say $\mathfrak{p} = \mathfrak{C}(\alpha)$, $\alpha \equiv 1 \pmod{\mathfrak{f}}$. Then

$$\Psi(\mathfrak{p}) = \Psi(\mathfrak{C})\Psi((\alpha)) = \Psi(\mathfrak{C})\alpha^{k-1}$$

and

$$p = N(\mathfrak{C})N((\alpha)).$$

Write $\alpha = a + b\sqrt{-D}$. Consider $p > \max\{N\mathfrak{C}_i\}$, so that $p \nmid N(\mathfrak{C})$. Thus, $p|N((\alpha))$.

$$p|a^2 + Db^2$$

$$a^2 \equiv -Db^2 \pmod p$$

$$\Psi(\bar{\mathfrak{p}}) = \Psi(\bar{\mathfrak{C}})\bar{\alpha}^{k-1}$$

$$p \mid a(p) \;=\; \Psi(\mathfrak{p}) + \Psi(\bar{\mathfrak{p}}) = \Psi(\mathfrak{C})\alpha^{k-1} + \Psi(\bar{\mathfrak{C}})\bar{\alpha}^{k-1}$$

This means

$$p \mid \Psi(\mathfrak{C})(a + b\sqrt{-D})^{k-1} + \Psi(\bar{\mathfrak{C}})(a - b\sqrt{-D})^{k-1}$$

Assume that

$$\Psi(\bar{\mathfrak{C}}) = \overline{\Psi(\mathfrak{C})}.$$

Then

$$\Psi(\mathfrak{C}) \sum_{j=0}^{k-1} \binom{k-1}{j} a^j b^{k-1-j} (\sqrt{-D})^{k-1-j} - \overline{\Psi(\mathfrak{C})} \sum_{j=0}^{k-1} \binom{k-1}{j} a^j b^{k-1-j} (\sqrt{-D})^{k-1-j} (-1)^{k-1-j} \equiv 0 (\text{mod } p).$$

Denote $\Psi(\mathfrak{C}) = c + di\sqrt{D}$.
We have $a^2 \;\equiv\; -Db^2 \pmod p$. Then $a \equiv \pm i\sqrt{D}b \pmod p$.
a) Assume $a \equiv i\sqrt{D}b \pmod p$.

$$\begin{aligned} a(p) \;&\equiv\; (c + di\sqrt{D}) \sum_{j=0}^{k-1} \binom{k-1}{j} (i\sqrt{D}b)^j b^{k-1-j} (\sqrt{-D})^{k-1-j} \;+ \\ &+\; (c - di\sqrt{D}) \sum_{j=0}^{k-1} \binom{k-1}{j} (i\sqrt{D}b)^j b^{k-1-j} (\sqrt{-D})^{k-1-j} (-1)^{k-1-j} \pmod p \;\equiv \\ &\equiv\; (c + di\sqrt{D}) \sum_{j=0}^{k-1} \binom{k-1}{j} (\sqrt{D}bi)^{k-1} + (c - di\sqrt{D}) \sum_{j=0}^{k-1} \binom{k-1}{j} (\sqrt{D}bi)^{k-1} (-1)^{k-1-j} \end{aligned}$$

Consider the two possible cases:
a1) Let k be odd: $k = 2l + 1$. Then $k - 1$ is even, and $(-1)^{k-1-j} = (-1)^{-j} = (-1)^j$.

$$a(p) \equiv (\sqrt{D}bi)^{k-1} \left(2c \sum_{\substack{j=0 \\ j\text{ - even}}}^{k-1} \binom{k-1}{j} + 2di\sqrt{D} \sum_{\substack{j=0 \\ j\text{ - odd}}}^{k-1} \binom{k-1}{j} \right).$$

Recall that

$$2 \sum_{\substack{j=0 \\ j\text{ - even}}}^{k-1} \binom{k-1}{j} = \sum_{j=0}^{k-1} \binom{k-1}{j} + \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j = (1+1)^{k-1} + (1-1)^{k-1} = 2^{k-1}$$

and

$$2 \sum_{\substack{j=0 \\ j\text{ - odd}}}^{k-1} \binom{k-1}{j} = \sum_{j=0}^{k-1} \binom{k-1}{j} - \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j = (1+1)^{k-1} - (1-1)^{k-1} = 2^{k-1}.$$

Thus, our expression becomes

$$a(p) \equiv (\sqrt{D}bi)^{k-1} \left(c2^{k-1} + di\sqrt{D}2^{k-1} \right) = 2^{k-1}(c + \sqrt{D}di)(\sqrt{D}bi)^{k-1}.$$

Since we consider the case $a(p) \in \mathbb{Z}$, and $k-1$ is even, we need to have $d=0$. Thus, $p \mid a(p)$ implies $p \mid b$ or $p \mid c$.
If $p \mid b$, then $p \mid a$.

$$b = pb_0,\ \ a = pa_0$$

$$a^2 + Db^2 = p$$

$$p^2 a_0^2 + Dp^2 b_0^2 = p$$

$$p(a_0^2 + Db_0^2) = 1$$

Contradiction.
The case $p \mid c$ is possible for only finitely many p's, so we can exclude them from consideration.

a2) Let k be even: $k = 2l$. Thus, $k-1$ is odd. Then $(-1)^{k-1-j} = -(-1)^j$, and so

$$\begin{aligned} a(p) &\equiv (\sqrt{D}bi)^{k-1}\left(2di\sqrt{D}\sum_{\substack{j=0\\ j\text{ - even}}}^{k-1}\binom{k-1}{j} + 2c\sum_{\substack{j=0\\ j\text{ - odd}}}^{k-1}\binom{k-1}{j}\right) = \\ &= (\sqrt{D}bi)^{k-1}2^{k-1}(c+\sqrt{D}di) = -(\sqrt{-D}b)^{k-2}2^{k-1}(Dbc + Dbdi) = (-Db^2)^{\frac{k-2}{2}}Db2^{k-1}(c+di) \end{aligned}$$

Since $a(p) \in \mathbb{Z}$, we have $d = 0$. Thus, if $p \mid a(p)$, then $p|D$ or $p|b$ or $p|c$.
The assumption $p \mid b$ leads to contradiction as before; $p|D$ or $p|c$ is possible for finitely many primes only.

b) Assume $a \equiv -i\sqrt{D}b \pmod{p}$.

$$\begin{aligned} a(p) &\equiv (c+di\sqrt{D})\sum_{j=0}^{k-1}\binom{k-1}{j}(-i\sqrt{D}b)^j b^{k-1-j}(\sqrt{-D})^{k-1-j} \\ &+ (c-di\sqrt{D})\sum_{j=0}^{k-1}\binom{k-1}{j}(-i\sqrt{D}b)^j b^{k-1-j}(\sqrt{-D})^{k-1-j}(-1)^{k-1-j} \pmod{p} \\ &\equiv (c+di\sqrt{D})\sum_{j=0}^{k-1}\binom{k-1}{j}(\sqrt{D}bi)^{k-1}(-1)^j + (c-di\sqrt{D})\sum_{j=0}^{k-1}\binom{k-1}{j}(\sqrt{D}bi)^{k-1}(-1)^{k-1} \end{aligned}$$

Consider the two possible cases:
b1) Let k be odd: $k = 2l+1$. Then $k-1$ is even, and $(-1)^{k-1} = 1$.

$$a(p) \equiv (\sqrt{D}bi)^{k-1}\left(2c\sqrt{D}\sum_{\substack{j=0\\ j\text{ - even}}}^{k-1}\binom{k-1}{j} - 2di\sqrt{D}\sum_{\substack{j=0\\ j\text{ - odd}}}^{k-1}\binom{k-1}{j}\right).$$

Again we use the equations:

$$2\sum_{\substack{j=0\\ j\text{ - even}}}^{k-1}\binom{k-1}{j}=2^{k-1},\ \text{and } 2\sum_{\substack{j=0\\ j\text{ - odd}}}^{k-1}\binom{k-1}{j}=2^{k-1}.$$

Thus, our expression becomes

$$a(p)\equiv(\sqrt{D}bi)^{k-1}\left(c2^{k-1}-di\sqrt{D}2^{k-1}\right)=2^{k-1}(c-\sqrt{D}di)(\sqrt{D}bi)^{k-1}.$$

Since we consider the case $a(p)\in\mathbb{Z}$, and $k-1$ is even, we need to have $d=0$. Thus, $p\mid a(p)$ implies $p\mid b$ or $p\mid c$.
If $p\mid b$, then $p\mid a$.

$$b=pb_0,\ a=pa_0$$

$$a^2+Db^2=p$$

$$p^2a_0^2+Dp^2b_0^2=p$$

$$p(a_0^2+Db_0^2)=1$$

Contradiction.
The case $p\mid c$ is possible for only finitely many ps, so we can exclude them from consideration.
b2) Let k be even: $k=2l$. Thus, $k-1$ is odd. Then $(-1)^{k-1}=-1$, and so

$$\begin{aligned}a(p) &\equiv (\sqrt{D}bi)^{k-1}\left(2\sqrt{D}di\sum_{\substack{j=0\\ j\text{ - even}}}^{k-1}\binom{k-1}{j}-2c\sum_{\substack{j=0\\ j\text{ - odd}}}^{k-1}\binom{k-1}{j}\right)=\\ &= (\sqrt{D}bi)^{k-1}2^{k-1}(-c+\sqrt{D}di)=(\sqrt{-D}b)^{k-2}2^{k-1}(-\sqrt{D}bci-Dbd)\\ &= -(-Db^2)^{\frac{k-2}{2}}\sqrt{D}b2^{k-1}(ci+d)\end{aligned}$$

Since $a(p)\in\mathbb{Z}$, we have $c=0$. Thus, if $p\mid a(p)$, then $p|D$ or $p|b$ or $p|d$.
The assumption $p\mid b$ leads to contradiction as before; $p|D$ or $p|d$ is possible for finitely many primes only.

Thus we have shown that for $p>\max\{N(\mathfrak{C}_l),c_l,d_l,D\}$, where $\Psi(\mathfrak{C}_l)=c_l+id_l$, we have $a(p)\not\equiv 0 \pmod p$.

We have proved the following proposition

Proposition 3.3.1. *Suppose Ψ is a Hecke character of the imaginary quadratic field K satisfying*

- $\Psi(\overline{\mathfrak{a}})=\overline{\Psi(\mathfrak{a})}$
- *Ψ takes values in K*
- *For any prime $\mathfrak{p}$ of K we have $\Psi(\mathfrak{p})+\Psi(\overline{\mathfrak{p}})\in\mathbb{Z}$.*

Then for p large enough

$$\Psi(\mathfrak{p})+\Psi(\overline{\mathfrak{p}})\not\equiv 0(\text{mod } p).$$

Chapter 4

Proof of Theorem 1.1.2

4.1 A sieve lemma

Lemma 4.1.1. *Let f be as in the previous section, a normalized Hecke eigenform of weight* ≥ 2 *with complex multiplication. Let* $y_1 = (\log\log x)^{1+\epsilon}$ *and set*

$$N_{y_1}(x) = \#\{n \leq x : \ q|n \Rightarrow q \geq y_1, a(n) \neq 0, q \notin Z_f\}. \tag{4.1}$$

Then

$$N_{y_1}(x) = \frac{U_f x}{(\log x \log\log\log x)^{\frac{1}{2}}} \ + \ O\left(\frac{x(\log\log\log x)^2}{(\log x)^{\frac{3}{2}}}\right), \tag{4.2}$$

where

$$U_f = u_f \mu_f c_f \prod_{i_f > 1} \left(1 - \frac{\xi_{p,d}(1)}{p}\right). \tag{4.3}$$

Note that the last two products are over the finite number of primes and

$$c_f = \prod_{\substack{p \leq y_1 \\ i_f(p) \geq 2}} \left(1 - \frac{1}{p}\right)^{-1} \prod_{\substack{p \leq y_1 \\ i_f(p) = 1}} \left(1 - \frac{1}{p^2}\right).$$

Proof. Set $P_{y_1} = \prod_{p < y_1} p$. By the principle of inclusion-exclusion, we have

$$N_{y_1}(x) = \sum_{d | P_{y_1}} \mu(d) M_{f,d}(x),$$

where μ is the Möbius function:

$$\mu(n) = \begin{cases} 1, & \text{if } n \text{ is a square-free positive integer with an even number of prime factors,} \\ -1, & \text{if } n \text{ is a square-free positive integer with an odd number of prime factors,} \\ 0, & \text{if } n \text{ is not square-free.} \end{cases}$$

$$\log P_{y_1} = \sum_{p < y_1} \log p < \pi(y_1) \cdot \log y_1 \ll \frac{y_1}{\log y_1} \cdot \log y_1 = y_1,$$

and so $P_{y_1} \ll e^{y_1}$. Thus for any $d | P_{y_1}$, we have $\log x \ll \log \frac{x}{d} \ll \log x$.

Now using Proposition 2.2.18, the right hand side is

$$= \frac{u_f x}{\sqrt{\log x}} \sum_{d|P_{y_1}} \frac{\mu(d)}{d} \left(\xi_d(1) + O\left(\frac{2^{\nu(d)}}{\log x} \right) \right).$$

The main term is

$$\begin{aligned}
&= \frac{u_f x}{\sqrt{\log x}} \sum_{d|P_{y_1}} \frac{\mu(d)\xi_d(1)}{d} = \frac{u_f x}{\sqrt{\log x}} \sum_{d|P_{y_1}} \prod_{p|d} \frac{(-1)\xi_{p,d}(1)}{p} \\
&= \frac{u_f x}{\sqrt{\log x}} \prod_{p<y_1} \left(1 - \frac{\xi_{p,d}(1)}{p}\right) \\
&= \frac{u_f x}{\sqrt{\log x}} \prod_{\substack{p<y_1 \\ i_f(p)=0}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p<y_1 \\ i_f(p)\geq 1}} \left(1 - \frac{\xi_{p,d}(1)}{p}\right) \\
&= \frac{u_f x}{\sqrt{\log x}} \prod_{\substack{p<y_1 \\ i_f(p)=0}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p<y_1 \\ i_f(p)=1}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p<y_1 \\ i_f(p)>1}} \left(1 - \frac{\xi_{p,d}(1)}{p}\right) \\
&= \frac{u_f x}{\sqrt{\log x}} \prod_{\substack{p<y_1 \\ a_f(p)\neq 0}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p<y_1 \\ i_f(p)=1}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p<y_1 \\ i_f(p)>1}} \left(\left(1 - \frac{\xi_{p,d}(1)}{p}\right)\left(1 - \frac{1}{p}\right)^{-1}\right).
\end{aligned}$$

Note that by Proposition 2.2.21 for $p \nmid N$, $p \notin Z_f$ we have: if $i_f(p) = 0$, then $\xi_{p,d}(1) = 1$. Also note that by Lemma 2.2.19, there are only finitely many primes p for which $i_f(p) > 1$, ensuring the convergence of

$$\prod_{i_f(p)>1} \left(\left(1 - \frac{\xi_{p,d}(1)}{p}\right)\left(1 - \frac{1}{p}\right)^{-1}\right).$$

Now using Proposition 2.1.1, we see that the above sum is

$$\frac{U_f x}{\sqrt{L_3(x)\log x}} \left(1 + O_f\left(\frac{1}{\log\log\log x}\right)\right).$$

The error term is

$$\ll \frac{x}{(\log x)^{3/2}} \sum_{d|P_{y_1}} \frac{|\mu(d)|}{d} 2^{\nu(d)}.$$

The sum over d is

$$\begin{aligned}
&\ll \prod_{l<y_1} \left(1 + \frac{2}{l}\right) \\
&\ll \prod_{l<y_1} \left(1 - \frac{1}{l}\right)^{-2} \\
&\ll L_3(x)^2.
\end{aligned}$$

This proves the result.

□

We record here a variant of the above result.

Lemma 4.1.2. *Suppose that $p \leq y_1$. We have*

$$\#\{n \leq x | \; p|n, a_f(n) \neq 0, q|n \Rightarrow q \geq p\}$$

$$\ll \frac{x}{p(\log x)^{\frac{1}{2}}} \prod_{l<p} \left(1 - \frac{1}{l}\right) + \frac{x}{(\log x)^{\frac{3}{2}}} e^{4\sqrt{p}} \frac{\log p}{p}.$$

4.2 Siegel zeros

Let $L/\mathbb{Q}$ be a Galois extension of number fields with group G and n_L, d_L be the degree and the absolute value of the discriminant of $L/\mathbb{Q}$ respectively. Suppose that Artin's conjecture on the holomorphy of Artin L-functions is known for $L/\mathbb{Q}$. Set

$$\log \mathcal{M} = 2\left(\sum_{p|d_L} \log p + \log n_L\right).$$

Also, denote by d the maximum degree and by $\mathcal{A}$ the maximum Artin conductor of an irreducible character G.
Let C be the set of elements in G that map to the Cartan subgroup and also have trace zero. Then C is stable under conjugation and thus C is a union of conjugacy classes. Denote by $\pi(x, C)$ the number of primes $p \leq x$ with $\mathrm{Frob}_p \in C$. Then, [8], Theorem 4.1 asserts that for

$$\log x \gg d^4(\log \mathcal{M}),$$

there is an absolute and effective constant $c > 0$ so that

$$\pi(x, C) = \frac{|C|}{|G|}\mathrm{Li}\ x - \frac{|C|}{|G|}\mathrm{Li}\ x^{\beta} + O\left(|C|^{\frac{1}{2}} x (\log x\mathcal{M})^2 exp\left\{\frac{-c\log x}{d^{3/2}\sqrt{d^3(\log \mathcal{A})^2 + \log x}}\right\}\right),$$

where

$$\mathrm{Li}\ x = \int_2^x \frac{dt}{\log t}.$$

The term involving β is present only if the Dedekind zeta function $\zeta_L(s)$ of L has a real zero β (the Siegel zero), in the strip

$$1 - \frac{1}{4\log d_L} \leq \Re(s) < 1.$$

Let L be the fixed field of the kernel of $\overline{\rho}_{p,f}$. (Recall that $\overline{\rho}_{p,f}$ was introduced in Section 2.) Now, let $G = \mathrm{Gal}(L/\mathbb{Q})$ (viewed as a subgroup of $\mathrm{GL}_2(\mathbb{Z}/p)$) and let C be the subset of elements of G of trace zero. It is known that the subgroup $H = \mathrm{Gal}(L/K)$ is Abelian and maps to a Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p)$. The image of G maps to the normalizer of this group. As G has an Abelian normal subgroup of index 2, it is well-known that all irreducible characters of G are monomial, and so Artin's holomorphy conjecture holds for it.
Thus we can appeal to the above version of the Chebotarev density theorem. The extension L/K is unramified outside of primes dividing pN, where N is the level of f. We have $d = 2$, and

$$\log \mathcal{M} \ll \log pN$$

as well as

$$\log \mathcal{A} \ll \log pN.$$

For p sufficiently large, it is known that G maps onto the normalizer of a Cartan subgroup, and hence

$$p^2 \ll |G| \ll p^2.$$

Moreover, the size of $|C|$ satisfies

$$p \ll |C| \ll p.$$

Thus, if we set $\delta(p) = |C|/|G|$, we have for p sufficiently large

$$\frac{1}{p} \ll \delta(p) \ll \frac{1}{p}.$$

Thus, we have the following result.

Let $\pi^*(x,p) := \#\{q \le x | a(q) \equiv 0 \pmod{p}, a(q) \neq 0\}$. Then $\pi^*(x,p) \le \pi(x,p)$.

Lemma 4.2.1. *(Lemma 3.2 from [7], or Theorem 6.1 from [3]): Let f be a CM-form of level N as before. If $\log x \gg (\log pN)^2$, then we have*

$$\pi^*(x,p) = \delta(p)\mathrm{Li}\ x - \delta(p)\mathrm{Li}\ x^{\beta} + O(x \cdot exp\{-c\sqrt{\log x}\}), \tag{4.4}$$

where $\frac{1}{p} \ll \delta(p) \ll \frac{1}{p}$ and the implied constant is absolute and effective.

From the discussion above, we know that the stated bounds on $\delta(p)$ hold for p sufficiently large. To deduce that they hold for all p, it suffices to show that $\delta(p) > 0$ holds for all p. This inequality follows from the fact the image of complex conjugation is an element of trace zero in the Galois group.

If the Dedekind zeta function $\zeta_K(s)$ of K has a Siegel zero β in the interval $1 - \frac{1}{4\log d_K} \le \Re(s) \le 1$, then by a result of Stark in ([15], p. 145) we know that there is a quadratic field M contained in L such that $\zeta_M(\beta) = 0$. Further, from [15], for such M

$$\beta \le 1 - \frac{1}{\sqrt{d_M}}.$$

Let $[L : M] = n$. Since $d_L \ge d_M^n$, we have

$$\beta \le 1 - \frac{1}{d_L^{1/2n}}.$$

Now, by inequality of Hensel [14], p. 129,

$$\log d_L \le 2n \log pn_L$$

and so

$$\frac{1}{2n} \log d_L \le \log pn_L.$$

Hence

$$\beta < 1 - \frac{1}{pn_L}. \tag{4.5}$$

4.3 Intermediate lemmas

Let $0 < \varepsilon < 1/2$, and set $y = L_2^{1-\varepsilon}(x)$.

Lemma 4.3.1. *Let $p < y$ be a fixed prime. Then we have*

$$\sum_{\substack{q \le x \\ a(q) \equiv 0 (\bmod p)}}^{*} \frac{1}{q} = \delta(p) L_2(x) + O(L_3(x)),$$

where $\sum_{q \le x}^{*}$ means that the summation is over all primes $q \le x$ for which $a(q) \neq 0$, and $\delta(p)$ is from Lemma 4.2.1.

Proof. By partial summation the sum is

$$\sum_{\substack{q \le x \\ a(q) \equiv 0 (\bmod p)}}^{*} \frac{1}{q} = \int_1^x \frac{1}{t} d(\pi^*(t,p)) = \frac{\pi^*(x,p)}{x} + \int_1^x \frac{\pi^*(t,p)}{t^2} dt.$$

The integral $\int_1^x \frac{\pi^*(t,p)}{t^2} dt$ can be written as a sum

$$\int_1^{(\log x)^\gamma} \frac{\pi^*(t,p)}{t^2} dt + \int_{(\log x)^\gamma}^x \frac{\pi^*(t,p)}{t^2} dt.$$

where γ is chosen in such a way that for $(\log x)^\gamma \le t \le x$, we have got $\log t \gg (\log pN)^2$. The first integral is

$$\le \int_1^{(\log x)^\gamma} \frac{\pi(t)}{t^2} dt \ll L_3(x), \text{ where } \pi(t) = \#\{p \le t \mid p \text{ prime}\}$$

and the second integral is

$$\int_{(\log x)^\gamma}^x \frac{1}{t^2} \left(\delta(p)\mathrm{Li}(t) - \delta(p)\mathrm{Li}(t^\beta) + O(te^{-c\sqrt{\log t}}) \right) dt, \text{ by Lemma 4.2.1.}$$

The first term is equal to

$$\delta(p) \int_{(\log x)^\gamma}^x \frac{dt}{t \log t} + O(L_3(x))$$
$$= \delta(p) L_2(x) + O(L_3(x)).$$

Next, consider the term with the Siegel zero. Since by (4.5), $\beta < 1 - \frac{1}{pn_L}$, therefore the second term is

$$\begin{aligned} \delta(p) \int_{(\log x)^\gamma}^x \frac{1}{t^2} \mathrm{Li}(t^\beta) dt &= \delta(p) \int_{(\log x)^\gamma}^x \frac{1}{t^2} \int_2^{t^\beta} \frac{du}{\log u} \, dt \\ &= \delta(p) \int_2^{x^\beta} \frac{1}{\log u} \int_{\max\left\{(\log x)^\gamma, u^{\frac{1}{\beta}}\right\}}^x \frac{dt}{t^2} \, du. \end{aligned}$$

We split the range of integration of u into two integrals:

$$\begin{array}{ll} (I) & 2 \le u \le (\log x)^{\gamma\beta}, \\ (II) & (\log x)^{\gamma\beta} \le u \le x^\beta. \end{array}$$

The first range gives rise to the integral

$$\delta(p)\int_2^{(\log x)^{\gamma\beta}} \frac{1}{\log u}\left\{\frac{1}{(\log x)^\gamma}-\frac{1}{x}\right\}\,du \ll \delta(p)(\log x)^{\gamma(\beta-1)} \ll 1.$$

The second range gives rise to the integral

$$\delta(p)\int_{(\log x)^{\gamma\beta}}^{x^\beta} \frac{1}{\log u}\left\{\frac{1}{u^{\frac{1}{\beta}}}-\frac{1}{x}\right\}\,du.$$

Set $v = u^{\frac{1}{\beta}}$. Then $v^\beta = u$ and $\beta\log v = \log u$. Moreover, $du = \beta v^{\beta-1}dv$. Hence the integral is

$$\begin{aligned}\delta(p)\int_{(\log x)^\gamma}^{x} \frac{\beta v^{\beta-1}}{\beta\log v}\left\{\frac{1}{v}-\frac{1}{x}\right\}\,dv \quad &\ll \quad \frac{\delta(p)}{(\log x)^{\gamma(1-\beta)}}\int_{(\log x)^\gamma}^{x}\frac{dv}{v\log v} \\ &\ll \quad \frac{\delta(p)L_2(x)}{(\log x)^{\frac{\gamma}{pn_L}}} \\ &\ll \quad \frac{\delta(p)L_2(x)}{e^{\frac{\gamma}{n_L}L_2(x)^\epsilon}} \ll 1.\end{aligned}$$

Finally, using the elementary estimate $e^{c\sqrt{u}} \gg u^2$, we deduce that the O-term is

$$\ll \int_{L_2(x)}^{\log x}\frac{du}{u^2} \ll 1.$$

The term $\dfrac{\pi^*(x,p)}{x}$ is of smaller order. This proves the lemma. □

Define $\nu(p,n) = \#\{q^m||n \mid a(q^m)\equiv 0(\bmod p)\}$, i.e. $\nu(p,n)$ counts the number of primes q that divide n and $a(q^m)\equiv 0(\text{mod } p)$, where m is defined by $q^m||n$. Note that for these primes q we have

$$a(q^m)\equiv 0(\bmod p) \Rightarrow a(n)\equiv 0(\bmod p),$$

and so $\sum_{p|n}\nu(p,n) = 0$ means that $(n,a(n)) = 1$.

Lemma 4.3.2. *Assume that $p < y$. Then we have*

$$\sum_{n\le x}^{*}\nu(p,n) = (1+o(1))\frac{u_f\delta(p)xL_2(x)}{\sqrt{\log x}} + O\left(\frac{xL_3(x)}{\sqrt{\log x}}\right),$$

where $\sum^*_{n\le x}$ means that the summation is over all natural numbers $n\le x$ such that $a(n)\neq 0$.

Proof. Interchanging summation, we see that

$$\begin{aligned}
\sideset{}{^*}\sum_{n\le x} \nu(p,n) &= \sideset{}{^*}\sum_{\substack{q^m\le x\\ a(q^m)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q^m||n}} 1\\
&= \sideset{}{^*}\sum_{\substack{q\le x\\ a(q)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q||n}} 1 + \sideset{}{^*}\sum_{\substack{q^m\le x,\ m\ge 2\\ a(q^m)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q^m||n}} 1.
\end{aligned}$$

The contribution of terms q^m with $m\ge 2$ is

$$\begin{aligned}
\sideset{}{^*}\sum_{\substack{q^m\le x\\ m\ge 2\\ a(q^m)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q^m||n}} 1 &= \sideset{}{^*}\sum_{\substack{q^m\le x^\epsilon\\ m\ge 2\\ a(q^m)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q^m||n}} 1 + \sideset{}{^*}\sum_{\substack{x^\epsilon\le q^m\le x\\ m\ge 2\\ a(q^m)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q^m||n}} 1\\
&\ll \sideset{}{^*}\sum_{\substack{q^m\le x^\epsilon\\ m\ge 2\\ a(q^m)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{n\le x/q^m} 1 + \sideset{}{^*}\sum_{\substack{x^\epsilon\le q^m\le x\\ m\ge 2}} \frac{x}{q^m}\\
&\ll \sideset{}{^*}\sum_{\substack{q^m\le x^\epsilon\\ m\ge 2\\ a(q^m)\equiv 0(\bmod p)}} \frac{\frac{x}{q^m}}{\sqrt{\log\left(\frac{x}{q^m}\right)}} + \sideset{}{^*}\sum_{\substack{x^\epsilon\le q^m\le x\\ m\ge 2}} \frac{x}{q^m},\ \text{by Proposition 2.2.18}\\
&\ll \frac{x}{\sqrt{\log x}} \sideset{}{^*}\sum_{\substack{q^m\le x^\epsilon\\ m\ge 2}} \frac{1}{q^m} + x\int_{x^\epsilon}^{x} \frac{dt}{t^2}\ \text{since } q^m\le x^\epsilon,\\
&\ll \frac{x}{\sqrt{\log x}} + \frac{x}{x^\epsilon} \ll \frac{x}{\sqrt{\log x}}.
\end{aligned}$$

Also, we have

$$\sideset{}{^*}\sum_{\substack{q\le x\\ a(q)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q||n}} 1 = \sideset{}{^*}\sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q||n}} 1 + \sideset{}{^*}\sum_{\substack{x^{1/\log\log x}\le q\le x\\ a(q)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q||n}} 1. \tag{4.6}$$

We show that the second double sum on the right of (4.6) contributes a negligible amount. For this we split it into two parts.

$$\sideset{}{^*}\sum_{\substack{x^{1/\log\log x}\le q\le x\\ a(q)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q||n}} 1 = \sideset{}{^*}\sum_{\substack{x^{1/\log\log x}\le q\le x^\epsilon\\ a(q)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q||n}} 1 + \sideset{}{^*}\sum_{\substack{x^\epsilon\le q\le x\\ a(q)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q||n}} 1$$

Consider first the sum:

$$\sideset{}{^*}\sum_{\substack{x^\epsilon\le q\le x\\ a(q)\equiv 0(\bmod p)}} \ \sideset{}{^*}\sum_{\substack{n\le x\\ q||n}} 1. \tag{4.7}$$

This is majorized by

$$\sideset{}{^*}\sum_{n\le x}\ \sideset{}{^*}\sum_{\substack{x^\epsilon\le q\le x\\ q||n}} 1.$$

Note that the inner sum is bounded because we can have at most $l=\left[\frac{1}{\epsilon}\right]$ distinct primes q such that $q|n$ and $q\ge x^\epsilon$. Indeed, assume $q_1,\dots,q_l$ are distinct primes that divide n. Then

$$\begin{aligned}
&x^{l\epsilon}\le q_1\dots q_l\le x\\
&l\epsilon\le 1\\
&l\le\frac{1}{\epsilon}
\end{aligned}$$

Since the inner sum is bounded by $\frac{1}{\epsilon}$, and by Proposition 2.2.18 we have

$$\sideset{}{^*}\sum_{n\le x}\ \sideset{}{^*}\sum_{\substack{x^\epsilon\le q\le x\\ q||n}} 1\le\frac{1}{\epsilon}\cdot\sideset{}{^*}\sum_{n\le x}1=\frac{1}{\epsilon}\cdot M_{f,1}(x)\ll\frac{x}{\sqrt{\log x}}$$

and so we see that (4.7) is

$$\ll\frac{x}{\sqrt{\log x}}.\tag{4.8}$$

Now, consider the quantity

$$\sideset{}{^*}\sum_{\substack{x^{1/\log\log x}\le q\le x^\epsilon\\ a(q)\equiv 0(\bmod p)}}\ \sideset{}{^*}\sum_{\substack{n\le x\\ q||n}} 1.\tag{4.9}$$

By Proposition 2.2.18, the inner sum is

$$\sideset{}{^*}\sum_{\substack{n\le x\\ q||n}} 1\ \le\ \sideset{}{^*}\sum_{\substack{n\le x\\ q|n}} 1\ =M_{f,q}(x)\ \ll\ \frac{x}{q\sqrt{\log x}}$$

Since

$$\sum_{x^{1/\log\log x}\le q\le x^\epsilon}\frac{1}{q}=\int_{x^{1/\log\log x}}^{x^\epsilon}\frac{1}{u}\,d(\pi(u))=\left.\frac{\pi(u)}{u}\right|_{x^{1/\log\log x}}^{x^\epsilon}+\int_{x^{1/\log\log x}}^{x^\epsilon}\frac{\pi(u)}{u^2}\,du=$$

$$=\left.\frac{\pi(u)}{u}\right|_{x^{1/\log\log x}}^{x^\epsilon}+\int_{x^{1/\log\log x}}^{x^\epsilon}\frac{1}{u\log u}\,du+O\left(\int_{x^{1/\log\log x}}^{x^\epsilon}\frac{1}{u(\log u)^2}\,du\right)=$$

$$=\left.\frac{\pi(u)}{u}\right|_{x^{1/\log\log x}}^{x^\epsilon}+\left.\log\log u\right|_{x^{1/\log\log x}}^{x^\epsilon}+O\left(\left.\frac{1}{\log u}\right|_{x^{1/\log\log x}}^{x^\epsilon}\right)=$$

$$=\log\log x^\epsilon-\log\log x^{1/\log\log x}+O(1)=\log(\epsilon\log x)-\log\left(\frac{1}{\log\log x}\log x\right)+O(1)=$$

$$=\log\epsilon+\log\log x-\log\log x+\log\log\log x+O(1)=\log\log\log x+O(1),$$

it follows that (4.9) is

$$\ll\frac{xL_3(x)}{\sqrt{\log x}}.\tag{4.10}$$

Putting (4.8) and (4.10) together, we deduce that

$$\sum_{\substack{q\le x\\ a(q)\equiv 0(\bmod p)}}^{*}\ \sum_{\substack{n\le x\\ q||n}}^{*} 1 = \sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}}^{*}\ \sum_{\substack{n\le x\\ q||n}}^{*} 1 + O\left(\frac{xL_3(x)}{\sqrt{\log x}}\right).$$

Now by Proposition 2.2.18, Lemma 2.2.17 (and the fact that in the sum $a^0(q)=1$),

$$\begin{aligned}
\sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}}^{*}\ \sum_{\substack{n\le x\\ q||n}}^{*} 1 \quad &\le \quad \sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}}^{*}\ \sum_{\substack{n\le x\\ q|n}}^{*} 1 = \sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}}^{*} M_{f,q} \ = \text{ by Proposition 2.2.18}\\
&= \quad \sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}}^{*} \left(\frac{u_f\xi_q(1)\frac{x}{q}}{(\log\frac{x}{q})^{\frac12}} + O\left(\frac{x2^{\nu(q)}}{q(\log\frac{x}{q})^{\frac32}}\right)\right) =\\
&= \quad u_f x \sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}}^{*} \frac{1}{\sqrt{\log\frac{x}{q}}}\left(\frac{\xi_q(1)}{q} + O\left(\frac{1}{q\log\frac{x}{q}}\right)\right) =\\
&= \quad u_f x \sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}}^{*} \frac{1}{\sqrt{\log\frac{x}{q}}}\left(\frac{1}{q} + O\left(\frac{1}{q^2}\right) + O\left(\frac{1}{q\log\frac{x}{q}}\right)\right) =\\
&= \quad u_f x \sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}}^{*} \frac{1}{q\sqrt{\log\frac{x}{q}}}\left(1 + O\left(\frac{1}{q}\right) + O\left(\frac{1}{\log\frac{x}{q}}\right)\right) =\\
&= \quad u_f x \sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}}^{*} \frac{1+o(1)}{q\sqrt{\log x}}\left(1 + O\left(\frac{1}{q}\right) + O\left(\frac{1}{\log\frac{x}{q}}\right)\right) =\\
&= \quad (1+o(1))\frac{u_f x}{\sqrt{\log x}} \sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}}^{*} \frac{1}{q}\left(1 + O\left(\frac{1}{q}\right) + O\left(\frac{1}{\log\frac{x}{q}}\right)\right)\\
&= \quad (1+o(1))\frac{u_f x}{\sqrt{\log x}} \sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0(\bmod p)}}^{*} \frac{1}{q} + O\left(\frac{x}{\sqrt{\log x}}\right).
\end{aligned}$$

Now applying Lemma 4.3.1, we see that this is

$$= (1+o(1))\frac{u_f\delta(p)xL_2(x)}{\sqrt{\log x}} + O\left(\frac{xL_3(x)}{\sqrt{\log x}}\right).$$

This proves the lemma. □

Lemma 4.3.3. *Assume that $p < y$. Then*

$$\sum_{n\le x}^{*} \nu(p,n)^2 = (1+o(1))\frac{u_f\delta^2(p)xL_2^2(x)}{\sqrt{\log x}} + O\left(\frac{xL_2(x)L_3(x)}{\sqrt{\log x}}\right).$$

Proof. The sum in question is equal to

$$\sum^{*}_{\substack{q_1^{m_1}\le x\\ a(q_1^{m_1})\equiv 0(\bmod p)}} \quad \sum^{*}_{\substack{q_2^{m_2}\le x\\ a(q_2^{m_2})\equiv 0(\bmod p)}} \quad \sum^{*}_{\substack{n\le x\\ q_1^{m_1}||n,q_2^{m_2}||n}} 1.$$

By a small modification to the argument given in the proof of Lemma 4.3.2 , we find that the contribution of terms with $q_1 = q_2$ is

$$\ll \frac{xL_2}{\sqrt{\log x}}.$$

Next, we consider the contribution S (say) of terms with $q_1^{m_1}q_2^{m_2} > x^\epsilon$. For estimating this, we may suppose that $q_1^{m_1} > q_2^{m_2}$. Since $q_2 > 2$, we may suppose that $x/2 \ge q_1^{m_1} \ge x^{\epsilon/2} = z$ (say). Denote by S_1 the contribution of terms for which $z \le q_1^{m_1} \le \sqrt{x}$ and by S_2 the contribution of all the remaining terms in S. Then by Proposition 2.2.18, we have

$$\begin{aligned} S_1 &\ll x \sum^{*}_{z\le q_1^{m_1}\le\sqrt{x}} \frac{1}{q_1^{m_1}} \sum_{q_2^{m_2}\le q_1^{m_1}} \frac{1}{q_2^{m_2}\sqrt{\log\frac{x}{q_1^{m_1}q_2^{m_2}}}} \\ &\ll x \sum_{z\le q_1^{m_1}\le\sqrt{x}} \frac{1}{q_1^{m_1}\sqrt{\log\frac{x}{q_1^{2m_1}}}} \log\log(q_1^{m_1}) \\ &\ll xL_2(x)\int_z^{\sqrt{x}} \frac{dt}{t\log t\sqrt{\log x/t^2}} \ll \frac{xL_2(x)}{\sqrt{\log x}}. \end{aligned}$$

Next, we observe that

$$S_2 \ll \sum_{\sqrt{x}<q_1^{m_1}\le\frac{x}{2}} \frac{1}{q_1^{m_1}} \sum^{*}_{n\le\frac{x}{q_1^{m_1}}} \nu(p,n)$$

and by Lemma 4.3.2, this is

$$\ll xL_2(x) \sum_{\sqrt{x}<q_1^{m_1}\le\frac{x}{2}} \frac{1}{q_1^{m_1}} \frac{1}{\sqrt{\log x/q_1^{m_1}}} \ll \frac{xL_2}{\sqrt{\log x}}.$$

It remains to estimate

$$\begin{aligned} &\sum^{*}_{\substack{q_1^{m_1}q_2^{m_2}\le x^\epsilon\\ a(q_1^{m_1})\equiv 0(\bmod p)\\ a(q_2^{m_2})\equiv 0(\bmod p)}} \quad \sum_{\substack{n\le x\\ q_1^{m_1}||n,q_2^{m_2}||n}} 1 \\ &= I + J, \text{ say} \end{aligned}$$

where in I we have the terms with $m_1 > 1$ or $m_2 > 1$ and in J we have the terms with $m_1 = m_2 = 1$. In order to estimate I, suppose without loss of generality that $m_1 \geq 2$. Then by Proposition 2.2.18, we have

$$\begin{aligned} I &\ll x \sideset{}{^*}\sum_{\substack{q_1^{m_1} \\ m_1 \geq 2}} \frac{1}{q_1^{m_1}} \sideset{}{^*}\sum_{\substack{q_2^{m_2} \\ q_1^{m_1} q_2^{m_2} \leq x^\epsilon}} \frac{1}{q_2^{m_2} \sqrt{\log \frac{x}{q_1^{m_1} q_2^{m_2}}}} \\ &\ll \frac{x}{\sqrt{\log x}} \sideset{}{^*}\sum_{\substack{q_1^{m_1} \\ m_1 \geq 2}} \frac{1}{q_1^{m_1}} \left(\sum_{q_2 \leq x^\epsilon} \frac{1}{q_2} + \sum_{\substack{q_2 \\ m_2 \geq 2}} \frac{1}{q_2^{m_2}} \right) \\ &\ll \frac{x L_2(x)}{\sqrt{\log x}}. \end{aligned}$$

Next, we consider

$$J = \sideset{}{^*}\sum_{\substack{q_1 q_2 \leq x^\epsilon \\ a(q_1) \equiv 0 (\bmod p) \\ a(q_2) \equiv 0 (\bmod p)}} \sum_{\substack{n \leq x \\ q_1 || n, q_2 || n}} 1.$$

By Proposition 2.2.18 and Proposition 2.2.21, we have

$$\begin{aligned} J &= (1 + o(1)) \frac{u_f x}{\sqrt{\log x}} \sideset{}{^*}\sum_{\substack{q_1 q_2 \leq x^\epsilon \\ a(q_1) \equiv 0 (\bmod p) \\ a(q_2) \equiv 0 (\bmod p) \\ q_1 \neq q_2}} \frac{1}{q_1 q_2} + O\left(\frac{x L_2(x)}{\sqrt{\log x}} \right) \\ &= (1 + o(1)) \frac{u_f x}{\sqrt{\log x}} \left(\sum_{\substack{q \leq x \\ a(q) \equiv 0 (\bmod p)}} \frac{1}{q} \right)^2 + O\left(\frac{x L_2(x)}{\sqrt{\log x}} \right) \\ &= (1 + o(1)) \frac{u_f x}{\sqrt{\log x}} (\delta(p) L_2(x) + O(L_3(x)))^2 + O\left(\frac{x L_2(x)}{\sqrt{\log x}} \right) \\ &= (1 + o(1)) \frac{u_f \delta^2(p) x L_2^2(x)}{\sqrt{\log x}} + O\left(\delta(p) \frac{x L_2(x) L_3(x)}{\sqrt{\log x}} \right). \end{aligned}$$

This proves the lemma. □

Lemma 4.3.4. *Suppose $p < y$. Then*

$$\sideset{}{^*}\sum_{n \leq x} (\nu(p, n) - \delta(p) L_2(x))^2 \ll \frac{\delta(p) x}{\sqrt{\log x}} L_2(x) L_3(x).$$

Proof. This follows from Lemma 4.3.2 and Lemma 4.3.3. □

Lemma 4.3.5. *Assume $p < y$. Then*

$$\#\{n \leq x | \nu(p, n) = 0\} \ll \frac{x L_3(x)}{\delta(p) \sqrt{\log x} L_2(x)}.$$

Proof. By Lemma 4.3.4 this is

$$\ll \frac{1}{\delta^2(p)L_2^2(x)}\left\{\delta(p)\frac{x}{\sqrt{\log x}}L_2(x)L_3(x)\right\} = \frac{xL_3(x)}{\delta(p)\sqrt{\log x}L_2(x)}.$$

□

4.4 Proof of Theorem 1.1.2

Break up the set $\{n \leq x |\ (n, a(n)) = 1\}$ into the sets $\{n \leq x, p|n, (n, a(n)) = 1, q|n \Rightarrow q \geq p\}$ for p-prime, i.e. for each prime p put together all n's such that p is their smallest prime divisor.
Denote by $G_p(x) := \#\{n \leq x, p|n, (n, a(n)) = 1,\ q|n \Rightarrow q \geq p, q \notin Z_f\}$. Note that for $p \in Z_f$ we have $G_p(x) = 0$, because $p|a(p) \Leftrightarrow p|a(p^m)$, so we get $p|n, p|a(n)$, which means that $(n, a(n)) \neq 1$. In fact, a stronger statement holds: $(n, a(n)) = 1$ implies that if $p|n$, then $p \notin Z_f$.
Then

$$\#\{n \leq x|\ (n, a(n)) = 1\} = \sum_{\substack{p \leq x \\ p-prime}} G_p(x) = A_1(x) + A_2(x) + A_3(x),$$

where

$$A_1(x) = \sum_{p \leq (\log\log x)^{\frac{1}{2}-\varepsilon}} G_p(x), \qquad A_2(x) = \sum_{(\log\log x)^{\frac{1}{2}-\varepsilon} < p < (\log\log x)^{1+\varepsilon}} G_p(x),$$

$$A_3(x) = \sum_{p \geq (\log\log x)^{1+\varepsilon}} G_p(x).$$

Estimating $A_1(x)$, $A_2(x)$ and $A_3(x)$ requires the intermediate lemmas from Chapter 4.3, so as in Chapter 4.3 let $0 < \epsilon < 1/2$ and set $y = L_2^{1-\epsilon}(x)$.

Now, using Lemma 4.3.5, we have

$$
\begin{aligned}
A_1(x) &\leq \sum_{p \leq L_2^{\frac{1}{2}-\epsilon}(x)} \#\{n \leq x \mid p|n, (n, a(n)) = 1\} \\
&= \sum_{p \leq L_2^{\frac{1}{2}-\epsilon}(x)} \#\{n \leq x \mid p|n, \ \sum_{p_1|n} \nu(p_1, n) = 0\} \\
&= \sum_{p \leq L_2^{\frac{1}{2}-\epsilon}(x)} \#\{n \leq x \mid p|n, \ \nu(p_1, n) = 0 \text{ for all } p_1|n\} \\
&\leq \sum_{p \leq L_2^{\frac{1}{2}-\epsilon}(x)} \#\{n \leq x \mid \nu(p, n) = 0\} \\
&\ll \frac{xL_3(x)}{(\log x)^{\frac{1}{2}} L_2(x)} \sum_{p \leq L_2^{\frac{1}{2}-\epsilon}(x)} \frac{1}{\delta(p)} \text{ by Lemma 4.3.5} \\
&\ll \frac{xL_3(x)}{(\log x)^{\frac{1}{2}} L_2(x)} \sum_{1 \ll p \leq L_2^{\frac{1}{2}-\epsilon}(x)} p, \text{ as } \delta(p) \gg \frac{1}{p} \\
&\ll \frac{x}{(\log x)^{\frac{1}{2}} L_2^{\epsilon}(x)} = o\left(\frac{x}{(L_3(x)\log x)^{\frac{1}{2}}} \right).
\end{aligned}
$$

Moreover, by Lemma 5.3.2 we have

$$
\begin{aligned}
A_2(x) &\leq \sum_{L_2^{\frac{1}{2}-\epsilon}(x) < p < L_2^{1+\epsilon}(x)} \#\{n \leq x \mid p|n, a(n) \neq 0, q|n \Rightarrow q \geq p\} \\
&\ll \frac{x}{\sqrt{\log x}} \sum_{L_2^{\frac{1}{2}-\epsilon}(x) < p < L_2^{1+\epsilon}(x)} \frac{1}{p} \prod_{\substack{l \leq p \\ l \text{ prime}}} \left(1 - \frac{1}{l}\right) \\
&\ll \frac{x}{\sqrt{\log x}} \sum_{L_2^{\frac{1}{2}-\epsilon}(x) < p < L_2^{1+\epsilon}(x)} \frac{1}{p \log p} \\
&\ll \frac{x}{\sqrt{\log x}} \cdot \frac{1}{\log(L_2^{\frac{1}{2}-\epsilon}(x))} \cdot \sum_{L_2^{\frac{1}{2}-\epsilon}(x) < p < L_2^{1+\epsilon}(x)} \frac{1}{p} \\
&\ll \frac{x}{\sqrt{\log x}} \cdot \frac{1}{L_3(x)} \cdot L_4(x) = o\left(\frac{x}{(L_3(x)\log x)^{\frac{1}{2}}} \right).
\end{aligned}
$$

Let $y_1 = L_2(x)^{1+\epsilon}$ and as in Lemma 4.1.1, denote by $N_{y_1}(x)$ the number of n's such that all their prime divisors q satisfy $q \geq y_1$:
$N_{y_1}(x) = \#\{n \leq x \mid q|n \Rightarrow q \geq y_1, a(n) \neq 0, q \nmid a(q)\}$.

Then

$$N_{y_1}(x) - \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ a(q_1^m) \equiv 0 (\bmod q_2)}} \sum^{**}_{\substack{n \le x \\ q_1^m || n, q_2 | n \\ q|n \Rightarrow q \nmid a(q)}} 1 \le A_3(x) \le N_{y_1}(x),$$

where $\sum^{**}_{n \le x}$ means that the summation is over all natural numbers $n \le x$ such that $a(n) \ne 0$ and $q|n$ implies that $q > y_1$.
Indeed, the inequality $A_3(x) \le N_{y_1}(x)$ is obvious, since $N_{y_1}(x)$ differs from $A_3(x)$ by having no requirement that $(a(n), n) = 1$. Now, denote

$$M_{y_1}(x) = \#\{(q_1, q_2, n) \mid q_1, q_2 \in [y_1, x], n \le x, q_1^m || n, q_2 | n, a(n) \ne 0, a(q_1^m) \equiv 0 (\bmod q_2)\} =$$

$$= \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ a(q_1^m) \equiv 0 (\bmod q_2)}} \sum^{**}_{\substack{n \le x \\ q_1^m || n, q_2 | n \\ q|n \Rightarrow q \nmid a(q)}} 1.$$

If we denote by A_3, N_{y_1} and M_{y_1} the sets that correspond to the numbers $A_3(x)$, $N_{y_1}(x)$ and $M_{y_1}(x)$, then

$$M_{y_1} \supseteq \{n \le x \mid q|n, \Rightarrow q \ge y_1,\ a(n) \ne 0,\ (a(n), n) \ne 1\} = N_{y_1} \setminus A_3,$$

$$N_{y_1} \setminus A_3 \subset M_{y_1},$$

and so (going back to numbers of the elements in the sets) we have

$$N_{y_1}(x) - M_{y_1}(x) \le N_{y_1}(x) - (N_{y_1}(x) - A_3(x)) = A_3(x).$$

By Lemma 4.1.1, to prove the theorem, it suffices to show that

$$\sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ a(q_1^m) \equiv 0 (\bmod q_2)}} \sum^{**}_{\substack{n \le x \\ q_1^m || n, q_2 | n \\ q|n \Rightarrow q \nmid a(q)}} 1 = o\left(\frac{x}{(L_3(x) \log x)^{\frac{1}{2}}}\right). \tag{4.11}$$

In order to prove (4.11), let us write

$$\sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ a(q_1^m) \equiv 0 (\bmod q_2) \\ q_1 \ne q_2}} \sum^{**}_{\substack{n \le x \\ q_1^m || n, q_2 | n}} 1 = \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ a(q_1^m) \equiv 0 \pmod{q_2} \\ q_1 \ne q_2 \\ m \ge 2}} \sum^{**}_{\substack{n \le x \\ q_1^m || n, q_2 | n}} 1 + \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ q_1 \ne q_2 \\ a(q_1) \equiv 0 \pmod{q_2}}} \sum^{**}_{\substack{n \le x \\ q_1 || n, q_2 | n}} 1 = B_1 + B_2.$$

Let us consider B_1 first.

$$B_1 = \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ a(q_1^m) \equiv 0 \pmod{q_2} \\ q_1 \ne q_2 \\ m \ge 2}} \sum^{**}_{\substack{n \le x \\ q_1^m || n, q_2 | n}} 1 = \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ a(q_1^m) \equiv 0 \pmod{q_2} \\ q_1 \ne q_2 \\ m \ge 2 \\ q_1^m q_2 \ge (\log x)\sqrt{x} y_1^2}} \sum^{**}_{\substack{n \le x \\ q_1^m || n, q_2 | n}} 1 + \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ a(q_1^m) \equiv 0 \pmod{q_2} \\ q_1 \ne q_2 \\ m \ge 2 \\ q_1^m q_2 < (\log x)\sqrt{x} y_1^2}} \sum^{**}_{\substack{n \le x \\ q_1^m || n, q_2 | n}} 1 =: B_{11} + B_{12}$$

$$B_{11} = \sum\nolimits^*_{\substack{y_1\le q_1,q_2\le x\\ a(q_1^m)\equiv 0 \pmod{q_2}\\ q_1\ne q_2, m\ge 2\\ q_1^m q_2\ge (\log x)\sqrt{x}y_1^2}} \sum\nolimits^{**}_{\substack{n\le x\\ q_1^m||n, q_2|n}} 1 \ll \sum\nolimits^*_{\substack{y_1\le q_1,q_2\le x\\ a(q_1^m)\equiv 0 \pmod{q_2}\\ m\ge 2\\ q_1^m q_2\ge (\log x)\sqrt{x}y_1^2}} \frac{x}{q_1^m q_2}\prod_{p<y_1} \ll$$

$$\ll \frac{x}{L_3(x)} \sideset{}{^*}\sum_{\substack{y_1\le q_1,q_2\le x\\ a(q_1^m)\equiv 0 \pmod{q_2}\\ m\ge 2\\ q_1^m q_2\ge (\log x)\sqrt{x}y_1^2}} \frac{1}{q_1^m q_2} = \frac{x}{L_3(x)}\left(\sideset{}{^*}\sum_{\substack{y_1\le q_1,q_2\le x\\ q_2\le\sqrt{x}\\ a(q_1^m)\equiv 0 \pmod{q_2}\\ m\ge 2\\ q_1^m q_2\ge (\log x)\sqrt{x}y_1^2}} \frac{1}{q_1^m q_2} + \sideset{}{^*}\sum_{\substack{y_1\le q_1,q_2\le x\\ q_2>\sqrt{x}\\ a(q_1^m)\equiv 0 \pmod{q_2}\\ m\ge 2\\ q_1^m q_2\ge (\log x)\sqrt{x}y_1^2}} \frac{1}{q_1^m q_2}\right) =: B_{111} + B_{112}.$$

$$\begin{aligned} B_{111} &= \frac{x}{L_3(x)} \sideset{}{^*}\sum_{\substack{y_1\le q_1,q_2\le x\\ q_2\le\sqrt{x}\\ a(q_1^m)\equiv 0 \pmod{q_2}\\ m\ge 2\\ q_1^m q_2\ge (\log x)\sqrt{x}y_1^2}} \frac{1}{q_1^m q_2} \le \frac{x}{L_3(x)} \sum_{q_2\le\sqrt{x}} \frac{1}{q_2} \sum_{\substack{y_1\le q_1\le\sqrt{x}\\ q_1^m q_2\le x\\ q_1^m q_2\ge (\log x)\sqrt{x}y_1^2}} \frac{1}{q_1^m} = \\ &= \frac{x}{L_3(x)} \sum_{q_2\le\sqrt{x}} \frac{1}{q_2} \sum_{\substack{y_1\le q_1\le\sqrt{x}\\ q_1^m q_2\le x\\ q_1^m q_2\ge (\log x)\sqrt{x}y_1^2\\ q_1\ge \frac{(\log x)\sqrt{x}y_1^2}{q_2}}} \frac{1}{q_1^m} + \frac{x}{L_3(x)} \sum_{q_2\le\sqrt{x}} \frac{1}{q_2} \sum_{\substack{y_1\le q_1\le\sqrt{x}\\ q_1^m q_2\le x\\ q_1^m q_2\ge (\log x)\sqrt{x}y_1^2\\ q_1< \frac{(\log x)\sqrt{x}y_1^2}{q_2}}} \frac{1}{q_1^m} =: B_{1111} + B_{1112}. \end{aligned}$$

$$\begin{aligned} B_{1111} &= \frac{x}{L_3(x)} \sum_{q_2\le\sqrt{x}} \frac{1}{q_2} \sum_{\substack{y_1\le q_1\le\sqrt{x}\\ q_1^m q_2\le x\\ q_1^m q_2\ge (\log x)^2\sqrt{x}y_1^2\\ q_1\ge \frac{(\log x)\sqrt{x}y_1^2}{q_2}}} \frac{1}{q_1^m} \ll \frac{x}{L_3(x)} \sum_{q_2\le\sqrt{x}} \frac{1}{q_2}\cdot\frac{1}{\frac{(\log x)\sqrt{x}y_1^2}{q_2}} = \\ &= \frac{x}{L_3(x)} \sum_{q_2\le\sqrt{x}} \frac{1}{(\log x)\sqrt{x}y_1^2} = \frac{\sqrt{x}}{(\log x)y_1^2 L_3(x)} \sum_{q_2\le\sqrt{x}} 1 \ll \frac{x}{(\log x)^2 y_1^2 L_3(x)}. \end{aligned}$$

$$\begin{aligned} B_{1112} &= \frac{x}{L_3(x)} \sum_{q_2\le\sqrt{x}} \frac{1}{q_2} \sum_{\substack{y_1\le q_1\le\sqrt{x}\\ q_1^m q_2\le x\\ q_1^m q_2\ge (\log x)\sqrt{x}y_1^2\\ q_1 q_2< (\log x)\sqrt{x}y_1^2}} \frac{1}{q_1^m} \le \frac{x}{L_3(x)} \sum_{y_1\le q_1\le\sqrt{x}} \frac{1}{q_1^m} \sum_{\substack{y_1\le q_2\le\sqrt{x}\\ \frac{(\log x)\sqrt{x}y_1^2}{q_1^m}< q_2< \frac{(\log x)\sqrt{x}y_1^2}{q_1}}} \frac{1}{q_2} \ll \\ &\ll \frac{x}{L_3(x)} \sum_{y_1\le q_1\le\sqrt{x}} \frac{1}{q_1^m}\cdot\frac{q_1^m}{(\log x)\sqrt{x}y_1^2}\cdot \#\{q_2 \mid y_1\le q_2\le\sqrt{x}, q_2|a(q_1^m)\} \ll \\ &\ll \frac{x}{L_3(x)} \sum_{y_1\le q_1\le\sqrt{x}} \frac{1}{q_1^m}\cdot\frac{q_1^m}{(\log x)\sqrt{x}y_1^2}\cdot\log x \ll \frac{\sqrt{x}}{L_3(x)}\cdot\frac{1}{y_1^2}\cdot\frac{\sqrt{x}}{\log x} = \frac{x}{(\log x)L_3(x)y_1^2}. \end{aligned}$$

Thus,

$$B_{111} \leq B_{1111} + B_{1112} = o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right).$$

$$B_{112} = \frac{x}{L_3(x)} \sideset{}{^*}\sum_{\substack{y_1 \leq q_1, q_2 \leq x \\ q_2 > \sqrt{x} \\ a(q_1^m) \equiv 0 \pmod{q_2} \\ m \geq 2 \\ q_1^m q_2 \geq (\log x)^2 \sqrt{x} y_1^2}} \frac{1}{q_1^m q_2} \ll \frac{x}{L_3(x)} \sideset{}{^*}\sum_{\substack{y_1 \leq q_1 \leq x \\ m \geq 2}} \frac{1}{q_1^m} \cdot \frac{1}{\sqrt{x}} \ll \frac{\sqrt{x}}{y_1 L_3(x)} = o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right).$$

Thus,

$$B_{11} \ll B_{111} + B_{112} = o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right).$$

For the remaining terms sum B_{12} we use Proposition 2.2.18:

$$\begin{aligned}
B_{12} &\leq \sum_{\substack{y_1 \leq q_1, q_2 \leq x \\ q_1^m q_2 \leq (\log x)\sqrt{x} y_1^2 \\ m \geq 2}} \#\{n \leq x \mid q_1^m q_2 | n, a(n) \neq 0, q | a(q) \Rightarrow q \notin \mathbb{Z}\} = \\
&= \sum_{\substack{y_1 \leq q_1, q_2 \leq x \\ q_1^m q_2 \leq (\log x)\sqrt{x} y_1^2 \\ m \geq 2}} M_{f, q_1^m q_2}(x) \ll \\
&\ll \sum_{\substack{y_1 \leq q_1, q_2 \leq x \\ q_1^m q_2 \leq (\log x)\sqrt{x} y_1^2 \\ m \geq 2}} \frac{x}{\sqrt{\log x} q_1^m q_2} \ll \\
&\ll \frac{x}{(\log x)^{\frac{1}{2}}} \sum_{y_1 \leq q_2 \leq x} \frac{1}{q_2} \sum_{\substack{y_1 \leq q_1^m \\ m \geq 2}} \frac{1}{q_1^m} \\
&\ll \frac{x}{y_1 (\log x)^{\frac{1}{2}}} \sum_{y_1 \leq q_2 \leq x} \frac{1}{q_2} \\
&\ll \frac{x L_2(x)}{y_1 (\log x)^{\frac{1}{2}}} = \frac{x}{(\log x)^{\frac{1}{2}} L_2^{\epsilon}(x)}.
\end{aligned}$$

To estimate the sum B_2 we observe that if $a(q_1) \neq 0$ and $a(q_1) \equiv 0 \pmod{q_2}$, then $q_2 \leq |a(q_1)| \leq 2(q_1)^{\frac{k-1}{2}}$. Hence $q_1 \geq \left(\frac{q_2}{2}\right)^{\frac{2}{k-1}}$ and so $q_1 q_2 \geq \frac{q_2^{\frac{k+1}{k-1}}}{2^{\frac{2}{k-1}}}$. Hence for the inner sum to be nonempty, we need $q_2 \leq 2^{\frac{2}{k+1}} x^{\frac{k-1}{k+1}}$. Thus

$$
\begin{aligned}
B_2 &= \sideset{}{^*}\sum_{\substack{y_1 \le q_1 \le x \\ y_1 \le q_2 \le 2^{\frac{2}{k+1}} x^{\frac{k-1}{k+1}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \sideset{}{^{**}}\sum_{\substack{n \le x \\ q_1 || n, q_2 | n}} 1 = \sideset{}{^*}\sum_{\substack{y_1 \le q_1 \le x \\ y_1 \le q_2 \le 2 q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \sideset{}{^{**}}\sum_{\substack{n \le x \\ q_1 || n, q_2 | n}} 1 \\
&= \sideset{}{^*}\sum_{\substack{y_1 \le q_1 \le x^{\frac{2}{k+2}} \\ y_1 \le q_2 \le 2 q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \sideset{}{^{**}}\sum_{\substack{n \le x \\ q_1 || n, q_2 | n}} 1 + \sideset{}{^*}\sum_{\substack{x^{\frac{2}{k+2}} < q_1 \le x \\ y_1 \le q_2 \le 2 q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \sideset{}{^{**}}\sum_{\substack{n \le x \\ q_1 || n, q_2 | n}} 1 \\
&= D_1 + D_2.
\end{aligned}
$$

Then by Proposition 2.2.18, and because $q_1 q_2 \le 2 q_1^{\frac{k-1}{2}+1} \ll x^{\frac{2}{k+2} \cdot \frac{k+1}{2}} = x^{\frac{k+1}{k+2}}$, we have

$$
\begin{aligned}
D_1 &\le \sideset{}{^*}\sum_{\substack{y_1 \le q_1 \le x^{\frac{2}{k+2}} \\ y_1 \le q_2 \le 2 q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} M_{f, q_1 q_2}(x) \ll \sideset{}{^*}\sum_{\substack{y_1 \le q_1 \le x^{\frac{2}{k+2}} \\ y_1 \le q_2 \le 2 q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{u_f x \xi_{q_1 q_2}(1)}{q_1 q_2 (\log \frac{x}{q_1 q_2})^{\frac{1}{2}}} \\
&\ll \sideset{}{^*}\sum_{\substack{y_1 \le q_1 \le x^{\frac{2}{k+2}} \\ y_1 \le q_2 \le 2 q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{u_f x \xi_{q_1 q_2}(1) \sqrt{k+2}}{q_1 q_2 (\log x)^{\frac{1}{2}}} \\
&\ll \frac{x}{(\log x)^{\frac{1}{2}}} \sideset{}{^*}\sum_{\substack{y_1 \le q_1 \le x^{\frac{2}{k+2}} \\ y_1 \le q_2 \le 2 q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_1 q_2} = \frac{x}{(\log x)^{\frac{1}{2}}} \sideset{}{^*}\sum_{\substack{y_1 \le q_2 \le 2 x^{\frac{k-1}{k+2}} \\ \left(\frac{q_2}{2}\right)^{\frac{2}{k-1}} \le q_1 \le x^{\frac{2}{k+2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_1 q_2} = \\
&= \frac{x}{(\log x)^{\frac{1}{2}}} \left\{ \sideset{}{^*}\sum_{\substack{y_1 \le q_2 \le 2 x^{\frac{k-1}{k+2}} \\ \left(\frac{q_2}{2}\right)^{\frac{2}{k-1}} \le q_1 < q_2^2 \log q_2 \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_1 q_2} + \sideset{}{^*}\sum_{\substack{y_1 \le q_2 \le 2 x^{\frac{k-1}{k+2}} \\ q_2^2 \log q_2 \le q_1 \le x^{\frac{2}{k+2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_1 q_2} \right\} = \frac{x}{(\log x)^{\frac{1}{2}}} \left(Q_1 + Q_2 \right).
\end{aligned}
$$

We changed the order of summation in Q_1 as follows. $q_1 \ge \left(\frac{q_2}{2}\right)^{\frac{2}{k-1}} \ge \left(\frac{y_1}{2}\right)^{\frac{2}{k-1}}$. Also, $q_1 \le q_2^2 \log q_2$ implies $q_2 \ge \sqrt{\frac{q_1}{\log q_1}}$. Indeed, if we assume $q_2 < \sqrt{\frac{q_1}{\log q_1}}$, then

$$
q_1 \le q_2^2 \log q_2 < \frac{q_1}{\log q_1} \cdot \log \sqrt{\frac{q_1}{\log q_1}} = \frac{1}{2} \cdot q_1 \frac{\log q_1 - \log \log q_1}{\log q_1} < q_1.
$$

Contradiction. Thus, $q_2 \ge \sqrt{\frac{q_1}{\log q_1}}$.

The sum Q_1 is then majorized by

$$\ll \sum_{\left(\frac{y_1}{2}\right)^{\frac{2}{k-1}} \le q_1 \le x} \frac{1}{q_1} \sideset{}{^*}\sum_{\substack{\sqrt{\frac{q_1}{\log q_1}} \le q_2 \le 2q_1^{\frac{k-1}{2}} \\ a(q_1)\equiv 0 (\mathrm{mod}\ q_2)}} \frac{1}{q_2}.$$

We note that the inner sum over q_2 is bounded. In fact, with $0 < |a(q_1)| \le 2q_1^{\frac{k-1}{2}}$, there exists at most $2k+2$ primes $q_2 \ge \sqrt{\frac{q_1}{\log q_1}}$ which divide $a(q_1)$. Indeed, assume that there are s such primes $q_{21}, q_{22}, \ldots, q_{2s}$ counting multiplicities. Then,

$$\left(\sqrt{\frac{q_1}{\log q_1}}\right)^s \le q_{21}q_{22}\cdots q_{2s} \le |a(q_1)| \le 2q_1^{\frac{k-1}{2}}.$$

Use the fact that $\log q_1 \le \sqrt{q_1}$ for all $q_1 > 0$

$$q_1^{\frac{s}{2}-\frac{s}{4}} \le \left(\frac{q_1^{\frac{1}{2}}}{(\log q_1)^{\frac{1}{2}}}\right)^s \le 2q_1^{\frac{k-1}{2}} \le q_1^{\frac{k+1}{2}}$$

Thus,

$$\frac{s}{4} \le \frac{k+1}{2},$$
$$s \le 2k+2.$$

Thus, the right hand side is

$$\ll \sum_{\left(\frac{y_1}{2}\right)^{\frac{2}{k-1}} \le q_1 \le x} \frac{\sqrt{\log q_1}}{q_1^{3/2}} \ll \left(\frac{2}{y_1}\right)^{\frac{1}{2(k-1)}} \sum_{\left(\frac{y_1}{2}\right)^{\frac{2}{k-1}} \le q_1 \le x} \frac{\sqrt{\log q_1}}{q_1^{5/4}} \ll \frac{1}{(L_2(x))^{\frac{1}{2(k-1)}}},$$

because $\displaystyle\sum_{\left(\frac{y_1}{2}\right)^{\frac{2}{k-1}} \le q_1 \le x} \frac{\sqrt{\log q_1}}{q_1^{5/4}}$ is convergent.

And so,

$$Q_1 \ll \frac{1}{(L_2(x))^{\frac{1}{2(k-1)}}} = o\left(\frac{1}{(\log\log\log x)^{\frac{1}{2}}}\right).$$

Let us estimate Q_2. For Lemma 4.2.1 to be applicable we need $x \cdot exp\{-c\sqrt{\log x}\} = o\left(\frac{1}{p}\ \mathrm{Li}\ x\right)$, or, equivalently,

$$p = o\left(\frac{e^{c\sqrt{\log x}}}{\log x}\right). \tag{4.12}$$

Also, the condition $\log x \gg (\log pN)^2$ is equivalent to $\log x \gg (\log p)^2$, which is the same as $x \gg e^{(\log p)^2}$ or

$$p \ll e^{\sqrt{\log x}}. \tag{4.13}$$

Since $\log x \ll e^{A\sqrt{\log x}}$ for any $A > 0$, there exists $\varepsilon_1 > 0$ such that $e^{\varepsilon_1\sqrt{\log x}} = o\left(\frac{e^{c\sqrt{\log x}}}{\log x}\right)$ and $e^{\varepsilon_1\sqrt{\log x}} =$

$o\left(e^{\sqrt{\log x}}\right)$. Thus, if we take $0 < \varepsilon_1 < \min\{c, 1\}$ and

$$p \le e^{\varepsilon_1 \sqrt{\log x}}, \tag{4.14}$$

then both conditions (4.12) and (4.13) will be satisfied.

We need to estimate the following sum:

$$\sum_{\substack{p^2 \log p \le q \le x \\ a(q) \equiv 0 \pmod p}}^{*} \frac{1}{q},$$

where $\sum^{*}$ means that the summation is over those q that $a(q) \neq 0$.

Lemma 4.4.1.

$$\sum_{\substack{p^2 \log p \le q \le x \\ a(q) \equiv 0 \pmod p}}^{*} \frac{1}{q} \ll \frac{1}{p} \log\log x + \int_{p^2 \log p}^{e^{\frac{1}{\varepsilon_1^2}(\log p)^2}} \pi^*(t,p) \frac{1}{t^2} dt. \tag{4.15}$$

Proof. This proof is similar to the proof of Lemma 4.1 in [7].

$$\sum_{\substack{p^2 \log p \le q \le x \\ a(q) \equiv 0 \pmod p}}^{*} \frac{1}{q} \ll \int_{p^2 \log p}^{x} \frac{1}{t} d(\pi^*(t,p)) \ll \left.\frac{\pi^*(t,p)}{t}\right|_{p^2 \log p}^{x} + \int_{p^2 \log p}^{x} \pi^*(t,p) \frac{1}{t^2} dt$$

The first term is small:

$$\left.\frac{\pi^*(t,p)}{t}\right|_{p^2 \log p}^{x} \ll \frac{\pi^*(x,p)}{x} = o\left(\frac{1}{p}\log\log x\right).$$

Estimate for the integral goes as follows. To use formula (4.4) we need condition (4.14) which is equivalent to $t \ge e^{\frac{1}{\varepsilon_1^2}(\log p)^2}$. We split the range of integration into two segments:

$$\int_{p^2 \log p}^{x} \pi^*(t,p) \frac{1}{t^2} dt = \int_{p^2 \log p}^{e^{\gamma(\log p)^2}} \pi^*(t,p) \frac{1}{t^2} dt + \int_{e^{\gamma(\log p)^2}}^{x} \pi^*(t,p) \frac{1}{t^2} dt$$

where $\gamma := \frac{1}{\varepsilon_1^2}$.
Estimate for the second integral:

$$\int_{e^{\gamma(\log p)^2}}^{x} \pi^*(t,p) \frac{1}{t^2} dt \ll \frac{1}{p} \int_{e^{\gamma(\log p)^2}}^{x} \frac{\mathrm{Li}\ t}{t^2} dt + \frac{1}{p} \int_{e^{\gamma(\log p)^2}}^{x} \frac{\mathrm{Li}\ t^{\beta}}{t^2} dt + O\left(\int_{e^{\gamma(\log p)^2}}^{x} \frac{1}{t^2} t e^{-c\sqrt{\log t}} dt\right)$$

1)

$$\frac{1}{p} \int_{e^{\gamma(\log p)^2}}^{x} \frac{\mathrm{Li}\ t}{t^2} dt \ll \frac{1}{p} \int_{e^{\gamma(\log p)^2}}^{x} \frac{1}{t \log t} dt \le \frac{1}{p} \log\log x$$

2) Next, consider the term with the Siegel zero. We use a corollary from [7] of the result of Stark that says that if there is an exceptional zero β, then it satisfies

$$\beta \le 1 - \frac{1}{p^{c_2}}$$

for some constant c_2 depending on N. Using this bound, we see that the term containing the exceptional zero is

$$\ll \frac{1}{p}\int_{e^{\gamma(\log p)^2}}^{x} \frac{1}{t^{2-\beta}\log t}dt \leq \frac{1}{p}\int_{e^{\gamma(\log p)^2}}^{x} \frac{1}{t\log t}\cdot \exp\left\{-\frac{\log t}{p^{c_1}}\right\}dt.$$

This is

$$\leq \frac{1}{p}\log\log x \cdot \exp\left\{-\frac{\gamma(\log p)^2}{p^{c_1}}\right\} \ll \frac{1}{p}\log\log x.$$

3) Finally, we deal with the integral coming from the O term:

$$\begin{aligned}\int_{e^{\gamma(\log p)^2}}^{x} \frac{1}{t^2} te^{-c\sqrt{\log t}}dt &= \int_{e^{\gamma(\log p)^2}}^{x} \frac{1}{t}e^{-c\sqrt{\log t}}dt = \\ &= \int_{\gamma(\log p)^2}^{\log x} e^{-c\sqrt{u}}du \ll \int_{\gamma(\log p)^2}^{\log x} e^{-c\sqrt{u}}\left(1-\frac{1}{c\sqrt{u}}\right)du = \\ &= -\frac{1}{c}\sqrt{u}e^{-c\sqrt{u}}\Big|_{\gamma(\log p)^2}^{\log x} \leq \frac{\sqrt{\gamma}}{c}\cdot\frac{\log p}{e^{c\sqrt{\gamma}\log p}} = \frac{\sqrt{\gamma}}{c}\cdot\frac{\log p}{p^{c\sqrt{\gamma}}} \ll \frac{1}{p},\end{aligned}$$

since earlier we choose $\epsilon_1 = \dfrac{1}{\sqrt{\gamma}} < c$, where c is from (4.4) .

Note, that here we used the following:

$$\left(\sqrt{u}e^{-c\sqrt{u}}\right)' = \frac{e^{-c\sqrt{u}}}{2\sqrt{u}} - \frac{c}{2}e^{-c\sqrt{u}}$$

implies that

$$\int\left(e^{-c\sqrt{u}} - \frac{e^{-c\sqrt{u}}}{c\sqrt{u}}\right)du = -\frac{2}{c}\sqrt{u}e^{-c\sqrt{u}} + C;$$

also $e^{-c\sqrt{u}} \ll e^{-c\sqrt{u}} - \dfrac{1}{c}\dfrac{e^{-c\sqrt{u}}}{\sqrt{u}}$.

This implies

$$\int_{e^{\gamma(\log p)^2}}^{x} \tfrac{1}{t}e^{-c\sqrt{\log t}}dt = \int_{\gamma(\log p)^2}^{\log x} e^{-c\sqrt{u}}du \ll$$

$$\ll \int_{\gamma(\log p)^2}^{\log x}\left(e^{-c\sqrt{u}} - \frac{1}{2c}\cdot\frac{e^{-c\sqrt{u}}}{\sqrt{u}}\right)du = \frac{1}{c}\sqrt{u}e^{-c\sqrt{u}}\Big|_{\gamma(\log p)^2}^{\log x} \leq$$

$$\leq \frac{\sqrt{\gamma}}{c}\cdot\frac{\log p}{p^{c\sqrt{\gamma}}} text \ll text \frac{1}{p} text \ll text \frac{1}{p}\log\log x,$$

since $\gamma = \frac{1}{\epsilon_1^2}$ and we chose ϵ_1 so that $\epsilon_1 \leq c$, which makes $c\sqrt{\gamma} > 1$.
Thus,

$$\int_{e^{\gamma(\log p)^2}}^{x} \pi^*(t,p)\frac{1}{t^2}dt \ll \frac{1}{p}\log\log x,$$

and so,

$$\sum^{*}_{\substack{p^2 \log p \le q \le x \\ a(q) \equiv 0 \pmod{p}}} \frac{1}{q} \ll \frac{1}{p} \log\log x + \int_{p^2 \log p}^{e^{(\log p)^2}} \pi^*(t,p) \frac{1}{t^2} dt.$$

□

We split our sum into two parts:

$$Q_2 = \sum^{*}_{\substack{y_1 \le q_2 \le 2x^{\frac{k-1}{k+2}} \\ q_2^2 \log q_2 \le q_1 \le x^{\frac{2}{k+2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_1 q_2} = \sum^{*}_{\substack{y_1 \le q_2 \le e^{\varepsilon_1 \sqrt{\log x}} \\ q_2^2 \log q_2 \le q_1 \le x^{\frac{2}{k+2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_1 q_2} + \sum^{*}_{\substack{e^{\varepsilon_1 \sqrt{\log x}} < q_2 \le 2x^{\frac{k-1}{k+2}} \\ q_2^2 \log q_2 \le q_1 \le x^{\frac{2}{k+2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_1 q_2} =$$

$$= Q_{21} + Q_{22}$$

For the first part we use the Lemma 4.4.1.

$$\begin{aligned} Q_{21} &= \sum^{*}_{\substack{y_1 \le q_2 \le e^{\varepsilon_1 \sqrt{\log x}} \\ q_2^2 \log q_2 \le q_1 \le x^{\frac{2}{k+2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_1 q_2} = \sum_{y_1 \le q_2 \le e^{\varepsilon_1 \sqrt{\log x}}} \frac{1}{q_2} \sum^{*}_{\substack{q_2^2 \log q_2 \le q_1 \le x^{\frac{2}{k+2}} \\ a(q_1) \equiv 0 \pmod{q_2}}} \frac{1}{q_1} \ll \\ &\ll \sum_{y_1 \le q_2 \le e^{\varepsilon_1 \sqrt{\log x}}} \frac{1}{q_2} \cdot \left(\frac{1}{q_2} \log\log x^{\frac{2}{k+2}} + \int_{q_2^2 \log q_2}^{e^{\frac{1}{\varepsilon_1^2} (\log q_2)^2}} \pi^*(t,q_2) \frac{1}{t^2} dt \right) = \\ &= Q_{211} + Q_{212} \end{aligned}$$

$$\begin{aligned} Q_{211} &= \sum_{y_1 \le q_2 \le e^{\varepsilon_1 \sqrt{\log x}}} \frac{1}{q_2} \cdot \frac{1}{q_2} \log\log x^{\frac{2}{k+2}} \\ &\ll \sum_{y_1 \le q_2 \le e^{\varepsilon_1 \sqrt{\log x}}} \frac{1}{q_2^2} \log\log x \ll \frac{1}{y_1} \log\log x = \frac{1}{(\log\log x)^{\varepsilon}}. \end{aligned}$$

Consider the expression

$$Q_{212} = \sum_{y_1 \le q_2 \le e^{\varepsilon_1 \sqrt{\log x}}} \frac{1}{q_2} \int_{q_2^2 \log q_2}^{e^{\frac{1}{\varepsilon_1^2} (\log q_2)^2}} \pi^*(t,q_2) \frac{1}{t^2} dt.$$

Switch the order of integration and summation:

$$\sum_{y_1 \le q_2 \le e^{\varepsilon_1 \sqrt{\log x}}} \frac{1}{q_2} \int_{q_2^2 \log q_2}^{e^{\frac{1}{\varepsilon_1^2}(\log q_2)^2}} \pi^*(t, q_2) \frac{1}{t^2} dt \ll \int_{y_1^2 \log y_1}^{x} \frac{1}{t^2} \sum_{e^{\varepsilon_1 \sqrt{\log t}} \le q_2 \le \sqrt{\frac{t}{\log t}}} \frac{\pi^*(t, q_2)}{q_2} dt \le$$

$$\le \int_{y_1^2 \log y_1}^{x} \frac{1}{t^2} \sum_{q_1 \le t}^{*} \sum_{\substack{q_2 | a(q_1) \\ q_2 \ge e^{\varepsilon_1 \sqrt{\log t}}}} \frac{1}{q_2} dt \ll \int_{y_1^2 \log y_1}^{x} \frac{1}{t^2} \cdot \frac{t}{\log t} \sqrt{\log t} \frac{1}{e^{\varepsilon_1 \sqrt{\log t}}} dt =$$

$$= \int_{y_1^2 \log y_1}^{x} \frac{1}{\sqrt{\log t}} \frac{1}{t} \cdot \frac{1}{e^{\varepsilon_1 \sqrt{\log t}}} dt = \int_{\log(y_1^2 \log y_1)}^{\log x} \frac{1}{\sqrt{u}} \cdot \frac{1}{e^{\varepsilon_1 \sqrt{u}}} du = -\frac{2}{\epsilon_1} \cdot \frac{1}{e^{\epsilon_1 \sqrt{u}}} \Bigg|_{\log(y_1^2 \log y_1)}^{\log x} \le$$

$$\le \frac{2}{\epsilon_1} \cdot \frac{1}{e^{\epsilon_1 \sqrt{\log(y_1^2 \log y_1)}}} \ll \frac{1}{e^{\sqrt{L_3(x)}}} = o\left(\frac{1}{\sqrt{L_3(x)}} \right).$$

The bounds for summation and integration were changed as follows:
$q_2^2 \log q_2 \le t \le e^{\frac{1}{\varepsilon_1^2}(\log q_2)^2}$, where q_2 lies in the range $[y_1; e^{\varepsilon_1 \sqrt{\log x}}]$. Thus, $t \ge y_1^2 \log y_1$, and $t \le e^{\frac{1}{\varepsilon_1^2}\left(\log\left(e^{\varepsilon_1 \sqrt{\log x}}\right)\right)^2} = e^{\log x} = x$.
For t we had the following bounds: $t \ge q_2^2 \log q_2$ and $t \le e^{\frac{1}{\varepsilon_1^2}(\log q_2)^2}$. The first inequality is satisfied if $q_2 \le \sqrt{\frac{t}{\log t}}$, and the second inequality is equivalent to $q_2 \ge e^{\varepsilon_1 \sqrt{\log t}}$.

Here we estimated the number of the prime divisors $q_2 \ge e^{\varepsilon_1 \sqrt{\log t}}$ of $a(q_1)$ as follows:

$$a(q_1) \le 2 q_1^{\frac{k-1}{2}} \le 2 t^{\frac{k-1}{2}}.$$

Let $a(q_1) = p_1 \cdot \dots \cdot p_m$ be the decomposition of $a(q_1)$ into the product of primes (not necessarily distinct). Then, since $p_i \ge e^{\varepsilon_1 \sqrt{\log t}}$, we have:

$$\left(e^{\varepsilon_1 \sqrt{\log t}}\right)^m \le a(q_1) \ll t^{\frac{k-1}{2}} = e^{\frac{k-1}{2} \log t}$$

$$e^{m \varepsilon_1 \sqrt{\log t}} \ll e^{\frac{k-1}{2} \log t}$$

$$m \ll \sqrt{\log t}.$$

Thus,

$$Q_{212} \ll \frac{1}{(\log\log\log x)^A},$$

and so

$$Q_{21} \ll \frac{1}{(\log\log\log x)^A}.$$

For the second sum Q_{22}, namely the sum over the primes in the interval

$e^{\varepsilon_1\sqrt{\log x}} < q_2 \le 2x^{\frac{k-1}{k+2}}$, we switch the order of summation:

$$Q_{22} = \sideset{}{^*}\sum_{\substack{e^{\varepsilon_1\sqrt{\log x}}<q_2\le 2x^{\frac{k-1}{k+2}} \\ q_2^2\log q_2\le q_1\le x^{\frac{2}{k+2}} \\ a(q_1)\equiv 0 \pmod{q_2}}} \frac{1}{q_1q_2} \le \sideset{}{^*}\sum_{\substack{e^{\varepsilon_1\sqrt{\log x}}<q_2\le 2x^{\frac{k-1}{k+2}} \\ q_1\le x^{\frac{2}{k+2}} \\ a(q_1)\equiv 0 \pmod{q_2}}} \frac{1}{q_1q_2} =$$

$$= \sideset{}{^*}\sum_{q_1\le x^{\frac{2}{k+2}}} \frac{1}{q_1} \sum_{\substack{e^{\varepsilon_1\sqrt{\log x}}<q_2\le 2x^{\frac{k-1}{k+2}} \\ a(q_1)\equiv 0 \pmod{q_2}}} \frac{1}{q_2} \ll \sideset{}{^*}\sum_{q_1\le x} \frac{1}{q_1}\cdot\frac{\log(2x^{\frac{k-1}{k+2}})}{e^{\varepsilon_1\sqrt{\log x}}\sqrt{\log x}} \ll \frac{\sqrt{\log x}\cdot\log\log x}{e^{\varepsilon_1\sqrt{\log x}}} \ll \frac{1}{(\log\log x)^{\varepsilon}}.$$

Put these two estimates together to obtain:

$$Q_2 = Q_{21}+Q_{22} = \sideset{}{^*}\sum_{\substack{y_1\le q_2\le 2x^{\frac{k-1}{k+2}} \\ q_2^2\log q_2\le q_1\le x^{\frac{2}{k+2}} \\ a(q_1)\equiv 0 \pmod{q_2}}} \frac{1}{q_1q_2} = o\left(\frac{1}{\sqrt{L_3(x)}}\right).$$

And so

$$Q_1+Q_2 = o\left(\frac{1}{\sqrt{L_3(x)}}\right).$$

Then use this to obtain:

$$D_1 \ll \frac{x}{(\log x)^{\frac{1}{2}}}(Q_1+Q_2) = o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right).$$

In order to estimate D_2, we write

$$\begin{aligned}
D_2 &= \sideset{}{^*}\sum_{\substack{x^{\frac{2}{k+2}}<q_1\le x \\ y_1\le q_2\le 2q_1^{\frac{k-1}{2}} \\ a(q_1)\equiv 0 \pmod{q_2}}} \sideset{}{^{**}}\sum_{\substack{n\le x \\ q_1\|n,q_2|n}} 1 \le \sideset{}{^*}\sum_{\substack{y_1\le q_2\le 2x^{\frac{k-1}{2}} \\ x^{\frac{2}{k+2}}<q_1\le\frac{x}{q_2} \\ a(q_1)\equiv 0 \pmod{q_2}}} \sideset{}{^{**}}\sum_{\substack{n\le x \\ q_1\|n,q_2|n}} 1 = \\
&= \sideset{}{^*}\sum_{\substack{y_1\le q_2\le e^{\sqrt{\log x}} \\ x^{\frac{2}{k+2}}\le q_1\le\frac{x}{2q_2} \\ a(q_1)\equiv 0 (\mathrm{mod}\ q_2)}} \sideset{}{^{**}}\sum_{\substack{n\le x \\ q_1\|n,q_2|n}} 1 + \sideset{}{^*}\sum_{\substack{y_1\le q_2\le e^{\sqrt{\log x}} \\ \frac{x}{2q_2}\le q_1\le\frac{x}{q_2} \\ a(q_1)\equiv 0 (\mathrm{mod}\ q_2)}} \sideset{}{^{**}}\sum_{\substack{n\le x \\ q_1\|n,q_2|n}} 1 + \\
&+ \sideset{}{^*}\sum_{\substack{e^{\sqrt{\log x}}<q_2\le x \\ x^{\frac{2}{k+2}}\le q_1\le\frac{x}{q_2} \\ a(q_1)\equiv 0 (\mathrm{mod}\ q_2)}} \sideset{}{^{**}}\sum_{\substack{n\le x \\ q_1\|n,q_2|n}} 1 \\
&= J_1+J_2+J_3.
\end{aligned}$$

Here

$$
\begin{aligned}
J_3 &= \sum^{*}_{\substack{e^{\sqrt{\log x}}<q_2\le 2x^{\frac{k-1}{2}}\\ x^{\frac{2}{k+2}}\le q_1\le \frac{x}{q_2}\\ a(q_1)\equiv 0 (\mathrm{mod}\ q_2)}} \sum^{**}_{\substack{n\le x\\ q_1||n, q_2|n}} 1 \ll \\
&\ll \sum^{*}_{x^{\frac{2}{k+2}}\le q_1\le x} \sum^{*}_{\substack{q_2|a(q_1)\\ e^{\sqrt{\log x}}\le q_2\le 2x^{\frac{k-1}{2}}}} \frac{x}{q_1 q_2} = x \sum^{*}_{x^{\frac{2}{k+2}}\le q_1\le x} \frac{1}{q_1} \sum^{*}_{\substack{q_2|a(q_1)\\ e^{\sqrt{\log x}}\le q_2\le 2x^{\frac{k-1}{2}}}} \frac{1}{q_2} \ll \\
&\ll \frac{x}{e^{\sqrt{\log x}}} \sum^{*}_{\sqrt{x}\le q_1\le x} \frac{1}{q_1}\cdot \#\left\{q_2 \mid q_2\ge e^{\sqrt{\log x}},\ q_2|a(q_1),\ 0\ne a(q_1)\le 2x^{\frac{k-1}{2}}\right\} \ll \\
&\ll \frac{x\sqrt{\log x}}{e^{\sqrt{\log x}}} \sum_{q_1\le x}\frac{1}{q_1} \ll \frac{x\sqrt{\log x}L_2(x)}{e^{\sqrt{\log x}}} = o\left(\frac{x}{(\log x)^{\frac{1}{2}}(\log\log x)^{\frac{1}{2}}}\right).
\end{aligned}
$$

Here we used the following estimate for the number

$$
\#\left\{q_2 \mid q_2\ge e^{\sqrt{\log x}},\ q_2|a(q_1),\ 0\ne a(q_1)\le 2x^{\frac{k-1}{2}}\right\} \le \frac{k+1}{2}\sqrt{\log x}. \tag{4.16}
$$

Indeed, for each fixed q_1 we are estimating the number of distinct primes q_2 such that $q_2 \ge e^{\sqrt{\log x}}$ and $q_2|a(q_1)$. Thus, if we denote the biggest possible such number by s, we get

$$
e^{s\sqrt{\log x}} \le q_{21}q_{22}\cdots q_{2s} \le |a(q_1)| \le 2x^{\frac{k-1}{2}} \le x^{\frac{k+1}{2}}
$$

$$
s\sqrt{\log x} \le \frac{k+1}{2}\log x
$$

and so

$$
s \le \frac{k+1}{2}\sqrt{\log x},
$$

which leads to the estimate (4.16).

In order to estimate J_1 and J_2, we write

$$
\begin{aligned}
J_1 &= \sum^{*}_{\substack{y_1\le q_2\le e^{\sqrt{\log x}}\\ x^{\frac{2}{k+2}}\le q_1\le \frac{x}{2q_2}\\ a(q_1)\equiv 0(\mathrm{mod}\ q_2)}} \sum^{**}_{\substack{n\le x\\ q_1||n, q_2||n}} 1 + \sum^{*}_{\substack{y_1\le q_2\le e^{\sqrt{\log x}}\\ x^{\frac{2}{k+2}}\le q_1\le \frac{x}{2q_2}\\ a(q_1)\equiv 0(\mathrm{mod}\ q_2)}} \sum^{**}_{\substack{n\le x\\ q_1||n\\ q_2^m||n, m\ge 2}} 1 \\
&= J_{11} + J_{12},
\end{aligned}
$$

and

$$J_2 = \sum^{*}_{\substack{y_1\le q_2\le e^{\sqrt{\log x}}\\ \frac{x}{2q_2}\le q_1\le\frac{x}{q_2}\\ a(q_1)\equiv 0(\mathrm{mod}\ q_2)}} \sum^{**}_{\substack{n\le x\\ q_1||n, q_2||n}} 1 + \sum^{*}_{\substack{y_1\le q_2\le e^{\sqrt{\log x}}\\ \frac{x}{2q_2}\le q_1\le\frac{x}{q_2}\\ a(q_1)\equiv 0(\mathrm{mod}\ q_2)}} \sum^{**}_{\substack{n\le x\\ q_1||n\\ q_2^m|n, m\ge 2}} 1$$

$$= J_{21} + J_{22}.$$

We show that J_{11} and J_{21} are $o\left(\frac{x}{(L_3(x)L_1(x))^{1/2}}\right)$. Similarly, one can show that J_{12} and J_{22} are $o\left(\frac{x}{(L_3(x)L_1(x))^{1/2}}\right)$. We can write

$$J_{11} \ll x \sum^{*}_{y_1\le q_2\le e^{\sqrt{\log x}}} \frac{1}{q_2} \sum^{*}_{\substack{x^{\frac{2}{k+2}}\le q_1\le\frac{x}{2q_2}\\ a(q_1)\equiv 0(\mathrm{mod}\ q_2)}} \frac{1}{q_1\left(\log\frac{x}{q_1q_2}\right)^{1/2}}$$

$$\ll x \sum^{*}_{y_1\le q_2\le e^{\sqrt{\log x}}} \frac{1}{q_2} \int_{x^{\frac{2}{k+2}}}^{x/(2q_2)} \frac{d\pi^*(t,q_2)}{t\left(\log\frac{x}{q_2t}\right)^{1/2}}$$

$$\ll x \sum^{*}_{y_1\le q_2\le e^{\sqrt{\log x}}} \frac{1}{q_2} \left[\left\{\frac{\pi^*(t,q_2)}{t\left(\log\frac{x}{q_2t}\right)^{1/2}}\right\}_{t=x^{\frac{2}{k+2}}}^{t=x/(2q_2)} + \int_{x^{\frac{2}{k+2}}}^{x/(2q_2)} \frac{\pi^*(t,q_2)dt}{t^2\left(\log\frac{x}{q_2t}\right)^{1/2}}\right]$$

Then by using Lemma 4.2.1, we have

$$J_{11} \ll x \sum^{*}_{y_1\le q_2\le e^{\sqrt{\log x}}} \frac{1}{q_2^2} \left[\left\{\frac{1}{\log t\left(\log\frac{x}{q_2t}\right)^{1/2}}\right\}_{t=x^{\frac{2}{k+2}}}^{t=x/(2q_2)} + \int_{x^{\frac{2}{k+2}}}^{x/(2q_2)} \frac{dt}{t\log t\left(\log\frac{x}{q_2t}\right)^{1/2}}\right]$$

$$\ll \frac{x}{\log x} \sum^{*}_{y_1\le q_2\le e^{\sqrt{\log x}}} \frac{1}{q_2^2} \left[1 + \int_{x^{\frac{2}{k+2}}}^{x/(2q_2)} \frac{dt}{t\left(\log\frac{x}{q_2t}\right)^{1/2}}\right]$$

$$\ll \frac{x}{(\log x)^{1/2}} \sum^{*}_{y_1\le q_2\le e^{\sqrt{\log x}}} \frac{1}{q_2^2} \ll \frac{x}{y_1(\log x)^{1/2}}.$$

Since for each pair of primes q_1, q_2 with $y_1\le q_2\le e^{\sqrt{\log x}}$, $x/(2q_2)\le q_1\le x/q_2$, there are at most two

$n \le x$ with $q_1 q_2 | n$ (either $\frac{x}{2} < q_1 q_2 < x$ and $n = q_1 q_2$, or $\frac{x}{2} = q_1 q_2$ and $n_1 = q_1 q_2$, $n_2 = 2q_1 q_2$), we have

$$\begin{aligned} J_{21} &\ll \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \sideset{}{^*}\sum_{\substack{\frac{x}{2q_2} \le q_1 \le \frac{x}{q_2} \\ a(q_1) \equiv 0 (\mathrm{mod}\ q_2)}} 1 \\ &\ll \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \pi^*(x/q_2, q_2) \ll \frac{x}{\log x} \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \frac{1}{q_2^2}, \text{ by Lemma 4.2.1} \\ &\ll \frac{x}{y_1 \log x}. \end{aligned}$$

Hence

$$D_2 \ll J_1 + J_2 + J_3 = o\left(\frac{x}{(\log x)^{\frac{1}{2}} L_3(x)} \right).$$

Now we put all the estimates together to conclude that

$$B_1 + B_2 \ll B_1 + D_1 + D_2 = o\left(\frac{x}{(\log x)^{\frac{1}{2}} (\log\log\log x)^{\frac{1}{2}}} \right).$$

which gives us the estimate (4.11).

Chapter 5

Proof of Theorem 1.2.2

5.1 Good primes

Let us now estimate

$$\#\{n \le x \mid (n, a(n)) \text{ is prime}\}.$$

We call the primes p for which $p \nmid a(p)$ *good primes*.

Lemma 5.1.1. *If p is a good prime, then $p \nmid a(p^m)$ for any m.*

Proof. Let p be a good prime. For $m \ge 2$

$$a(p^m) = a(p)a(p^{m-1}) - p^{k-1}a(p^{m-2}). \tag{5.1}$$

Thus,

$$p|a(p^m) \Rightarrow p|a(p)a(p^{m-1}) \Rightarrow p|a(p^{m-1}) \text{ since } p \text{ is a good prime.}$$

We have

$$p|a(p^m) \Rightarrow p|a(p^{m-1}) \Rightarrow p|a(p^{m-2}) \Rightarrow \cdots \Rightarrow p|a(p).$$

Contradiction. □

In fact, the converse statement is also true: $p|a(p) \Rightarrow p|a(p^m)$ for all $m \in \mathbb{N}$, since $p|a(p^m)$ if and only if $p|a(p)$ or $p|a(p^{m-1})$.
But we see that for $m \ge 3$ we have $p|a(p) \Rightarrow p^m|a(p^m)$ from equation (5.1). Thus, we have proved the following lemma.

Lemma 5.1.2. *For weight $k \ge 3$ we have $p|a(p)$ if and only if $p^m|a(p^m)$ for any m.*

$$\sum_{\substack{n \le x \\ (n,a(n)) \text{ is a prime}}} 1 = \sum_{\substack{n \le x \\ n \text{ is squarefree} \\ (n,a(n)) \text{ is a prime}}} 1 + \sum_{\substack{n \le x \\ n \text{ is non-squarefree} \\ (n,a(n)) \text{ is a prime}}} 1 =: \Sigma' + \Sigma''.$$

We split the first sum in two parts, depending on whether the prime that is the $(n, a(n))$ is a good or a bad prime.

$$\Sigma' = \sum_{\substack{n\le x \\ n \text{ is squarefree} \\ (n,a(n)) \text{ is a prime}}} 1 = \sum_{\substack{l\le x \\ l- \text{ prime}}} \sum_{\substack{n\le x \\ n \text{ is squarefree} \\ (n,a(n))=l}} 1$$

$$= \sum_{\substack{l\le x \\ l- \text{ prime} \\ l|a(l)}} \sum_{\substack{n\le x \\ n \text{ is squarefree} \\ (n,a(n))=l}} 1 + \sum_{\substack{l\le x \\ l- \text{ prime} \\ l\nmid a(l)}} \sum_{\substack{n\le x \\ n \text{ is squarefree} \\ (n,a(n))=l}} 1.$$

Write $n = ml$. Since n is squarefree, we have $l \nmid m$. Thus, $a(ml) = a(m)a(l)$.
1. Consider the $l|a(l)$ case: $(lm, a(l)a(m)) = l$ implies that $(m, \frac{a(l)}{l} \cdot a(m)) = 1$, in particular $(m, a(m)) = 1, (m, a(l)) = 1$. Thus

$$\sum_{\substack{l\le x \\ l- \text{ prime} \\ l|a(l)}} \sum_{\substack{n\le x \\ (n,a(n))=l}} 1 = \sum_{\substack{l\le x \\ l- \text{ prime} \\ l|a(l)}} \sum_{\substack{m\le \frac{x}{l} \\ l\nmid m \\ \left(m, \frac{a(l)}{l}\cdot a(m)\right)=1}} 1$$

$$\le \sum_{\substack{l\le x \\ l- \text{ prime} \\ l|a(l)}} \sum_{\substack{m\le \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1}} 1 =: \Sigma_1.$$

2. Consider the $l \nmid a(l)$ case: $(lm, a(l)a(m)) = l$ implies $l|a(m)$.

$$\sum_{\substack{l\le x \\ l- \text{ prime} \\ l\nmid a(l)}} \sum_{\substack{n\le x \\ (n,a(n))=l}} 1 = \sum_{\substack{l\le x \\ l- \text{ prime} \\ l\nmid a(l)}} \sum_{\substack{m\le \frac{x}{l} \\ l\nmid m \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l|a(m)}} 1 \le \sum_{\substack{l\le x \\ l- \text{ prime} \\ l\nmid a(l)}} \sum_{\substack{m\le \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l|a(m)}} 1 =: \Sigma_2$$

5.2 Contribution of non-squarefree numbers

Let us estimate the number of non-squarefree numbers Σ''.

$$\Sigma'' = \sum_{\substack{n\le x \\ n \text{ is non-squarefree} \\ (n,a(n)) \text{ is a prime}}} 1 = \sum_{\substack{l\le x \\ l- \text{ prime}}} \sum_{\substack{n\le x \\ (n,a(n))=l \\ l^2|n}} 1 + \sum_{\substack{l\le x \\ l- \text{ prime}}} \sum_{\substack{n\le x \\ n \text{ is non-squarefree} \\ (n,a(n))=l \\ l||n}} 1 \le$$

$$\le \sum_{\substack{l\le x \\ l- \text{ prime}}} \sum_{\substack{m\le \frac{x}{l} \\ l|m \\ (ml,a(ml))=l}} 1 + \sum_{\substack{l\le x \\ l- \text{ prime}}} \sum_{\substack{m\le \frac{x}{l} \\ p^2|m \text{ for some prime } p \\ (m,a(m))=1,(m,a(l))=1 \\ l|a(m)}} 1 =: \Sigma_3 + \Sigma_4.$$

Σ_3 counts the non-squarefree numbers n for which l comes from the non-squarefree part: $l^2|n$. Since $l^2|n$, $n \le x$, we have $l \le \sqrt{x}$.
Note that if $l|a(l)$, then $l^2|n$ implies $l^2|a(n)$, and so $l^2|(n, a(n))$. Thus, in the estimate of Σ_3 we can sum over $l \nmid a(l)$.

$$\Sigma_3 = \sum_{\substack{l\le x\\ l-\text{ prime}}} \sum_{\substack{m\le \frac{x}{l}\\ l|m\\ (ml,a(ml))=l}} 1 = \sum_{\substack{l\le \sqrt{x}\\ l-\text{ prime}\\ l\nmid a(l)}} \sum_{\substack{m\le \frac{x}{l}\\ l|m\\ (ml,a(ml))=l}} 1 = \sum_{\substack{l\le \sqrt{x}\\ l-\text{ prime}\\ l\nmid a(l)}} \sum_{\substack{m_0\le \frac{x}{l^2}\\ l||a(l^2m_0)\\ (l^2m_0,a(l^2m_0))=l}} 1$$

Let $m = l^k m_1$, where $l \nmid m_1$ (i.e. $n = l^{k+1}m_1$). Since $l|m$, we have $k \geq 1$. Then

$$a(n) = a(l^{k+1}m_1) = a(l^{k+1})a(m_1),$$

$$l||a(n) \;\Leftrightarrow\; l||a(m_1) \text{ , because } l \nmid a(l^{k+1}).$$

Since $l \nmid m_1$ and $l \nmid a(l^k)$, we have

$$\big(l^k m_1, a(l^k)a(m_1)\big) = l \;\Rightarrow\; \begin{cases} (m_1, a(m_1)) = 1 \\ l||a(m_1) \end{cases} . \tag{5.2}$$

For each m_1 that satisfies condition (5.2), there are $\ll \log \dfrac{x}{m_1}$ such m 's, that are counted in the sum Σ_3. Indeed, for each m_1 we have

$$m_1 < m_1 \cdot l < m_1 \cdot l^2 < \ldots < m_1 \cdot l^i \leq \frac{x}{l}$$

$$m_1 \cdot l^{i+1} \leq x$$

$$l^{i+1} \leq \frac{x}{m_1}$$

$$i \ll \log \frac{x}{m_1}.$$

Thus, our sum Σ_3 can be estimated as follows

$$\Sigma_3 \ll \sum_{\substack{l\le \sqrt{x}\\ l-\text{ prime}\\ l\nmid a(l)}} \sum_{\substack{m_1\le \frac{x}{l^2}\\ l||a(m_1)\\ (m_1,a(m_1))=1}} \log \frac{x}{m_1} \leq \sum_{\substack{l\le \sqrt{x}\\ l-\text{ prime}}} \sum_{\substack{m_1\le \frac{x}{l^2}\\ (m_1,a(m_1))=1}} \log \frac{x}{m_1}.$$

To estimate this sum, we will break it up into two pieces:

$$\sum_{\substack{l\le \sqrt{x}\\ l-\text{ prime}}} \sum_{\substack{m_1\le \frac{x}{l^2}\\ (m_1,a(m_1))=1}} \log \frac{x}{m_1} = \sum_{\substack{l\le x^{\frac{1}{2}-\alpha}\\ l-\text{ prime}}} \sum_{\substack{m_1\le \frac{x}{l^2}\\ (m_1,a(m_1))=1}} \log \frac{x}{m_1} + \sum_{\substack{x^{\frac{1}{2}-\alpha}<l\le \sqrt{x}\\ l-\text{ prime}}} \sum_{\substack{m_1\le \frac{x}{l^2}\\ (m_1,a(m_1))=1}} \log \frac{x}{m_1} =: \Sigma_{31} + \Sigma_{32}.$$

Let us estimate the inner sum of Σ_{31}:

$$\sum_{\substack{m_1 \le \frac{x}{l^2} \\ (m_1, a(m_1))=1}} \log \frac{x}{m_1} \ll \int_{c_1}^{x/l^2} \log \frac{x}{t} d\mu(t) =$$

where $\mu(t) = \#\{m_1 \le t \mid (m_1, a(m_1)) = 1\}$ and $c_1 = e^{e^e}$, since we want to make sure that the $L_3(t)$ that appears in the estimate for $\mu(t)$ is well defined,

$$= \log \frac{x}{t} \cdot \mu(t)\Big|_{c_1}^{x/l^2} - \int_{c_1}^{x/l^2} \mu(t) \left(\log \frac{x}{t}\right)' dt =$$

$$= \log \frac{x}{t} \cdot \mu(t)\Big|_{c_1}^{x/l^2} - \int_{c_1}^{x/l^2} \mu(t) \frac{t}{x} \cdot \left(-\frac{x}{t^2}\right) dt =$$

$$= \log \frac{x}{t} \cdot (1 + o(1)) \frac{U_f t}{\sqrt{L_1(t) L_3(t)}}\Bigg|_{c_1}^{x/l^2} + \int_{c_1}^{x/l^2} \frac{U_f(1 + o(1))}{\sqrt{L_1(t) L_3(t)}} dt \ll$$

$$\ll \log l^2 \frac{x/l^2}{\sqrt{L_1(x/l^2) L_3(x/l^2)}} - \frac{c_1}{e^e} \log \frac{x}{c_1} + \int_{c_1}^{x/l^2} \frac{U_f}{\sqrt{L_1(t) L_3(t)}} dt \ll$$

$$\ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \cdot \frac{\log l}{l^2} + \frac{x/l^2}{\sqrt{L_1(x/l^2) L_3(x/l^2)}} \ll$$

since for $l \le x^{\frac{1}{2} - \alpha}$ the first term dominates the second one

$$\ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \cdot \frac{\log l}{l^2}.$$

Here we used the following:

$$\int_{c_1}^{x/l^2} \frac{1}{\sqrt{L_1(t) L_3(t)}} dt = \int_{c_1}^{\sqrt{x/l^2}} \frac{1}{\sqrt{L_1(t) L_3(t)}} dt + \int_{\sqrt{x/l^2}}^{x/l^2} \frac{1}{\sqrt{L_1(t) L_3(t)}} dt \ll$$

$$\ll \sqrt{x/l^2} + \frac{1}{\sqrt{L_1(\sqrt{x/l^2}) L_3(\sqrt{x/l^2})}} (x/l^2 - \sqrt{x/l^2}) \ll \frac{x/l^2}{\sqrt{L_1(x/l^2) L_3(x/l^2)}}$$

Then,

$$\Sigma_{31} \ll \sum_{l \le x^{\frac{1}{2} - \alpha}} \frac{x}{\sqrt{L_1(x) L_3(x)}} \cdot \frac{\log l}{l^2} = \frac{x}{\sqrt{L_1(x) L_3(x)}} \cdot \sum_{l \le x^{\frac{1}{2} - \alpha}} \frac{\log l}{l^2} \ll \frac{x}{\sqrt{L_1(x) L_3(x)}}$$

Let us estimate Σ_{32}:

The inner sum can be estimated:

$$\sum_{\substack{m_1\le \frac{x}{l^2}\\(m_1,a(m_1))=1}} \log\frac{x}{m_1} \sim \int_1^{x/l^2}\log\frac{x}{t}dt = t\cdot\log\frac{x}{t}\Big|_1^{x/l^2} - \int_1^{x/l^2} t\cdot\frac{1}{\frac{x}{t}}\cdot\left(-\frac{x}{t^2}\right)dt =$$

$$= \frac{x}{l^2}\log l^2 - \log x + \int_1^{x/l^2} dt \ll$$

$$\ll \frac{x}{l^2}\log l + \frac{x}{l^2} \ll \frac{x}{l^2}\log l.$$

Thus,

$$\Sigma_{32} = \sum_{\substack{x^{\frac{1}{2}-\alpha}\le l\le\sqrt{x}\\ l-\text{ prime}}}\ \sum_{\substack{m_1\le\frac{x}{l^2}\\(m_1,a(m_1))=1}}\log\frac{x}{m_1} \ll \sum_{\substack{x^{\frac{1}{2}-\alpha}\le l\le\sqrt{x}\\ l-\text{ prime}}} x\cdot\frac{\log l}{l^2} \le$$

$$\le \frac{x}{x^{1/2-\alpha}}\cdot\sum_{\substack{x^{\frac{1}{2}-\alpha}\le l\le\sqrt{x}\\ l-\text{ prime}}}\frac{\log l}{l} \ll x^{\frac{1}{2}+\alpha}\cdot(\sqrt{x})^{\beta} = x^{\frac{1}{2}+\alpha+\frac{\beta}{2}} = o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right).$$

Here we used the following estimate:

$$\frac{\log t}{t} \ll \frac{1}{t^{1-\beta}} \text{ for any } 0<\beta<1$$

$$\sum_{l=x^{1/2-\alpha}}^{\sqrt{x}}\frac{\log l}{l} \ll \sum_{l=x^{1/2-\alpha}}^{\sqrt{x}}\frac{1}{l^{1-\beta}} \ll \int_1^{\sqrt{x}}\frac{1}{t^{1-\beta}}dt \sim \frac{1}{\beta}x^{\frac{\beta}{2}}$$

choose α and β so that $\frac{1}{2}+\alpha+\frac{\beta}{2}<1$.

Thus,

$$\Sigma_3 = \Sigma_{31}+\Sigma_{32} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} = o\left(\frac{xL_4^2(x)}{\sqrt{L_1(x)L_3(x)}}\right).$$

Now let us estimate Σ_4, which is a bound for those non-squarefree numbers n for which $l=(n,a(n))$ comes from the squarefree part of n:

$$\Sigma_4 = \sum_{\substack{l\le x\\ l-\text{ prime}}}\ \sum_{\substack{m\le\frac{x}{l}\\ p^2|m \text{ for some prime } p\\(m,a(m))=1\\(m,a(l))=1\\ l|a(m)}} 1 =$$

$$= \sum_{l\le e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}}\ \sum_{\substack{m\le\frac{x}{l}\\ p^2|m \text{ for some prime } p\\(m,a(m))=1\\(m,a(l))=1\\ l|a(m)}} 1 + \sum_{e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}<l\le x}\ \sum_{\substack{m\le\frac{x}{l}\\ p^2|m \text{ for some prime } p\\(m,a(m))=1\\(m,a(l))=1\\ l|a(m)}} 1 =: \Sigma_{41}+\Sigma_{42},$$

where $0 < \epsilon < 1$, and ϵ_1 is chosen so that $\epsilon_1 < \min\{c, 1\}$ with c from Lemma 4.2.1.

$$
\begin{aligned}
\Sigma_{42} &= \sum_{e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}} \le l \le x} \sum_{\substack{p^s \le \frac{x}{l} \\ s \ge 2}} \sum_{\substack{m_0 \le \frac{x}{lp^s} \\ (m_0, a(m_0))=1 \\ l | a(p^s)a(m_0)}} 1 \le \\
&\le \sum_{x^{e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \le l \le x} \sum_{\substack{p^s \le \frac{x}{l} \\ s \ge 2}} \left(\sum_{\substack{m_0 \le \frac{x}{lp^s} \\ (m_0, a(m_0))=1 \\ l | a(p^s)}} 1 + \sum_{\substack{m_0 \le \frac{x}{lp^s} \\ (m_0, a(m_0))=1 \\ l | a(m_0)}} 1 \right) =: \Sigma_{421} + \Sigma_{422}.
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_{421} &= \sum_{e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}} \le l \le x} \sum_{\substack{p^s \le \frac{x}{l} \\ s \ge 2 \\ l | a(p^s)}} \sum_{\substack{m_0 \le \frac{x}{lp^s} \\ (m_0, a(m_0))=1}} 1 \le \sum_{e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}} \le l \le x} \sum_{\substack{p^s \le \frac{x}{l} \\ s \ge 2 \\ l | a(p^s)}} \frac{x}{lp^s} \le \\
&\le x \sum_{\substack{p^s \le \frac{x}{l} \\ s \ge 2}} \frac{1}{p^s} \sum_{\substack{e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}} \le l \le x \\ l | a(p^s)}} \frac{1}{l} \ll
\end{aligned}
$$

since the number of l's in the second sum is estimated as follows:
$\left(e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}\right)^r \le l_1 l_2 \dots l_r \le |a(p^s)| \le (s+1)\,(p^s)^{\frac{k-1}{2}} \ll (p^s)^{\frac{k-1}{2}+\alpha} \le x^{\frac{k-1}{2}+\alpha}$,
so $r \ll \dfrac{\log x}{\sqrt{\log x}} = \sqrt{\log x}$

$$
\ll x \sum_{\substack{p^s \le \frac{x}{l} \\ s \ge 2}} \frac{1}{p^s} \cdot \frac{1}{e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \cdot \sqrt{\log x} \ll \frac{x}{\log x} = o\left(\frac{x}{L_1(x) L_3(x)}\right).
$$

$$\Sigma_{422} = \sum_{e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}\le l\le x}\ \sum_{\substack{p^s\le\frac{x}{l}\\ s\ge 2}}\ \sum_{\substack{m_0\le\frac{x}{lp^s}\\ (m_0,a(m_0))=1\\ l|a(p^s)}} 1 \le \sum_{e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}\le l\le x}\ \sum_{\substack{p^s\le\frac{x}{l}\\ s\ge 2}}\ \sum_{\substack{q^i\le\frac{x}{lp^s}\\ i\ge 1\\ l|a(q^i)}}\ \sum_{\substack{m_1\le\frac{x}{lp^sq^i}\\ (m_1,a(m_1))=1}} 1 \le$$

$$\le \sum_{e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}\le l\le x}\ \sum_{\substack{p^s\le\frac{x}{l}\\ s\ge 2}}\ \sum_{\substack{q^i\le\frac{x}{lp^s}\\ i\ge 1\\ l|a(q^i)}} \frac{x}{lp^sq^i} \le x\sum_{\substack{p^s\le x\\ s\ge 2}}\frac{1}{p^s}\sum_{\substack{q^i\le x\\ i\ge 1}}\frac{1}{q^i}\sum_{\substack{e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}\le l\le x\\ l|a(q^i)}}\frac{1}{l} \ll$$

The number of l's in the inner sum is estimated as in the estimate of the previous sum Σ_{421} : it is $\ll \dfrac{\log x}{\sqrt{\log x}} = \sqrt{\log x}$

$$\ll x\sum_{\substack{p^s\le\frac{x}{l}\\ s\ge 2}}\frac{1}{p^s}\sum_{\substack{q^i\le x\\ i\ge 1}}\frac{1}{q^i}\cdot\frac{1}{e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}}\cdot\sqrt{\log x} \ll$$

$$\ll \frac{x\sqrt{\log x}L_2(x)}{e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \ll \frac{x}{\log x} = o\left(\frac{x}{L_1(x)L_3(x)}\right).$$

$$\Sigma_{41} = \sum_{l\le e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}}\ \sum_{\substack{p^s\le\frac{x}{l}\\ s\ge 2}}\ \sum_{\substack{m_0\le\frac{x}{lp^s}\\ (m_0,a(m_0))=1\\ l|a(p^s)a(m_0)}} 1 \le$$

$$\le \sum_{l\le e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}}\ \sum_{\substack{p^s\le\frac{x}{l}\\ s\ge 2}}\left(\sum_{\substack{m_0\le\frac{x}{lp^s}\\ (m_0,a(m_0))=1\\ l|a(p^s)}} 1 + \sum_{\substack{m_0\le\frac{x}{lp^s}\\ (m_0,a(m_0))=1\\ l|a(m_0)}} 1\right) =: \Sigma_{411}+\Sigma_{412}.$$

$$\Sigma_{411} = \sum_{l\le e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}}\left(\sum_{\substack{p^s\le\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\ge 2}}\ \sum_{\substack{m_0\le\frac{x}{lp^s}\\ (m_0,a(m_0))=1\\ l|a(p^s)}} 1 + \sum_{\substack{\left(\frac{x}{l}\right)^{1-\epsilon}<p^s\le\frac{x}{l}\\ s\ge 2}}\ \sum_{\substack{m_0\le\frac{x}{lp^s}\\ (m_0,a(m_0))=1\\ l|a(p^s)}} 1\right) =: \Sigma_{4111}+\Sigma_{4112}$$

$$\Sigma_{4111} = \sum_{l\le e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}}\ \sum_{\substack{p^s\le\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\ge 2}}\ \sum_{\substack{m_0\le\frac{x}{lp^s}\\ (m_0,a(m_0))=1\\ l|a(p^s)}} 1 =$$

$$= \sum_{l\le y_1}\ \sum_{\substack{p^s\le\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\ge 2}}\ \sum_{\substack{m_0\le\frac{x}{lp^s}\\ (m_0,a(m_0))=1\\ l|a(p^s)}} 1 + \sum_{y_1<l\le e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}}\ \sum_{\substack{p^s\le\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\ge 2}}\ \sum_{\substack{m_0\le\frac{x}{lp^s}\\ (m_0,a(m_0))=1\\ l|a(p^s)}} 1 =: \Sigma_{41111}+\Sigma_{41112}$$

$$\Sigma_{41112} = \sum_{y_1<l\le e^{\epsilon_1 \epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s\le\left(\frac{x}{l}\right)^{1-\epsilon} \\ s\ge 2 \\ l|a(p^s)}} \sum_{\substack{m_0\le \frac{x}{lp^s} \\ (m_0,a(m_0))=1}} 1 \ll$$

$$\ll \sum_{y_1<l\le e^{\epsilon_1 \epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s\le\left(\frac{x}{l}\right)^{1-\epsilon} \\ s\ge 2 \\ l|a(p^s)}} \frac{x}{lp^s\sqrt{L_1\left(\frac{x}{lp^s}\right)L_3\left(\frac{x}{lp^s}\right)}} \ll$$

since $\frac{x}{lp^s} \ge \left(\frac{x}{lp^s}\right)^{\epsilon} \ge x^{\epsilon^2}$ implies $\frac{x}{lp^s\sqrt{L_1\left(\frac{x}{lp^s}\right)L_3\left(\frac{x}{lp^s}\right)}} \ll \frac{x}{lp^s\sqrt{L_1(x)L_3(x)}}$

$$\ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{l\le e^{\epsilon_1 \epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l} \sum_{\substack{p^s\le x \\ s\ge 2 \\ l|a(p^s)}} \frac{1}{p^s} \le$$

switch the order of summation here;

$$\le \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{\substack{p^s\le x \\ s\ge 2}} \frac{1}{p^s} \sum_{l|a(p^s)} \frac{1}{l} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{\substack{p^s\le x \\ s\ge 2}} \frac{1}{p^s}\log\log(p^s) \ll$$

$$\ll \frac{x}{\sqrt{L_1(x)L_3(x)}},$$

since

$$\sum_{\substack{p^s\le x \\ s\ge 2}} \frac{1}{p^s}\log\log(p^s) \ll \sum_{p\le\sqrt{x}} \sum_{s\ge 2} \frac{1}{p^s}\log\log p \ll \sum_{p}\left(\frac{1}{p^2}\log\log p\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right)\right) \ll 1.$$

$$\begin{aligned}
\Sigma_{41111} &= \sum_{l \le y_1} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2 \\ l | a(p^s)}} \sum_{\substack{m_0 \le \frac{x}{lp^s} \\ (m_0, a(m_0))=1}} 1 \le \\
&\le \sum_{l \le y_1} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \sum_{\substack{m_0 \le \frac{x}{lp^s} \\ (m_0, a(m_0))=1}} 1 \ll \\
&\ll \sum_{l \le y_1} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \frac{x}{lp^s \sqrt{L_1\left(\frac{x}{lp^s}\right) L_3\left(\frac{x}{lp^s}\right)}} \ll \\
&\ll \sum_{l \le y_1} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \frac{x}{lp^s \sqrt{L_1\left(\frac{x}{lp^s}\right) L_3\left(\frac{x}{lp^s}\right)}} \ll \\
&\ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{l \le y_1} \frac{1}{l} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \frac{1}{p^s} \ll
\end{aligned}$$

since the sum $\displaystyle\sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \frac{1}{p^s}$ is convergent

$$\ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \cdot \log\log(y_1) = \frac{x L_4^2(x)}{\sqrt{L_1(x) L_3(x)}}.$$

This completes the estimate for Σ_{4111}.

$$\Sigma_{4112} = \sum_{l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{\left(\frac{x}{l}\right)^{1-\epsilon} < p^s \le \frac{x}{l} \\ s \ge 2 \\ l | a(p^s)}} \sum_{\substack{m_0 \le \frac{x}{lp^s} \\ (m_0, a(m_0))=1}} 1 \le \sum_{l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{\left(\frac{x}{l}\right)^{1-\epsilon} < p^s \le \frac{x}{l} \\ s \ge 2}} \sum_{\substack{m_0 \le \frac{x}{lp^s} \\ (m_0, a(m_0))=1}} 1 \ll$$

$$\ll \sum_{l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{\left(\frac{x}{l}\right)^{1-\epsilon} < p^s \le \frac{x}{l} \\ s \ge 2}} \frac{x}{lp^s} \ll x \sum_{l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l} \sum_{\substack{x^{\epsilon(1-\epsilon)} < p^s \le x \\ s \ge 2}} \frac{1}{p^s},$$

since $\displaystyle\frac{1}{l} \ge \frac{1}{e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \ge \frac{1}{x^{1-\epsilon}}$ implies $\displaystyle\left(\frac{x}{l}\right)^{1-\epsilon} \ge \left(\frac{x}{x^{1-\epsilon}}\right)^{1-\epsilon} = x^{\epsilon(1-\epsilon)}$.

Let us estimate the inner sum $\displaystyle\sum_{\substack{x^{\epsilon(1-\epsilon)} < p^s \le x \\ s \ge 2}} \frac{1}{p^s}$. For this we break it up into two sums with $0 < \alpha < \epsilon(1-\epsilon)$.

$$\sum_{\substack{x^{\epsilon(1-\epsilon)} < p^s \le x \\ s \ge 2}} \frac{1}{p^s} = \sum_{\substack{p \ge x^\alpha \\ x^{\epsilon(1-\epsilon)} < p^s \le x \\ s \ge 2}} \frac{1}{p^s} + \sum_{\substack{p < x^\alpha \\ x^{\epsilon(1-\epsilon)} < p^s \le x \\ s \ge 2}} \frac{1}{p^s} =: \Sigma_{41121} + \Sigma_{41122}.$$

The first sum is bounded by

$$\sum_{\substack{p\geq x^{\alpha}\\ x^{\epsilon(1-\epsilon)}<p^s\leq x\\ s\geq 2}} \frac{1}{p^s} \leq \sum_{p\geq x^{\alpha}} \frac{1}{p^2}\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right) \ll \sum_{p\geq x^{\alpha}} \frac{1}{p^2} \ll \frac{1}{x^{\alpha}\log(x^{\alpha})} \ll \frac{1}{x^{\alpha}\log x}.$$

The second sum is bounded by

$$\sum_{\substack{p<x^{\alpha}\\ x^{\epsilon(1-\epsilon)}<p^s\leq x\\ s\geq 2}} \frac{1}{p^s} \leq \frac{1}{x^{\epsilon(1-\epsilon)}} \sum_{\substack{s\geq 2\\ p^s\leq x}} 1 \sum_{\substack{p-\text{ prime}\\ p\leq x^{\alpha}}} 1 \ll \frac{1}{x^{\epsilon(1-\epsilon)}}\cdot\log x\cdot\frac{x^{\alpha}}{\log x^{\alpha}} \ll \frac{1}{x^{\epsilon(1-\epsilon)-\alpha}},$$

where $\sum_{\substack{s\geq 2\\ p^s\leq x}} 1 \leq \log x$, since $2^s\leq p^s\leq x$.

Thus, since $\epsilon(1-\epsilon)-\alpha>0$, both of these sums, Σ_{41121} and Σ_{41122}, are bounded by $\frac{1}{x^{\epsilon}}$ for some $\epsilon>0$. So we have

$$\Sigma_{4112} \ll x \sum_{l\leq e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l}\left(\Sigma_{41121}+\Sigma_{41122}\right) \ll x\cdot\frac{1}{x^{\epsilon}}\cdot\log\log x \ll x^{1-\epsilon}$$

This completes the estimate for Σ_{411}.
The sum Σ_{412} is the same as the sum Σ_{411}, only the condition $l|a(p^s)$ is replaced with $l|a(m_0)$. When estimating the sum Σ_{411} we used this condition $l|a(p^s)$ in only one place, namely, in the estimate of the sum Σ_{41112}. Thus, when estimating sum Σ_{412} everything will stay exactly the same, except that in the estimate of the corresponding piece Σ_{41212} we have to use the condition $l|a(m_0)$ instead of $l|a(p^s)$.

$$\begin{aligned}
\Sigma_{41212} &= \sum_{y_1<l\leq e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s\leq\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\geq 2}} \sum_{\substack{m_0\leq\frac{x}{lp^s}\\ (m_0,a(m_0))=1\\ l|a(m_0)}} 1 \leq \\
&\leq \sum_{y_1<l\leq e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s\leq\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\geq 2}} \sum_{\substack{q^i\leq\frac{x}{lp^s}\\ l|a(q^i)\\ i\geq 1}} \sum_{\substack{m_1\leq\frac{x}{lp^sq^i}\\ (m_1,a(m_1))=1}} 1 = \\
&= \sum_{y_1<l\leq e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s\leq\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\geq 2}} \sum_{\substack{q^i\leq\left(\frac{x}{lp^s}\right)^{1-\epsilon}\\ l|a(q^i)\\ i\geq 1}} \sum_{\substack{m_1\leq\frac{x}{lp^sq^i}\\ (m_1,a(m_1))=1}} 1 \\
&+ \sum_{y_1<l\leq e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s\leq\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\geq 2}} \sum_{\substack{\left(\frac{x}{lp^s}\right)^{1-\epsilon}<q^i\leq\frac{x}{lp^s}\\ l|a(q^i)\\ i\geq 1}} \sum_{\substack{m_1\leq\frac{x}{lp^sq^i}\\ (m_1,a(m_1))=1}} 1 =: \Sigma_{412121}+\Sigma_{412122}
\end{aligned}$$

$$\Sigma_{412121} = \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \sum_{\substack{q^i \le \left(\frac{x}{lp^s}\right)^{1-\epsilon} \\ l | a(q^i) \\ i \ge 1}} \sum_{\substack{m_1 \le \frac{x}{lp^s q^i} \\ (m_1, a(m_1))=1}} 1 \ll$$

$$\ll \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \sum_{\substack{q^i \le \left(\frac{x}{lp^s}\right)^{1-\epsilon} \\ l | a(q^i) \\ i \ge 1}} \frac{x}{lp^s q^i \sqrt{L_1\left(\frac{x}{lp^s q^i}\right) L_3\left(\frac{x}{lp^s q^i}\right)}} \ll$$

$$\ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \sum_{\substack{q^i \le \left(\frac{x}{lp^s}\right)^{1-\epsilon} \\ l | a(q^i) \\ i \ge 1}} \frac{1}{lp^s q^i} =$$

$$= \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \left(\sum_{\substack{q \le \left(\frac{x}{lp^s}\right)^{1-\epsilon} \\ l | a(q)}} \frac{1}{lp^s q} + \sum_{\substack{q^i \le \left(\frac{x}{lp^s}\right)^{1-\epsilon} \\ l | a(q^i) \\ i \ge 2}} \frac{1}{lp^s q^i} \right) =:$$

$$=: \Sigma_{4121211} + \Sigma_{4121212}$$

$$\Sigma_{4121212} = \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \sum_{\substack{q^i \le \left(\frac{x}{lp^s}\right)^{1-\epsilon} \\ l | a(q^i) \\ i \ge 2}} \frac{1}{lp^s q^i} \ll$$

$$\ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{\substack{p^s \le x \\ s \ge 2}} \frac{1}{p^s} \sum_{\substack{q^i \le x \\ i \ge 2}} \frac{1}{q^i} \log\log(q^i) \ll \frac{x}{\sqrt{L_1(x) L_3(x)}}.$$

$$\Sigma_{4121211} \le \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l} \sum_{\substack{p^s \le x \\ s \ge 2}} \frac{1}{p^s} \sum_{\substack{q \le x \\ l | a(q)}} \frac{1}{q} \ll$$

$$\ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l} \sum_{\substack{p^s \le x \\ s \ge 2}} \frac{1}{p^s} \left(\frac{1}{l} L_2(x) + \right.$$

$$\left. + \int_{l^2 \log l}^{e^{\frac{1}{\epsilon_1^2}(\log l)^2}} \pi^*(t, l) \frac{1}{t^2}\, dt + \sum_{\substack{q \le l^2 \log l \\ l | a(q)}} \frac{1}{q} \right) =:$$

$$=: \Sigma_{41212111} + \Sigma_{41212112} + \Sigma_{41212113}$$

$$\Sigma_{41212111} = \frac{xL_2(x)}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l^2} \sum_{\substack{p^s \le x \\ s \ge 2}} \frac{1}{p^s} \ll \frac{xL_2(x)}{y_1 \sqrt{L_1(x)L_3(x)}} = o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right)$$

$$\begin{aligned} \Sigma_{41212112} &= \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l} \sum_{\substack{p^s \le x \\ s \ge 2}} \frac{1}{p^s} \int_{l^2 \log l}^{e^{\frac{1}{\epsilon_1^2}(\log l)^2}} \pi^*(t,l) \frac{1}{t^2}\, dt = \\ &= \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{\substack{p^s \le x \\ s \ge 2}} \frac{1}{p^s} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l} \int_{l^2 \log l}^{e^{\frac{1}{\epsilon_1^2}(\log l)^2}} \pi^*(t,l) \frac{1}{t^2}\, dt \end{aligned}$$

Let us estimate the inner part of this sum by switching the order of summation and integration:

$$\sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l} \int_{l^2 \log l}^{e^{\frac{1}{\epsilon_1^2}(\log l)^2}} \pi^*(t,l) \frac{1}{t^2}\, dt \le \int_{y_1^2 \log y_1}^{x} \frac{1}{t^2} \sum_{e^{\epsilon_1 \sqrt{\log t}} < l \le \sqrt{\frac{t}{\log t}}} \frac{\pi^*(t,l)}{l}\, dt \le$$

$$\le \int_{y_1^2 \log y_1}^{x} \frac{1}{t^2} \sideset{}{^*}\sum_{q_1 \le t} \sum_{\substack{l > e^{\epsilon_1 \sqrt{\log t}} \\ l \mid a(q_1)}} \frac{1}{q_1}\, dt \ll \int_{y_1^2 \log y_1}^{x} \frac{1}{t^2} \cdot \frac{t}{\log t} \cdot \sqrt{\log t} \frac{1}{e^{\epsilon_1 \sqrt{\log t}}}\, dt =$$

$$= \int_{y_1^2 \log y_1}^{x} \frac{1}{t \log t \; e^{\epsilon_1 \sqrt{\log t}}}\, dt = -\frac{2}{\epsilon_1} \left. \frac{1}{e^{\epsilon_1 \sqrt{\log t}}} \right|_{y_1^2 \log y_1}^{x} = o(1).$$

Thus,

$$\begin{aligned} \Sigma_{41212112} &= \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{\substack{p^s \le x \\ s \ge 2}} \frac{1}{p^s} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l} \int_{l^2 \log l}^{e^{\frac{1}{\epsilon_1^2}(\log l)^2}} \pi^*(t,l) \frac{1}{t^2}\, dt = \\ &= o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right). \end{aligned}$$

$$
\begin{aligned}
\Sigma_{41212113} &= \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{\substack{p^s \le x \\ s\ge 2}} \frac{1}{p^s} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l} \sum_{\substack{q \le l^2 \log l \\ l \mid a(q)}} \frac{1}{q} \le \\
&\le \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{\substack{p^s \le x \\ s\ge 2}} \frac{1}{p^s} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l} \sum_{\substack{q \le l^3 \\ l \mid a(q)}} \frac{1}{q} \le \\
&\le \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{\substack{p^s \le x \\ s\ge 2}} \frac{1}{p^s} \sum_{q \le x} \frac{1}{q} \sum_{\substack{l \ge q^{\frac{1}{3}} \\ l \mid a(q)}} \frac{1}{l} \ll \\
&\ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{\substack{p^s \le x \\ s\ge 2}} \frac{1}{p^s} \sum_{q \le x} \frac{1}{q} \cdot \frac{1}{q^{\frac{1}{3}}} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}.
\end{aligned}
$$

This finishes the estimate for the sum $\Sigma_{4121211}$.

$$
\begin{aligned}
\Sigma_{4121212} &= \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \frac{1}{l} \sum_{\substack{p^s \le x \\ s\ge 2}} \frac{1}{p^s} \sum_{\substack{q^i \le x \\ i \ge 2 \\ l \mid a(q^i)}} \frac{1}{q^i} \le \\
&\le \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{\substack{p^s \le x \\ s\ge 2}} \frac{1}{p^s} \sum_{\substack{q^i \le x \\ i \ge 2}} \frac{1}{q^i} \sum_{\substack{l \le x \\ l \mid a(q^i)}} \frac{1}{l} \ll \\
&\ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{\substack{p^s \le x \\ s\ge 2}} \frac{1}{p^s} \sum_{\substack{q^i \le x \\ i \ge 2}} \frac{1}{q^i} \log\log(q^i) \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}.
\end{aligned}
$$

This finishes the estimate for the sum $\Sigma_{4121212}$, and so also the estimate for Σ_{412121}.

$$
\begin{aligned}
\Sigma_{412122} &\ll \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \sum_{\substack{\left(\frac{x}{lp^s}\right)^{1-\epsilon} \le q^i \le \frac{x}{lp^s} \\ l \mid a(q^i) \\ i \ge 1}} \frac{x}{lp^s q^i \sqrt{L_1\left(\frac{x}{lp^s q^i}\right) L_3\left(\frac{x}{lp^s q^i}\right)}} \le \\
&\le \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \sum_{\substack{\left(\frac{x}{lp^s}\right)^{1-\epsilon} \le q \le \frac{x}{lp^s} \\ l \mid a(q)}} \frac{x}{lp^s q \sqrt{L_1\left(\frac{x}{lp^s q}\right) L_3\left(\frac{x}{lp^s q}\right)}} + \\
&+ \sum_{y_1 < l \le e^{\epsilon_1 \epsilon \sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ s \ge 2}} \sum_{\substack{\left(\frac{x}{lp^s}\right)^{1-\epsilon} \le q^i \le \frac{x}{lp^s} \\ l \mid a(q^i) \\ i \ge 2}} \frac{x}{lp^s q^i \sqrt{L_1\left(\frac{x}{lp^s q^i}\right) L_3\left(\frac{x}{lp^s q^i}\right)}} =: \\
&=: \Sigma_{4121221} + \Sigma_{4121222}.
\end{aligned}
$$

For the estimate of $\Sigma_{4121221}$ we will need the following lemma:

Lemma 5.2.1. *For $u \geq x^{\alpha}$ and $l \leq e^{\epsilon_1\sqrt{\alpha}\sqrt{1-\epsilon}\sqrt{\log x}}$ we have*

$$\sum_{\substack{u^{1-\epsilon}\leq q\leq \frac{u}{e^{e^e}}\\ l|a(q)}} \frac{u}{q\sqrt{L_1\left(\frac{u}{q}\right)L_3\left(\frac{u}{q}\right)}} \ll \frac{1}{l}\cdot\frac{u}{\sqrt{L_1(u)\,L_3(u)}}. \tag{5.3}$$

Proof. This proof is similar to the one of Lemma 4.4.1, which in turn is similar to the proof of Lemma 4.1 in [7]. We will use Lemma 4.2.1 here.

$$\sum_{\substack{u^{1-\epsilon}\leq q\leq \frac{u}{e^{e^e}}\\ l|a(q)}} \frac{u}{q\sqrt{L_1\left(\frac{u}{q}\right)L_3\left(\frac{u}{q}\right)}} =$$

$$= \int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}} \frac{u}{t\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}d(\pi^*(t,l)) =$$

$$= \frac{u}{t\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\pi^*(t,l)\Bigg|_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}} - \int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}}\pi^*(t,l)\left(\frac{u}{t\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\right)_t' dt$$

Let us estimate the first summand:

$$\frac{u}{t\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\pi^*(t,l) = \frac{u}{t\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\left(\frac{1}{l}\mathrm{Li}(t) - \frac{1}{l}\mathrm{Li}(t^{\beta}) + O\left(\frac{t}{e^{c\sqrt{\log t}}}\right)\right),$$

where $\mathrm{Li}(t) = \int_2^t \frac{ds}{\log s} \ll \frac{t}{\log t}$.

The first term is

$$\frac{u\mathrm{Li}(t)}{tl\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}} \ll \frac{u}{l}\cdot\frac{1}{\log t},$$

the second term is

$$\frac{u\mathrm{Li}(t^{\beta})}{tl\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}} \ll \frac{u}{lt}\cdot\frac{t^{\beta}}{\beta\log t} \ll \frac{u}{l}\cdot\frac{1}{\log t},$$

and the third term is

$$\ll \frac{t}{e^{c\sqrt{\log t}}} = o\left(\frac{t}{\log t}\right).$$

Thus,

$$\frac{u}{t\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\pi^*(t,l)\Bigg|_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}} \ll \frac{u}{l}\cdot\frac{1}{\log t}\Bigg|_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}} \ll \frac{u}{l\log u} = o\left(\frac{uL_2(u)}{l\sqrt{L_1(u)L_3(u)}}\right).$$

Estimate for the main term goes as follows:

First, we compute the derivative

$$\left(\frac{u}{t\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\right)'_t =$$

$$= u(-1)\frac{1}{t^2L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}\cdot\left(t\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}\right)'_t =$$

$$= -\frac{u}{t^2L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}\cdot\left(\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}+\right.$$

$$\left.+ t\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\cdot\left(\frac{t}{u}\cdot\left(-\frac{u}{t^2}\right)L_3\left(\frac{u}{t}\right)+\frac{1}{L_2\left(\frac{u}{t}\right)}\cdot\frac{1}{L_1\left(\frac{u}{t}\right)}\frac{t}{u}\cdot\left(-\frac{u}{t^2}\right)L_1\left(\frac{u}{t}\right)\right)\right) =$$

$$= -\frac{u}{t^2\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}+\frac{u}{2t^2\left(L_1\left(\frac{u}{t}\right)\right)^{\frac{3}{2}}\sqrt{L_3\left(\frac{u}{t}\right)}}+\frac{u}{t^3L_1\left(\frac{u}{t}\right)L_2\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}.$$

Thus, the main term is:

$$- \int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}}\pi^*(t,l)\left(\frac{u}{t\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\right)'_t dt =$$

$$= \int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}}\pi^*(t,l)\left(\frac{u}{t^2\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}-\frac{u}{2t^2\left(L_1\left(\frac{u}{t}\right)\right)^{\frac{3}{2}}\sqrt{L_3\left(\frac{u}{t}\right)}}-\frac{u}{t^3L_1\left(\frac{u}{t}\right)L_2\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}\right)dt \le$$

$$\le \int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}}\pi^*(t,l)\frac{u}{t^2\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\,dt =$$

since $u \ge x^\alpha$ implies $l \le e^{\epsilon_1\sqrt{\alpha}\sqrt{1-\epsilon}\sqrt{\log x}} \le e^{\epsilon_1\sqrt{1-\epsilon}\sqrt{\log u}}$,
which is equivalent to $u^{1-\epsilon} \ge e^{\frac{1}{\epsilon_1^2}(\log l)^2}$, and so we can use Lemma 4.2.1

$$= \int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}}\left(\frac{1}{l}\mathrm{Li}(t)-\frac{1}{l}\mathrm{Li}(t^\beta)+O\left(\frac{t}{e^{c\sqrt{\log t}}}\right)\right)\frac{u}{t^2\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\,dt =: I_1+I_2+I_3.$$

Let us estimate these three integrals:

$$
\begin{aligned}
I_1 &= \frac{u}{l}\int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}} \frac{\mathrm{Li}(t)}{t^2\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\,dt \ll \frac{u}{l}\int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}} \frac{1}{t\log t\sqrt{L_1\left(\frac{u}{t}\right)L_3\left(\frac{u}{t}\right)}}\,dt =
\end{aligned}
$$

$$
\text{substitution:}\quad \begin{array}{ll} v\log u = \log t & t\in[u^{1-\epsilon}, \frac{u}{e^{e^e}}] \\ \log u\ dv = \frac{1}{t}\ dt & v\in[1-\epsilon, 1-\frac{e^e}{\log u}] \end{array}
$$

$$
\begin{aligned}
&= \frac{u}{l}\int_{1-\epsilon}^{1-\frac{e^e}{\log u}} \frac{\log u}{v\log u\sqrt{\log u}\sqrt{1-v}\sqrt{L_2(\log u(1-v))}}\,dv = \\
&= \frac{u}{l}\int_{1-\epsilon}^{1-\frac{e^{(L_2(u))^{c_1}}}{\log u}} \frac{1}{v\sqrt{\log u}\sqrt{1-v}\sqrt{L_2((\log u)(1-v))}}\,dv + \\
&+ \frac{u}{l}\int_{1-\frac{e^{(L_2(u))^{c_1}}}{\log u}}^{1-\frac{e^e}{\log u}} \frac{1}{v\sqrt{\log u}\sqrt{1-v}\sqrt{L_2((\log u)(1-v))}}\,dv \;=:\; I_{11}+I_{12},
\end{aligned}
$$

where c_1 is any number that satisfies $0 < c_1 < 1$.

Note that for $v \le 1 - \frac{e^{(L_2(u))^{c_1}}}{\log u}$ we have

$$
\begin{aligned}
&1-v \ge \frac{e^{(L_2(u))^{c_1}}}{\log u} \\
&L_2((\log u)(1-v)) \ge L_2\left(e^{(L_2(u))^{c_1}}\right) = c_1 L_3(u) \\
&\text{and so } \frac{1}{\sqrt{L_2((\log u)(1-v))}} \le \frac{1}{\sqrt{c_1 L_3(u)}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_{11} &\ll \frac{u}{l\sqrt{L_1(u)L_3(u)}}\int_{1-\epsilon}^{1-\frac{e^{(L_2(u))^{c_1}}}{\log u}} \frac{1}{v\sqrt{1-v}}\,dv \;\le \\
&\le \frac{u}{l\sqrt{L_1(u)L_3(u)}}\int_{1-\epsilon}^{1} \frac{1}{v\sqrt{1-v}}\,dv \;\ll \frac{u}{l\sqrt{L_1(u)L_3(u)}}.
\end{aligned}
$$

$$
\begin{aligned}
I_{12} &\ll \frac{u}{l\sqrt{\log u}}\int_{1-\frac{e^{(L_2(u))^{c_1}}}{\log u}}^{1} \frac{1}{v\sqrt{1-v}}\,dv \;\le \frac{u}{l\sqrt{\log u}}\cdot 2\sqrt{\frac{e^{(L_2(u))^{c_1}}}{\log u}} \ll \\
&\ll \frac{u}{l(\log u)^{1-\alpha}} = o\left(\frac{u}{l\sqrt{L_1(u)L_3(u)}}\right).
\end{aligned}
$$

In the estimate of the integral we used the following substitution:

$$\int_{1-\frac{e^{(L_2(u))^{c_1}}}{\log u}}^{1} \frac{1}{v\sqrt{1-v}}\, dv \;=$$

$$\text{substitution:} \quad \begin{array}{ll} \sqrt{1-v} = z & v \in [1 - \frac{e^{(L_2(u))^{c_1}}}{\log u}, 1] \\ dz = -\frac{1}{2\sqrt{1-v}} & z \in [\sqrt{\frac{e^{(L_2(u))^{c_1}}}{\log u}}, 0] \end{array}$$

$$= 2\int_0^{\sqrt{\frac{e^{(L_2(u))^{c_1}}}{\log u}}} \frac{1}{1-z^2}\, dz \;= (\log(1+z) - \log(1-z))\Big|_0^{\sqrt{\frac{e^{(L_2(u))^{c_1}}}{\log u}}} =$$

$$= \log\left(1 + \sqrt{\frac{e^{(L_2(u))^{c_1}}}{\log u}}\right) - \log\left(1 - \sqrt{\frac{e^{(L_2(u))^{c_1}}}{\log u}}\right) \ll 2\sqrt{\frac{e^{(L_2(u))^{c_1}}}{\log u}}, \text{ as } u \to \infty.$$

$$I_3 \;\ll\; \int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}} \frac{t}{e^{c\sqrt{\log t}}} \frac{u}{t^2\sqrt{L_1\left(\frac{u}{t}\right) L_3\left(\frac{u}{t}\right)}}\, dt = \int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}} \frac{u}{te^{c\sqrt{\log t}}\sqrt{L_1\left(\frac{u}{t}\right) L_3\left(\frac{u}{t}\right)}}\, dt \ll$$

since ϵ_1 was chosen so that $\epsilon_1 < c,$ which implies $e^{c\sqrt{\log t}} \gg e^{\epsilon_1\sqrt{\log t}} \log t$

$$\ll \int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}} \frac{u}{t\log t\; e^{\epsilon_1\sqrt{\log t}}\sqrt{L_1\left(\frac{u}{t}\right) L_3\left(\frac{u}{t}\right)}}\, dt \le$$

$$\le \frac{u}{e^{\epsilon_1\sqrt{1-\epsilon}\sqrt{\log u}}} \int_{u^{1-\epsilon}}^{\frac{u}{e^{e^e}}} \frac{u}{t\log t\sqrt{L_1\left(\frac{u}{t}\right) L_3\left(\frac{u}{t}\right)}}\, dt \ll$$

$$\ll \frac{u}{e^{\epsilon_1\sqrt{1-\epsilon}\sqrt{\log u}}\sqrt{L_1(u)L_3(u)}} \le \frac{u}{l\sqrt{L_1(u)L_3(u)}},$$

since $l \le e^{\epsilon_1\sqrt{\alpha}\sqrt{1-\epsilon}\sqrt{\log x}} \le e^{\epsilon_1\sqrt{1-\epsilon}\sqrt{\log u}}$

This finishes the proof of lemma. □

We will also need the following variant of the above lemma:

Lemma 5.2.2.

$$\sum_{x^{1-\epsilon}\le q\le x} \frac{x}{q\sqrt{L_1\left(\frac{x}{q}\right) L_3\left(\frac{x}{q}\right)}} \;\ll\; \frac{x}{\sqrt{L_1(x)\, L_3(x)}}. \tag{5.4}$$

Proof. The proof is done in the same way as the proof of the previous of Lemma. We will use Lemma 4.2.1 here.

$$\sum_{x^{1-\epsilon}\leq q\leq x} \frac{x}{q\sqrt{L_1\left(\frac{x}{q}\right)L_3\left(\frac{x}{q}\right)}} \ll$$

$$\ll \int_{x^{1-\epsilon}}^{\frac{x}{e^{e^e}}} \frac{x}{t\sqrt{L_1\left(\frac{x}{t}\right)L_3\left(\frac{x}{t}\right)}} d(\pi(t)) =$$

$$= \frac{x}{t\sqrt{L_1\left(\frac{x}{t}\right)L_3\left(\frac{x}{t}\right)}}\pi(t)\Bigg|_{x^{1-\epsilon}}^{\frac{x}{e^{e^e}}} - \int_{x^{1-\epsilon}}^{\frac{x}{e^{e^e}}} \pi(t)\left(\frac{x}{t\sqrt{L_1\left(\frac{x}{t}\right)L_3\left(\frac{x}{t}\right)}}\right)'_t dt$$

Let us estimate the first summand:

$$\frac{x}{t\sqrt{L_1\left(\frac{x}{t}\right)L_3\left(\frac{x}{t}\right)}}\pi(t)\Bigg|_{x^{1-\epsilon}}^{\frac{x}{e^{e^e}}} \ll \frac{x}{\log t\sqrt{L_1\left(\frac{x}{t}\right)L_3\left(\frac{x}{t}\right)}}\Bigg|_{x^{1-\epsilon}}^{\frac{x}{e^{e^e}}} \ll \frac{x}{\log x}$$

Estimate for the main term goes as follows. We use the derivative computed in the previous proof.

$$- \int_{x^{1-\epsilon}}^{\frac{x}{e^{e^e}}} \pi(t)\left(\frac{x}{t\sqrt{L_1\left(\frac{x}{t}\right)L_3\left(\frac{x}{t}\right)}}\right)'_t dt \leq$$

$$\leq \int_{x^{1-\epsilon}}^{\frac{x}{e^{e^e}}} \pi(t)\frac{x}{t^2\sqrt{L_1\left(\frac{x}{t}\right)L_3\left(\frac{x}{t}\right)}}\, dt =$$

$$\ll \int_{x^{1-\epsilon}}^{\frac{x}{e^{e^e}}} \frac{x}{t\log t\sqrt{L_1\left(\frac{x}{t}\right)L_3\left(\frac{x}{t}\right)}}\, dt =$$

$$\text{substitution:} \quad \begin{array}{ll} v\log x = \log t & t\in[x^{1-\epsilon}, \frac{x}{e^{e^e}}] \\ \log x\; dv = \frac{1}{t}\; dt & v\in[1-\epsilon, 1-\frac{e^e}{\log x}] \end{array}$$

$$= x\int_{1-\epsilon}^{1-\frac{e^e}{\log x}} \frac{1}{v\sqrt{\log x}\sqrt{1-v}\sqrt{L_2(\log x(1-v))}}\, dv =$$

$$\ll \frac{x}{\sqrt{L_1(x)L_3(x)}},$$

where the last integral is the same as in the proof of Lemma 5.2.1. This finishes the proof of lemma. □

Use Lemma 5.2.1 for the sum $\Sigma_{4121221}$:

$$\begin{aligned}
\Sigma_{4121221} &\ll \sum_{y_1<l\le e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s\le\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\ge 2}} \frac{x}{l^2p^s\sqrt{L_1\left(\frac{x}{lp^s}\right)L_3\left(\frac{x}{lp^s}\right)}} \ll \\
&\ll \frac{x}{\sqrt{L_1(x)L_3(x)}}\sum_{y_1<l}\frac{1}{l^2}\sum_{\substack{p^s\le\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\ge 2}}\frac{1}{p^s} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}\cdot\frac{1}{y_1\log y_1} = \\
&= o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}}\right).
\end{aligned}$$

$$\begin{aligned}
\Sigma_{4121222} &= \sum_{y_1<l\le e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s\le\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\ge 2}} \sum_{\substack{\left(\frac{x}{lp^s}\right)^{1-\epsilon}\le q^i\le\frac{x}{lp^s}\\ l\mid a(q^i)\\ i\ge 2}} \frac{x}{lp^sq^i\sqrt{L_1\left(\frac{x}{lp^sq^i}\right)L_3\left(\frac{x}{lp^sq^i}\right)}} \le \\
&\le \sum_{y_1<l\le e^{\epsilon_1\epsilon\sqrt{1-\epsilon}\sqrt{\log x}}} \sum_{\substack{p^s\le\left(\frac{x}{l}\right)^{1-\epsilon}\\ s\ge 2}} \sum_{\substack{\left(\frac{x}{lp^s}\right)^{1-\epsilon}\le q^i\le\frac{x}{lp^s}\\ l\mid a(q^i)\\ i\ge 2}} \frac{x}{lp^sq^i} \le \\
&\le x\sum_{\substack{p^s\le(x)^{1-\epsilon}\\ s\ge 2}}\frac{1}{p^s}\sum_{\substack{(x)^{\epsilon^2(1-\epsilon)}\le q^i\le x\\ i\ge 2}}\frac{1}{q^i}\sum_{\substack{l\le x\\ l\mid a(q^i)}}\frac{1}{l} \ll \\
&\ll x\sum_{\substack{p^s\le(x)^{1-\epsilon}\\ s\ge 2}}\frac{1}{p^s}\sum_{\substack{(x)^{\epsilon^2(1-\epsilon)}\le q^i\le x\\ i\ge 2}}\frac{1}{q^i}\log\log(q^i) = \\
&= x\sum_{\substack{p^s\le(x)^{1-\epsilon}\\ s\ge 2}}\frac{1}{p^s}\left(\sum_{\substack{(x)^{\epsilon^2(1-\epsilon)}\le q^i\le x\\ i\ge 2\\ q\ge\log x}}\frac{1}{q^i}\log\log(q^i) + \sum_{\substack{(x)^{\epsilon^2(1-\epsilon)}\le q^i\le x\\ i\ge 2\\ q<\log x}}\frac{1}{q^i}\log\log(q^i)\right) = \\
&= \Sigma_{41212221}+\Sigma_{41212222}.
\end{aligned}$$

$$\Sigma_{41212221} \quad = \quad x \sum_{\substack{p^s \leq (x)^{1-\epsilon} \\ s \geq 2}} \frac{1}{p^s} \sum_{\substack{(x)^{\epsilon^2(1-\epsilon)} \leq q^i \leq x \\ i \geq 2 \\ q \geq \log x}} \frac{1}{q^i} \log\log(q^i) \ll$$

$$\ll \quad x \sum_{\substack{p^s \\ s \geq 2}} \frac{1}{p^s} \sum_{\substack{q^i \\ i \geq 2 \\ q \geq \log x}} \frac{1}{q^{i-\alpha}} \ll x \sum_{\substack{p^s \\ s \geq 2}} \frac{1}{p^s} \sum_{q \geq \log x} \frac{1}{q^{2-\alpha}} \ll \frac{x}{(\log x)^{1-\alpha}} =$$

choose $\alpha < \frac{1}{2}$

$$= \quad o\left(\frac{x}{L_1(x)L_3(x)}\right).$$

$$\Sigma_{41212222} \quad = \quad x \sum_{\substack{p^s \leq (x)^{1-\epsilon} \\ s \geq 2}} \frac{1}{p^s} \sum_{\substack{(x)^{\epsilon^2(1-\epsilon)} \leq q^i \leq x \\ i \geq 2 \\ q \leq \log x}} \frac{1}{q^i} \log\log(q^i) \ll$$

$$\ll \quad x \sum_{\substack{p^s \leq (x)^{1-\epsilon} \\ s \geq 2}} \frac{1}{p^s} \frac{1}{x^{\epsilon^2(1-\epsilon)}} \log\log((\log x)^{\log x}) \cdot \#\{q \mid q \leq \log x\} \cdot \#\{i \mid q^i \leq x\} \ll$$

$$\ll \quad x \cdot \frac{1}{x^{\epsilon^2(1-\epsilon)}} L_2(x) \cdot \frac{\log x}{L_2(x)} \cdot \log x \ll x^{1-\beta} = o\left(\frac{x}{L_1(x)L_3(x)}\right).$$

This finishes the estimate of $\Sigma_{4121222}$, which together with the estimate for $\Sigma_{4121221}$ gives us the estimate for Σ_{412122}.
This, in turn, finishes the estimate for Σ_{41212}, and so also the estimate for Σ_4.

5.3 Contribution of squarefree numbers

Let us estimate the number Σ' of squarefree numbers n such that $(n, a(n)) = 1$.

$$\Sigma' = \Sigma_1 + \Sigma_2 \quad = \sum_{\substack{l \leq x \\ l-\text{ prime} \\ l \mid a(l)}} \sum_{\substack{m \leq \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1}} 1 + \sum_{\substack{l \leq x \\ l-\text{ prime} \\ l \nmid a(l)}} \sum_{\substack{m \leq \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l \mid a(m)}} 1 =$$

$$= \sum_{\substack{l \leq x \\ l-\text{ prime}}} \sum_{\substack{m \leq \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l \mid a(m)}} 1 + \sum_{\substack{l \leq x \\ l \mid a(l)}} \sum_{\substack{m \leq \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l \nmid a(m)}} 1 =: F + F_4$$

$$
\begin{aligned}
F &= \sum_{\substack{l\le y_1\\ l-\text{ prime}}} \sum_{\substack{m\le \frac{x}{l}\\ (m,a(m))=1\\ (m,a(l))=1\\ l|a(m)}} 1 + \sum_{\substack{y_1<l\le x\\ l-\text{ prime}}} \sum_{\substack{m\le \frac{x}{l}\\ (m,a(m))=1\\ (m,a(l))=1\\ l|a(m)}} 1 = \\
&= \sum_{l\le y_1} \sum_{\substack{m\le \frac{x}{l}\\ (m,a(m))=1\\ l|a(m)}} \sum_{\substack{d|m\\ d|a(l)}} \mu(d) + \sum_{\substack{y_1<l\le x\\ l-\text{ prime}}} \sum_{\substack{m\le \frac{x}{l}\\ (m,a(m))=1\\ (m,a(l))=1\\ l|a(m)}} 1 \le \\
&\le \sum_{l\le y_1} \sum_{d|a(l)} \mu(d) \left\{ \sum_{\substack{m\le x/l\\ m\equiv 0(\bmod d)\\ (m,a(m))=1}} 1 \right\} + \sum_{y_1<l\le x} \sum_{\substack{m\le \frac{x}{l}\\ (m,a(m))=1\\ (m,a(l))=1\\ l|a(m)}} 1 = \\
&= \sum_{l\le y_1} \sum_{d|a(l)} \mu(d) f\left(\frac{x}{l}, d\right) + \sum_{y_1<l\le x} \sum_{\substack{m\le \frac{x}{l}\\ (m,a(m))=1\\ (m,a(l))=1\\ l|a(m)}} 1 = \\
&= \sum_{l\le y_1} \sum_{\substack{d|a(l)\\ d\le \left(\frac{x}{l}\right)^{\frac{1}{2(k+2)}}}} \mu(d) f\left(\frac{x}{l}, d\right) + \sum_{l\le y_1} \sum_{\substack{d|a(l)\\ d> \left(\frac{x}{l}\right)^{\frac{1}{2(k+2)}}}} \mu(d) f\left(\frac{x}{l}, d\right) + \sum_{y_1<l\le x} \sum_{\substack{m\le \frac{x}{l}\\ (m,a(m))=1\\ (m,a(l))=1\\ l|a(m)}} 1 = \\
&=: F_1 + F_2 + F_3.
\end{aligned}
$$

Let us estimate F_2:

$$
\begin{aligned}
F_2 &= \sum_{l\le y_1} \sum_{\substack{d|a(l)\\ d> \left(\frac{x}{l}\right)^{\frac{1}{2(k+2)}}}} \mu(d) f\left(\frac{x}{l}, d\right) \le \sum_{l\le y_1} \sum_{\substack{d|a(l)\\ d> \left(\frac{x}{l}\right)^{\frac{1}{2(k+2)}}}} \frac{x}{ld} \le \\
&\le \sum_{l\le y_1} \frac{\frac{x}{l}}{\left(\frac{x}{l}\right)^{\frac{1}{2(k+2)}}} \cdot \#\{d-\text{squarefree }, d|a(l)\} \ll \\
&\ll x^{1-\frac{1}{2(k+2)}} \sum_{l\le y_1} \frac{1}{l^{1-\frac{1}{2(k+2)}}} \cdot l \ll x^{1-\frac{1}{2(k+2)}} \cdot y_1^2 \ll x^{1-\epsilon}.
\end{aligned}
$$

The estimate for the number of squarefree divisors of $|a(l)|$ was done as follows. First, we estimate $\omega(|a(l)|)$ – the number of distinct prime divisors of $|a(l)|$ using a trivial estimate:

$$\omega(|a(l)|) \le \log |a(l)| \ll \log l.$$

Then, since the number of squarefree divisors of $|a(l)|$ is equal to $2^{\omega(|a(l)|)}$, we have:

$$\#\{d - \text{squarefree }, d|a(l)\} \ll 2^{\log l} = l.$$

Let us estimate F_3.

$$F_3 = \sum_{y_1 < l \leq x} \sum_{\substack{m \leq \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l|a(m)}} 1 =$$

$$= \sum_{y_1 < l \leq e^{\epsilon_1 \sqrt{\log x}}} \sum_{\substack{m \leq \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l|a(m)}} 1 + \sum_{e^{\epsilon_1 \sqrt{\log x}} < l \leq x} \sum_{\substack{m \leq \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l|a(m)}} 1 =: F_{31} + F_{32}.$$

Since we are estimating the number of squarefree integers n, we have:

$$F_{31} = \sum_{y_1 < l \leq e^{\epsilon_1 \sqrt{\log x}}} \sum_{\substack{m \leq \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l|a(m)}} 1 \leq \sum_{y_1 < l \leq e^{\epsilon_1 \sqrt{\log x}}} \sum_{\substack{p < \frac{x}{l} \\ l|a(p)}} \sum_{\substack{m_1 \leq \frac{x}{lp} \\ (m_1,a(m_1))=1}} 1 =$$

$$= \sum_{y_1 < l \leq e^{\epsilon_1 \sqrt{\log x}}} \left(\sum_{\substack{p \leq \left(\frac{x}{l}\right)^{1-\epsilon} \\ l|a(p)}} \sum_{\substack{m_1 \leq \frac{x}{lp} \\ (m_1,a(m_1))=1}} 1 + \sum_{\substack{\left(\frac{x}{l}\right)^{1-\epsilon} < p < \frac{x}{l} \\ l|a(p)}} \sum_{\substack{m_1 \leq \frac{x}{lp} \\ (m_1,a(m_1))=1}} 1 \right) =: F_{311} + F_{312}.$$

We will use Lemma 5.2.1 for the estimate of F_{312} :

$$F_{312} = \sum_{y_1 \leq l \leq e^{\epsilon_1 \sqrt{\log x}}} \sum_{\substack{\left(\frac{x}{l}\right)^{1-\epsilon} < p < \frac{x}{l} \\ l|a(p)}} \sum_{\substack{m_1 \leq \frac{x}{l} \\ (m_1,a(m_1))=1}} 1 \ll$$

$$\ll \sum_{y_1 \leq l \leq e^{\epsilon_1 \sqrt{\log x}}} \sum_{\substack{\left(\frac{x}{l}\right)^{1-\epsilon} < p < \frac{x}{l} \\ l|a(p)}} \frac{x}{lp\sqrt{L_1\left(\frac{x}{lp}\right) L_3\left(\frac{x}{lp}\right)}} \ll$$

since $\frac{x}{l} \geq \frac{x}{e^{\epsilon_1 \sqrt{\log x}}} \ll x^{\alpha}$, we can use Lemma 5.2.1 here

$$\ll \sum_{y_1 \leq l \leq e^{\epsilon_1 \sqrt{\log x}}} \frac{x}{l^2 \sqrt{L_1\left(\frac{x}{l}\right) L_3\left(\frac{x}{l}\right)}} \ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{y_1 \leq l \leq e^{\epsilon_1 \sqrt{\log x}}} \frac{1}{l^2} \ll$$

$$\ll \frac{x}{\sqrt{L_1(x) L_3(x)}}.$$

$$
\begin{aligned}
F_{311} &= \sum_{y_1 \le l \le e^{\epsilon_1 \sqrt{\log x}}} \sum_{\substack{p \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ l \mid a(p)}} \sum_{\substack{m_1 \le \frac{x}{l} \\ (m_1, a(m_1))=1}} 1 \ll \\
&\ll \sum_{y_1 \le l \le e^{\epsilon_1 \sqrt{\log x}}} \sum_{\substack{p \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ l \mid a(p)}} \frac{x}{lp\sqrt{L_1\left(\frac{x}{lp}\right) L_3\left(\frac{x}{lp}\right)}} \ll \\
&\ll \sum_{y_1 \le l \le e^{\epsilon_1 \sqrt{\log x}}} \sum_{\substack{p \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ l \mid a(p)}} \frac{x}{lp\sqrt{L_1\left(\frac{x}{l}\right) L_3\left(\frac{x}{l}\right)}} \ll \\
& \text{since } l \le x^{1-\epsilon}, \\
&\ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{y_1 \le l \le e^{\epsilon_1 \sqrt{\log x}}} \frac{1}{l} \sum_{\substack{p \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ l \mid a(p)}} \frac{1}{p}
\end{aligned}
$$

Let us estimate the inner sum:

$$
\begin{aligned}
\sum_{\substack{p \le \left(\frac{x}{l}\right)^{1-\epsilon} \\ l \mid a(p)}} \frac{1}{p} &\le \sum_{\substack{p \le x \\ l \mid a(p)}} \frac{1}{p} = \sum_{\substack{l^2 \log l < p \le x \\ l \mid a(p)}} \frac{1}{p} + \sum_{\substack{p \le l^2 \log l \\ l \mid a(p)}} \frac{1}{p} \ll \\
& \text{by Lemma 8.1} \\
&\ll \frac{1}{l} L_2(x) + \int_{l^2 \log l}^{e^{\frac{1}{\epsilon_1^2}(\log l)^2}} \pi^*(t, l) \frac{1}{t^2} dt + \sum_{\substack{p \le l^2 \log l \\ l \mid a(p)}} \frac{1}{p}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
F_{311} &\ll \frac{x}{\sqrt{L_1(x) L_3(x)}} \sum_{y_1 \le l \le e^{\epsilon_1 \sqrt{\log x}}} \frac{1}{l} \left(\frac{1}{l} L_2(x) + \int_{l^2 \log l}^{e^{\frac{1}{\epsilon_1^2}(\log l)^2}} \pi^*(t, l) \frac{1}{t^2} dt + \sum_{\substack{p \le l^2 \log l \\ l \mid a(p)}} \frac{1}{p} \right) =: \\
&=: F_{3111} + F_{3112} + F_{3113}.
\end{aligned}
$$

$$
F_{3111} \ll \frac{x L_2(x)}{\sqrt{L_1(x) L_3(x)}} \cdot \frac{1}{y_1 \log y_1} = o\left(\frac{x}{\sqrt{L_1(x) L_3(x)}} \right)
$$

$$F_{3112} = \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{y_1 \le l \le e^{\epsilon_1 \sqrt{\log x}}} \frac{1}{l} \int_{l^2 \log l}^{e^{e^{\frac{1}{\epsilon_1^2}(\log l)^2}}} \pi^*(t,l) \frac{1}{t^2} dt \le$$

switch order of summation and integration

$$\le \frac{x}{\sqrt{L_1(x)L_3(x)}} \int_{2y_1^2 \log y_1}^{x} \frac{1}{t^2} \left(\sum_{e^{\epsilon_1 \sqrt{\log x}} \le l \le \sqrt{\frac{t}{\log t}}} \frac{1}{l} \pi^*(t,l) \right) dt \le$$

$$\le \frac{x)}{\sqrt{L_1(x)L_3(x)}} \int_{y_1^2 \log y_1}^{x} \frac{1}{t^2} \left(\sum_{q \le t} \sum_{\substack{e^{\epsilon_1 \sqrt{\log x}} \le l \\ l | a(q)}} \frac{1}{l} \right) dt \ll$$

since $\#\{l \ge e^{\epsilon_1 \sqrt{\log x}} \mid l|a(q)\} \ll \sqrt{\log x}$

$$\ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \int_{y_1^2 \log y_1}^{x} \frac{1}{t^2} \cdot \frac{t}{\log t} \cdot \sqrt{\log t} \frac{1}{e^{\epsilon_1 \sqrt{\log t}}} dt =$$

$$= \frac{x}{\sqrt{L_1(x)L_3(x)}} \int_{y_1^2 \log y_1}^{x} \frac{1}{t} \cdot \frac{1}{\sqrt{\log t}} \cdot \frac{1}{e^{\epsilon_1 \sqrt{\log t}}} dt =$$

substitution $u = e^{\epsilon_1 \sqrt{\log t}}$

$$= \frac{x}{\sqrt{L_1(x)L_3(x)}} \left(-\frac{2}{\epsilon_1} e^{\epsilon_1 \sqrt{\log t}} \right) \Bigg|_{y_1^2 \log y_1}^{x} \ll \frac{x}{\sqrt{L_1(x)L_3(x)}} \cdot \frac{1}{e^{\epsilon_1 \sqrt{\log(y_1^2 \log y_1)}}} =$$

$$= o\left(\frac{x}{\sqrt{L_1(x)L_3(x)}} \right)$$

$$
\begin{aligned}
F_{3113} &= \frac{x}{\sqrt{L_1(x)\,L_3(x)}} \sum_{y_1 \le l \le e^{\epsilon_1 \sqrt{\log x}}} \frac{1}{l} \sum_{\substack{p \le l^2 \log l \\ l \mid a(p)}} \frac{1}{p} \le \\
&\quad \text{switch order of summation} \\
&\le \frac{x}{\sqrt{L_1(x)\,L_3(x)}} \sum_{p \le \epsilon_1 \sqrt{\log x}\, e^{2\epsilon_1 \sqrt{\log x}}} \frac{1}{p} \sum_{\substack{l \ge \sqrt{\frac{p}{\log p}} \\ l \mid a(p)}} \frac{1}{l} \le \\
&\quad \text{note that for each } p \text{ the number of distinct } l'\text{s is estimated as follows:} \\
&\quad \#\{l \ge \sqrt{\frac{p}{\log p}} \mid l|a(p)\} \ll 1, \text{ since} \\
&\quad \sqrt{\frac{p}{\log p}}^{\,i} \le l_1 l_2 \cdot \ldots \cdot l_i \le |a(p)| \le p^{\frac{k-1}{2}+\epsilon}, \\
&\quad \text{and so } i \le \frac{\log p}{\log\left(\sqrt{\frac{p}{\log p}}\right)} \ll 1 \\
&\le \frac{x}{\sqrt{L_1(x)\,L_3(x)}} \sum_{p \le x} \frac{1}{p} \cdot \frac{1}{\sqrt{\frac{p}{\log p}}} \ll \frac{x}{\sqrt{L_1(x)\,L_3(x)}} \sum_{p \le x} \frac{1}{p^{\frac{3}{2}-\alpha}} \ll \\
&\ll o\left(\frac{x}{\sqrt{L_1(x) L_3(x)}}\right)
\end{aligned}
$$

The sum F_4 is expected to be small because of the hypothesis about the density of the primes l with the property $l|a(l)$:

$$
F_4 = \sum_{\substack{l \le x \\ l|a(l)}} \sum_{\substack{m \le \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l \nmid a(m)}} 1 = \sum_{\substack{l \le x^{1-\epsilon} \\ l|a(l)}} \sum_{\substack{m \le \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l \nmid a(m)}} 1 + \sum_{\substack{x^{1-\epsilon} < l \le x \\ l|a(l)}} \sum_{\substack{m \le \frac{x}{l} \\ (m,a(m))=1 \\ (m,a(l))=1 \\ l \nmid a(m)}} 1 =: F_{41} + F_{42}.
$$

$$
F_{41} \ll \sum_{\substack{l \le x^{1-\epsilon} \\ l|a(l)}} \frac{x}{l\sqrt{L_1\left(\frac{x}{l}\right) L_3\left(\frac{x}{l}\right)}} \ll \frac{x}{\sqrt{L_1(x)\,L_3(x)}} \sum_{\substack{l \le x^{1-\epsilon} \\ l|a(l)}} \frac{1}{l} \ll \frac{x}{\sqrt{L_1(x)\,L_3(x)}},
$$

where we used the fact that Hypothesis Z implies that

$$
\sum_{\substack{l \le x^{1-\epsilon} \\ l|a(l)}} \frac{1}{l}
$$

converges.
We use Lemma 5.2.2 for the estimate of F_{42}:

$$
F_{42} \ll \sum_{\substack{x^{1-\epsilon} < l \le x \\ l|a(l)}} \frac{x}{l\sqrt{L_1\left(\frac{x}{l}\right) L_3\left(\frac{x}{l}\right)}} \ll \frac{x}{\sqrt{L_1(x) L_3(x)}}.
$$

To estimate F_1 we need to estimate the function $f\left(\frac{x}{p}, d\right)$ for $d \le \left(\frac{x}{l}\right)^{\frac{1}{2(k+2)}}$.
We will work with the function

$$f(x,u) = \sum_{\substack{m \le x \\ m \equiv 0 (\bmod u) \\ (m,a(m))=1}} 1 \;=\; \#\{m \le x \mid u|m, (m,a(m)) = 1\},$$

and estimate it for $u \le x^{\frac{1}{2(k+2)}}$.
It was estimated before in [3] that

$$\sum_{\substack{m \le x \\ (m,a(m))=1}} 1 = (1+o(1)) \frac{U_f x}{\sqrt{L_1(x) L_3(x)}}. \tag{5.5}$$

We prove the following estimate:

Lemma 5.3.1. *For* $u \le x^{\frac{1}{2(k+2)}}$

$$\sum_{\substack{m \le x \\ m \equiv 0 (\bmod u) \\ (m,a(m))=1}} 1 = (1+o(1)) \frac{U_f x}{u\sqrt{L_1(x) L_3(x)}}.$$

Proof. To prove this, we use the same technique as in the proof of (5.5). We break up the set of all the m's that satisfy $m \le x$, $u|m$ and $(m,a(m)) = 1$ into the union of sets $\{m \le x \mid p|m, u|m, (m,a(m)) = 1, q|m \Rightarrow q \ge p\}$, i.e. we group them according to the smallest prime divisor of m (slightly abusing notation we denote the sets and the numbers of elements in these sets by the same letter). Denote

$$G_p(x,u) = \#\{m \le x \mid p|m, u|m, (m,a(m)) = 1, q|m \Rightarrow q \ge p\},$$

Note that some of these sets will be empty. If $m \in G_p(x,u)$, then $p|m$, $u|m$, and all the prime divisors of m are $\ge p$. Thus, all the prime divisors of u have to satisfy this condition, since $u|m$. So, if there exists q_1 such that $q_1|u, q_1 < p$, then $G_p(x,u) = 0$.

Denote by p_u the smallest prime divisor of u. Then $G_p(x,u) = 0$ if $p > p_u$, since in that case p cannot be the smallest divisor of n.

As before, we denote by $\nu(p,n) = \#\{q^m || n \mid a(q^m) \equiv 0 \ (\bmod p)\}$. Note that $\nu(p,n) = 0$ for all $p|n$ means that $(n,a(n)) = 1$.

$$\sum_{\substack{m \le x \\ m \equiv 0 (\bmod u) \\ (m,a(m))=1}} 1 \;=\; \sum_{p\text{ - prime}} G_p(x,u) \;=\; \sum_{\substack{p\text{ - prime} \\ p \le p_u}} G_p(x,u)$$

We split the sum $\sum_{p\text{ - prime}} G_p(x,u)$ into three parts:

$$\sum_{p\text{ - prime}} G_p(x,u) \;=\; A_1(x,u) + A_2(x,u) + A_3(x,u),$$

where

$$
\begin{aligned}
A_1(x,u) &= \sum_{p \le (\log\log x)^{\frac{1}{2}-\epsilon}} G_p(x,u) \\
A_2(x,u) &= \sum_{(\log\log x)^{\frac{1}{2}-\epsilon} \le p \le (\log\log x)^{1+\epsilon}} G_p(x,u) \\
A_3(x,u) &= \sum_{p \ge (\log\log x)^{1+\epsilon}} G_p(x,u)
\end{aligned}
$$

Note that if $p_u \le (\log\log x)^{1+\epsilon}$, then $A_3(x,u) = 0$, and if $p_u \le (\log\log x)^{\frac{1}{2}-\epsilon}$, then $A_2(x,u) = A_3(x,u) = 0$.

Estimate for $A_1(x,u)$:

$$
\begin{aligned}
A_1(x,u) &= \sum_{p \le (\log\log x)^{\frac{1}{2}-\epsilon}} G_p(x,u) = \\
&= \sum_{\substack{p \le (\log\log x)^{\frac{1}{2}-\epsilon} \\ p \nmid u}} G_p(x,u) + \sum_{\substack{p \le (\log\log x)^{\frac{1}{2}-\epsilon} \\ p \mid u}} G_p(x,u) \\
&= P_1 + P_2, \text{ say.}
\end{aligned}
$$

Note that $G_p(x,u) = 0$ if $p > p_u$, where p_u denotes the smallest prime divisor of u. Thus, the second sum P_2 will consist of at most one nonzero term $G_{p_u}(x,u)$, since $G_p(x,u) = 0$ for $p > p_u$. We have

$$
\begin{aligned}
P_2 &= \sum_{\substack{p \le (\log\log x)^{\frac{1}{2}-\epsilon} \\ p \mid u}} G_p(x,u) = \\
&= \begin{cases} G_{p_u}(x,u), & \text{if } p_u \le (\log\log x)^{\frac{1}{2}-\epsilon} \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

So, we need to estimate

$$
\begin{aligned}
G_{p_u}(x,u) &= \#\{m \le x \mid p_u|m, u|m, (m, a(m)) = 1,\ q|m \Rightarrow q \ge p_u\} = \\
&= \sum_{\substack{m_0 \le \frac{x}{u} \\ (m_0 u, a(m_0 u)) = 1 \\ q|m_0 \Rightarrow q \ge p_u}} 1 \ \le \\
&\le \sum_{\substack{m_0 \le \frac{x}{u} \\ (m_0 u, a(m_0 u)) = 1}} 1 \ \ll \frac{x}{u\sqrt{L_1\left(\frac{x}{u}\right) L_3\left(\frac{x}{u}\right)}} \ll \\
&\ll \frac{x}{u\sqrt{L_1(x) L_3(x)}}, \quad \text{since } u \le x^{\frac{1}{2(k+2)}}.
\end{aligned}
$$

the last step holds by Lemma 5.3.2 in the case $p = p_u$.

$$
\begin{aligned}
P_1 &\leq \sum_{\substack{p\leq(\log\log x)^{\frac{1}{2}-\epsilon}\\ p\nmid u}} \#\{n\leq x \mid p|n, (n,a(n))=1, u|n\} = \\
&= \sum_{\substack{p\leq(\log\log x)^{\frac{1}{2}-\epsilon}\\ p\nmid u}} \#\{n\leq x \mid pu|n, \nu(p_1,n)=0 \text{ for all } p_1|n\} = \\
&= \sum_{\substack{p\leq(\log\log x)^{\frac{1}{2}-\epsilon}\\ p\nmid u}} \#\left\{m\leq \frac{x}{u} \mid p|m, \nu(p_1,mu)=0 \text{ for all } p_1|mu\right\} = \\
&= \sum_{\substack{p\leq(\log\log x)^{\frac{1}{2}-\epsilon}\\ p\nmid u}} \#\left\{m\leq \frac{x}{u} \mid p|m, (u,m)=1, \nu(p_1,mu)=0 \text{ for all } p_1|mu\right\}
\end{aligned}
$$

Since we are working in the squarefree case, we have $(m,u)=1$, and so:

$$
\begin{aligned}
\nu(p,mu) &= \#\{q^k||mu \mid a(q^k)\equiv 0(\bmod p)\} = \\
&= \#\{q^k||m \mid a(q^k)\equiv 0(\bmod p)\} + \#\{q^k||u \mid a(q^k)\equiv 0(\bmod p)\} = \\
&= \nu(p,m)+\nu(p,u)
\end{aligned}
$$

$\nu(p,mu)=0$ means that $\nu(p,m)=0$ and $\nu(p,u)=0$.

Thus,

$$
\begin{aligned}
P_1 &\ll \sum_{\substack{p\leq(\log\log x)^{\frac{1}{2}-\epsilon}\\ p\nmid u}} \#\left\{m\leq \frac{x}{u} \mid \nu(p,mu)=0\right\} = \\
&= \sum_{\substack{p\leq(\log\log x)^{\frac{1}{2}-\epsilon}\\ p\nmid u}} \#\left\{m\leq \frac{x}{u} \mid \nu(p,m)=0, \nu(p,u)=0\right\} \leq \\
&\leq \sum_{\substack{p\leq(\log\log x)^{\frac{1}{2}-\epsilon}\\ p\nmid u}} \#\left\{m\leq \frac{x}{u} \mid \nu(p,m)=0\right\} = \text{ using the result from [3]} \\
&= o\left(\frac{x}{u\sqrt{L_1\left(\frac{x}{u}\right)L_3\left(\frac{x}{u}\right)}}\right) = o\left(\frac{x}{u\sqrt{L_1(x)L_3(x)}}\right),
\end{aligned}
$$

since this estimate is obtained under the condition $u\leq x^{\frac{1}{2(k+2)}}$.
Thus,

$$
A_1 \ll P_1+P_2 = o\left(\frac{x}{u\sqrt{L_1(x)L_3(x)}}\right).
$$

The estimate of $A_2(x,u)$ depends on a variant of the Lemma 5.2 from [3]:

Lemma 5.3.2. *Suppose that* $p\leq y_1$, $u\leq x^{\frac{1}{2(k+2)}}$, *and let* p_u *denote the smallest prime divisor of* u. *We* *have*

$$1)\quad \#\{n \le x \mid p|n, u|n, a_f(n) \ne 0, q|n \Rightarrow q \ge p\} \ll$$

$$\ll \frac{x}{pu(\log x)^{\frac{1}{2}}} \prod_{l<p} \left(1 - \frac{1}{l}\right) + \frac{x}{u(\log x)^{\frac{3}{2}}} \frac{2^{1+\nu(u)}(\log p)^2}{p}.$$

$$2)\quad \#\{n \le x \mid p_u|n, u|n, a_f(n) \ne 0, q|n \Rightarrow q \ge p_u\} \ll$$

$$\ll \frac{x}{u(\log x)^{\frac{1}{2}}} \prod_{l<p_u} \left(1 - \frac{1}{l}\right) + \frac{x \cdot 2^{\nu(u)}}{u(\log x)^{\frac{3}{2}}} \cdot (\log p_u)^2.$$

Then the estimate for $A_2(x,u)$ is done in the same way as before, only u appears in the denominator, which is exactly what we need.

$$\begin{aligned}
A_2(x) \quad &\le \sum_{L_2^{\frac{1}{2}-\epsilon}(x)<p<L_2^{1+\epsilon}(x)} \#\{n \le x \mid p|n, u|n, a(n) \ne 0, q|n \Rightarrow q \ge p\} \\
&\ll \frac{x}{u\sqrt{\log x}} \sum_{L_2^{\frac{1}{2}-\epsilon}(x)<p<L_2^{1+\epsilon}(x)} \frac{1}{p} \prod_{\substack{l \le p \\ l \text{ prime}}} \left(1 - \frac{1}{l}\right) \\
&\ll \frac{x}{u\sqrt{\log x}} \sum_{L_2^{\frac{1}{2}-\epsilon}(x)<p<L_2^{1+\epsilon}(x)} \frac{1}{p \log p} \\
&\ll \frac{x}{u\sqrt{\log x}} \cdot \frac{1}{\log(L_2^{\frac{1}{2}-\epsilon}(x))} \cdot \sum_{L_2^{\frac{1}{2}-\epsilon}(x)<p<L_2^{1+\epsilon}(x)} \frac{1}{p} \\
&\ll \frac{x}{u\sqrt{\log x}} \cdot \frac{1}{L_3(x)} \cdot L_4(x) = o\left(\frac{x}{u(L_3(x)\log x)^{\frac{1}{2}}}\right).
\end{aligned}$$

Let us estimate A_3.
Let $y_1 = L_2(x)^{1+\epsilon}$ and $N_{y_1}(x,u) = \#\{n \le x | q|n \Rightarrow q \ge y_1, a(n) \ne 0, u|n\}$. Note that all prime divisors of u have to be $\ge y_1$, otherwise $N_{y_1}(x,u) = 0$, because the set of numbers counted by $N_{y_1}(x,u)$ is empty.
Then

$$N_{y_1}(x,u) - \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ a(q_1^m) \equiv 0(\bmod q_2)}} \sum^{**}_{\substack{n \le x \\ q_1^m || n, q_2 | n, u | n}} 1 \le A_3(x,u) \le N_{y_1}(x,u).$$

Since we are working in the squarefree case, we can write

$$N_{y_1}(x,u) - \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ a(q_1) \equiv 0(\bmod q_2)}} \sum^{**}_{\substack{n \le x \\ q_1 || n, q_2 || n, u | n}} 1 \le A_3(x,u) \le N_{y_1}(x,u).$$

For this an analogue of Lemma 5.1 from [3] is proved, namely:

Lemma 5.3.3. *(Analogue of Lemma 5.1 for the case $u|n$)*

$$N_{y_1}(x,u) = \frac{U_f x}{u(\log x \log\log\log x)^{\frac{1}{2}}} + O\left(\frac{x(\log\log\log x)^2}{u(\log x)^{\frac{3}{2}}}\right).$$

Proof. As before, set $P_{y_1} = \prod_{p<y_1} p$.
Denote by $M_{f,d}(x,u)$ the number

$$M_{f,d}(x,u) = \#\{n \le x \mid a(n) \neq 0, d|n, u|n\}.$$

Then

$$N_{y_1}(x,u) = \sum_{d|P_{y_1}} \mu(d) M_{f,d}(x,u) \text{ by the principle of inclusion-exclusion.}$$

Note that $(d,u) = 1$ because all prime divisors of u are $\ge y_1$. Thus,

$$M_{f,d}(x,u) = M_{f,du}(x) \text{ in the notation of [3].}$$

By Proposition 4.18 from that paper (the number, though, refers to my write-up) we have

$$M_{f,du}(x) = \frac{u_f \xi_{du}(1)\frac{x}{du}}{\left(\log\frac{x}{du}\right)^{\frac{1}{2}}} + O\left(\frac{x2^{\nu(du)}}{du\left(\log\frac{x}{du}\right)^{\frac{3}{2}}}\right)$$

Here ξ_{du} satisfies the conditions 5.2 by Proposition 2.2.21.
Thus,

$$\begin{aligned} N_{y_1}(x,u) &= \sum_{d|P_{y_1}} \mu(d)\left(\frac{u_f \xi_{du}(1)\frac{x}{du}}{\left(\log\frac{x}{du}\right)^{\frac{1}{2}}} + O\left(\frac{x2^{\nu(du)}}{du\left(\log\frac{x}{du}\right)^{\frac{3}{2}}}\right)\right) = \\ &= \frac{u_f x}{u\sqrt{\log x}} \sum_{d|P_{y_1}} \frac{\mu(d)}{d}\left(\xi_{du}(1) + O\left(\frac{2^{\nu(du)}}{\log x}\right)\right), \end{aligned}$$

the last equality follows from the fact that $u \le x^{\frac{1}{2(k+2)}}$ and $d \le y_1 = (L_2(x))^{1+\epsilon}$.

The main term is

$$= \frac{u_f x}{u\sqrt{\log x}} \sum_{d|P_{y_1}} \frac{\mu(d)\xi_{du}(1)}{d} =$$

$$= \frac{u_f x}{u\sqrt{\log x}} \sum_{d|P_{y_1}} \prod_{p|d} \frac{(-1)\xi_{p,du}(1)}{p} =$$

$$= \frac{u_f x}{u\sqrt{\log x}} \prod_{p<y_1} \left(1 - \frac{\xi_{p,du}(1)}{p}\right) =$$

If $i_f(p) = 0$, then by Proposition 4.21 $\xi_{p,du}(1) = 1$; if $i_f(p) = 1$, and u is squarefree, then du is also squarefree, and by Proposition 4.21 $\xi_{p,du}(1) = \frac{1}{p}$. Thus

$$= \frac{u_f x}{u\sqrt{\log x}} \prod_{\substack{p<y_1 \\ i_f(p)=0}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p<y_1 \\ i_f(p)=1}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p<y_1 \\ i_f(p)>1}} \left(1 - \frac{\xi_{p,du}(1)}{p}\right) =$$

where the last product is finite

$$= \frac{u_f x}{u\sqrt{\log x}} \prod_{\substack{p<y_1 \\ a(p)\neq 0}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p<y_1 \\ i_f(p)=1}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p<y_1 \\ i_f(p)>1}} \left(\left(1 - \frac{\xi_{p,du}(1)}{p}\right)\left(1 - \frac{1}{p}\right)^{-1}\right) =$$

$$= \frac{u_f x}{u\sqrt{\log x}} \prod_{\substack{p<y_1 \\ a(p)\neq 0}} \left(1 - \frac{1}{p}\right) \cdot C_f, \text{ say,}$$

use Proposition 3.1 from [3]

$$= C_f \left(\frac{u_f x}{u\sqrt{\log x}} \frac{\mu_f}{\sqrt{\log y_1}} + O_f\left(\frac{1}{(\log y_1)^{3/2}} \frac{u_f x}{u\sqrt{\log x}}\right)\right) =$$

$$= \frac{u_f C_f x}{u\sqrt{L_1(x)L_3(x)}} \left(\frac{1}{\sqrt{1+\epsilon}} + O_f\left(\frac{1}{\log\log\log x}\right)\right).$$

□

By Lemma 5.3.3, to prove the theorem, it suffices to show that

$$\sum^{*}_{\substack{y_1 \leq q_1, q_2 \leq x \\ a(q_1) \equiv 0 (\bmod q_2)}} \sum^{**}_{\substack{n \leq x \\ q_1 || n, q_2 || n, u|n}} 1 = o\left(\frac{x}{u(L_3(x)\log x)^{\frac{1}{2}}}\right). \tag{5.6}$$

In order to prove (5.6) we use the same technique as for the estimate of the sum B_2 in the proof of Theorem 1.1.2. Our sum differs from the sum B_2 by having an additional condition $u|n$.

Since $u \le x^{\frac{1}{2(k+2)}} \Rightarrow q_1 q_2 u \ll x^{\alpha}$ for some $0 < \alpha < 1$, in the estimate for E_{12} we have

$$\frac{1}{\log\left(\frac{x}{q_1 q_2 u}\right)} \ll \frac{1}{\log x},$$

and the $\frac{1}{u}$ from the expression $\frac{x}{q_1 q_2 u}$ will factor out from the sums.

$$\sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ a(q_1) \equiv 0 (\bmod q_2)}} \sum^{**}_{\substack{n \le x \\ q_1 || n, q_2 || n, u | n}} 1 = \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ q_1 \nmid u, q_2 \nmid u \\ a(q_1) \equiv 0 (\bmod q_2)}} \sum^{**}_{\substack{n \le x \\ q_1 || n, q_2 || n, u | n}} 1 + \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ q_1 | u \text{ or } q_2 \nmid u \\ a(q_1) \equiv 0 (\bmod q_2)}} \sum^{**}_{\substack{n \le x \\ q_1 || n, q_2 || n, u | n}} 1 =: E_1 + E_2$$

$$E_1 = \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ q_1 \nmid u, q_2 \nmid u \\ a(q_1) \equiv 0 (\bmod q_2)}} \sum^{**}_{\substack{n \le x \\ q_1 || n, q_2 || n, u | n}} 1 \le \sum^{*}_{\substack{y_1 \le q_1, q_2 \le x \\ q_1 \nmid u, q_2 \nmid u, \\ a(q_1^m) \equiv 0 (\bmod q_2)}} \sum^{**}_{\substack{n \le x \\ q_1 q_2 u | n}} 1 = D_1 + D_2,$$

where D_1 and D_2 are defined by $y_1 \le q_1 \le x^{\frac{2}{k+2}}$ and $x^{\frac{2}{k+2}} < q_1 \le x$.

For D_1 we use Proposition 4.18 from [3]:

$$D_1 \ll \sum^{*}_{\substack{y_1 \le q_1 \le x^{\frac{2}{k+2}} \\ y_1 \le q_2 \le 2q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 (\bmod q_2) \\ q_1 \nmid u, q_2 \nmid u}} M_{f,q_1 q_2 u}(x) \ll \sum^{*}_{\substack{y_1 \le q_1 \le x^{\frac{2}{k+2}} \\ y_1 \le q_2 \le 2q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 (\bmod q_2)}} \frac{u_f x \xi_{q_1 q_2 u}(1)}{q_1 q_2 u (\log \frac{x}{q_1 q_2 u})^{\frac{1}{2}}} \le$$

we need to have $q_1 q_2 u \le x^{\alpha} < x$.

$q_1 q_2 \le 2x^{\frac{k+1}{k+2}}$

$q_1 q_2 u \le 2x^{\frac{k+1}{k+2}} u \le x^{\alpha} < x$

$u \le x^{\alpha - \frac{k+1}{k+2}}$

Take $\alpha = \frac{k+\frac{3}{2}}{k+2}$. Then with $u \le x^{\frac{1}{2}\frac{1}{k+2}}$ we have $q_1 q_2 u \le x^{\alpha}$.

Thus, $\dfrac{x}{q_1 q_2 u} \gg x^{1-\alpha} = x^{\frac{1}{2}\frac{1}{k+2}}$, and so
$\dfrac{1}{(\log \frac{x}{q_1 q_2 u})^{1/2}} \ll \dfrac{\sqrt{2(k+2)}}{(\log x)^{\frac{1}{2}}} \ll \dfrac{1}{(\log x)^{\frac{1}{2}}}$.

$$\le \sum^{*}_{\substack{y_1 \le q_1 \le x^{\frac{2}{k+2}} \\ y_1 \le q_2 \le 2q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 (\bmod q_2)}} \frac{u_f x \xi_{q_1 q_2 u}(1) \sqrt{2(k+2)}}{q_1 q_2 u (\log x)^{\frac{1}{2}}} \ll$$

$$\ll \frac{x}{u (\log x)^{\frac{1}{2}}} \sum^{*}_{\substack{y_1 \le q_1 \le x^{\frac{2}{k+2}} \\ y_1 \le q_2 \le 2q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 (\bmod q_2)}} \frac{1}{q_1 q_2} = o\left(\frac{x}{u \sqrt{L_1(x) L_3(x)}}, \right),$$

because the sum

$$\sum^{*}_{\substack{y_1 \le q_1 \le x^{\frac{2}{k+2}} \\ y_1 \le q_2 \le 2q_1^{\frac{k-1}{2}} \\ a(q_1) \equiv 0 (\bmod q_2)}} \frac{1}{q_1 q_2}$$

does not depend on u and was estimated in [3].

D_2 is estimated in the same way as before, because everywhere $\dfrac{x}{u q_1 q_2}$ will replace $\dfrac{x}{q_1 q_2}$, and so $\frac{1}{u}$ factors out from the sums.

The sum E_2 contains those cases when $q_1 | u$ or $q_2 | u$. We split it into three parts:

$$E_2 = \sum^{*}_{\substack{y_1\le q_1,q_2\le x\\ q_1|u,q_2\nmid u,\\ a(q_1)\equiv 0 \pmod{q_2}}} \sum^{**}_{\substack{n\le x,u|n\\ q_1||n,q_2||n}} 1 + \sum^{*}_{\substack{y_1\le q_1,q_2\le x\\ q_1\nmid u,q_2|u,\\ a(q_1)\equiv 0 \pmod{q_2}}} \sum^{**}_{\substack{n\le x,u|n\\ q_1||n,q_2||n}} 1 + \sum^{*}_{\substack{y_1\le q_1,q_2\le x\\ q_1|u,q_2|u,\\ a(q_1)\equiv 0 \pmod{q_2}}} \sum^{**}_{\substack{n\le x,u|n\\ q_1||n,q_2||n}} 1 =$$

recall that $u \le x^{\frac{1}{2(k+2)}}$

$$= \sum^{*}_{\substack{y_1\le q_1\le x^{\frac{1}{2(k+2)}}\\ y_1\le q_2\le x\\ q_1|u,q_2\nmid u,\\ a(q_1)\equiv 0 \pmod{q_2}}} \sum^{**}_{\substack{n\le x,u|n\\ q_1||n,q_2||n}} 1 + \sum^{*}_{\substack{y_1\le q_1\le x\\ y_1\le q_2\le x^{\frac{1}{2(k+2)}}\\ q_1\nmid u,q_2|u,\\ a(q_1)\equiv 0 \pmod{q_2}}} \sum^{**}_{\substack{n\le x,u|n\\ q_1||n,q_2||n}} 1 + \sum^{*}_{\substack{y_1\le q_1,q_2\le x^{\frac{1}{2(k+2)}}\\ q_1|u,q_2|u,\\ a(q_1)\equiv 0 \pmod{q_2}}} \sum^{**}_{\substack{n\le x,u|n\\ q_1||n,q_2||n}} 1 =$$

$$= E_{21} + E_{22} + E_{23}.$$

The estimate $\sum_{u\le x^{\frac{1}{2(k+2)}}} (E_{21}+E_{22}+E_{23}) = o\left(\frac{xL_4^2(x)}{\sqrt{L_1(x)L_3(x)}}\right)$ finishes the proof of Lemma 5.3.1. □

Thus,

$$f(x,u) \le (1+o(1))\frac{U_f x}{u\sqrt{L_1(x)L_3(x)}}$$

for $u \le x^{\frac{1}{2(k+2)}}$.
Then, we have

$$F_1 = \sum_{l\le y_1} \sum_{\substack{d|a(l)\\ d\le \left(\frac{x}{l}\right)^{\frac{1}{2(k+2)}}}} \mu(d) f\left(\frac{x}{l}, d\right) \ll$$

$$= \sum_{l\le y_1} \sum_{\substack{d|a(l)\\ d\le \left(\frac{x}{l}\right)^{\frac{1}{2(k+2)}}}} \frac{x\mu(d)U_f}{dl\sqrt{L_1(x)L_3(x)}} =$$

$$= \frac{x}{\sqrt{L_1(x)L_3(x)}} \sum_{l\le y_1} \frac{1}{l} \sum_{\substack{d|a(l)\\ d\le \left(\frac{x}{l}\right)^{\frac{1}{2(k+2)}}}} \frac{\mu(d)}{d} \ll$$

$$\ll \frac{xL_4^2(x)}{\sqrt{L_1(x)L_3(x)}},$$

because

$$\left|\sum_{d|a(l)} \frac{\mu(d)}{d}\right| \le \sum_{d|a(l)} \frac{1}{d} = \frac{1}{|a(l)|}\sum_{d|a(l)} \frac{|a(l)|}{d} = \frac{1}{|a(l)|}\sum_{d|a(l)} d \ll \log\log|a(l)| \ll L_4(x).$$

Here we used the fact that $\log\log|a(l)| \ll \log\log(2y_1^{\frac{k-1}{2}}) \ll L_4(x)$.

We also need to have the following estimate:

$$F_3 = o\left(\frac{xL_4^2(x)}{\sqrt{L_1(x)L_3(x)}}\right)$$

Then, since $\Sigma_1 + \Sigma_2 = F_1 + F_2 + F_3 + F_4$, we would get $\Sigma_1 + \Sigma_2 = (1 + o(1))\frac{xL_4^2(x)}{\sqrt{L_1(x)L_3(x)}}$. This combined with the estimate $\Sigma_3 \ll \frac{x}{\sqrt{L_1(x)L_3(x)}}$ would give us the desired estimate

$$\#\{n \leq x|(n, a(n)) \text{ is a prime}\} \leq \Sigma_1 + \Sigma_2 + \Sigma_3 = (1 + o(1))\frac{xL_4^2(x)}{\sqrt{L_1(x)L_3(x)}}$$

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