The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.

A Möbius Band

Good Luck!
Problem 1. (a) What is the homology of $S^1$? You need not prove this, just state the result.
   (b) Explain what the Mayer-Vietoris sequence is. Use it to compute the homology of $S^n$ for all $n > 1$, assuming your result from (a).

Problem 2. (a) Use Stokes’ theorem to compute the integral
   \[ \int_{S^2} xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \]
   Here
   \[ S^2 = \{ x, y, z \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}. \]
   (b) Let $\beta$ be the 2-form on $\mathbb{R}^3$ given by
   \[ \beta = -2ydx \wedge dy + 2zdx \wedge dz + dx \wedge dy - dy \wedge dz + yzdx \wedge dz + xzdy \wedge dz. \]
   Is there a 1-form $\alpha$ on $\mathbb{R}^3$ satisfying $\beta = d\alpha$? Give a proof justifying your answer.

Problem 3. Show that if $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for every $x \in S^2$, then $g$ is surjective.

Problem 4. Let $M$ be a compact $n$-manifold and $N$ a connected $n$-manifold. Show that an embedding $f : M \to N$ must be surjective.

Problem 5. Compute the homology groups $H_*(X, A)$ where $X = S^1 \times S^1$ and $A$ is a finite set of points in $X$. You’re allowed to take the homology of $X$ as known, yet if you use it, you should state what it is.

Problem 6. Let $X$ be the topological space obtained from $S^2$ by joining the north and the south poles by a straight line segment. Find the universal covering space of $X$. What is its group of covering transformations?

Good Luck!