Do not turn this page until instructed.

Math 1300 Geometry and Topology

Final Examination

University of Toronto, May 2, 2008

Solve 3 of the 4 problems in Part I and 3 of the 4 problems in Part II of this exam.

Each problem is worth 17 points.
You have three hours to write this test.

Notes.

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- Advance apology: It will take me around 3 weeks to grade this exam; sorry. If you have a strong reason why you must have your grade ready quicker than that (e.g., you need to graduate), you **MUST** indicate that to Dror **NOW** and he will grade your exam separately and sooner.

Good Luck!

Part I

Solve 3 of the following 4 problems. Each problem is worth 17 points. **Neatness counts! Language counts!**

Problem 1.

1. State and prove the theorem about the local structure of immersions.
2. A manifold N is embedded inside a manifold M. Prove that every smooth function on N can be extended to a smooth function on M, at least locally.

Tip. As always in math exams, when proving a theorem you may freely assume anything that preceded it but you may not assume anything that followed it.

Problem 2.

1. State the Van Kampen theorem.
2. By appropriately gluing a disk to the wedge of two circles (the "∞" space), construct a space X_{34} whose fundamental group is $\langle a, b \colon a^3 b^4 = 1 \rangle$.

Tip. Of course, you also need to *prove* that X_{34} has the desired property.

Problem 3. Let $p : \mathbb{R}^2_{x,y} = \mathbb{C}_z \longrightarrow \mathbb{C}_w - \{0\} = \mathbb{R}^2_{u,v} - \{0\}$ be given by $w = e^z$ (i.e., by $p(z) = e^z$).

1. Prove that there is a unique form $\omega \in \Omega^1(\mathbb{R}^2_{u,v} - \{0\})$ such that $p^\star \omega = dy$.
2. Find an explicit formula for ω, of the form $\omega = f(u,v)du + g(u,v)dv$.
3. Show that ω is closed but not exact.

Problem 4.

1. State precisely (but don't bother proving) the theorem about existence and uniqueness of lifts of maps $f : Y \to B$, where B is the basis of a covering $p : X \to B$.
2. Let $p_1 : X_1 \to B$ and $p_2 : X_2 \to B$ be coverings of a connected and locally connected space B, and assume that $p_{1\star}\pi_1(X_1) = p_{2\star}\pi_1(X_2)$. Prove that X_1 and X_2 are homeomorphic.

Part II

Solve 3 of the following 4 problems. Each problem is worth 17 points. **Neatness counts! Language counts!**

Problem 5 "Compute". Embed S^3 inside $\mathbb{C}^2$ as the subset $\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$ and consider the map $f : S^3 \to S^3$ given by $(z_1, z_2) \mapsto (z_1^3/|z_1|^2, |z_2|^6/z_2^5)$ (for the purpose of this definition, $\frac{0}{0} = 0$). Compute the degree $\deg f$.

Tip. "Compute", of course, really means "compute and justify your computation".

Problem 6 "Reproduce".

1. State the exactness axiom for a homology theory.
2. State the excision axiom for a homology theory.
3. Use the exact sequences for a sphere in a disk and for a disk in a sphere, and the excision axiom, to prove that $H_p(S^n) = H_{p-1}(S^{n-1})$ when both p and n are large (that is, don't worry about "basing the induction").

Problem 7 "Think". The suspension SX of a topological space X is defined to be $X \times [0, 1]$ with $X \times \{0\}$ identified to a point and $X \times \{1\}$ identified to (another) point. Prove that $\tilde{H}_{n+1}(SX) = \tilde{H}_n(X)$ for every n.

Problem 8 "Sketch". The "homotopy axiom" for a homology theory states that if $f \sim g : X \to Y$, then $f_\star = g_\star : H_\star(X) \to H_\star(Y)$. Sketch to the best of your understanding the proof of the homotopy axiom for singular homology.

Tip. A good thumb rule is that you can safely omit details whose completion would qualify as "mechanical exercises".

Good Luck!