DEPARTMENT OF MATHEMATICS  
University of Toronto  

Topology Exam (3 hours)  

Monday, September 12, 2005, 1-4 PM  

The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.  

Good Luck!
1. The finite topology on a set $X$ is the topology whose closed sets are the finite subsets of $X$. Verify that $\mathbb{C}$ with the finite topology is not Hausdorff, is not second countable, but is separable. Show that a polynomial with complex coefficients defines a continuous map from $\mathbb{C}$ to itself.

2. Let $X$ be a compact metric space with metric $d$ and let $f : X \to X$ be a contraction map; a map satifying $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$.
   (a) Prove that $f$ is continuous or find a counterexample.
   (b) Prove that $f$ has a fixed point; a point $x \in X$ for which $f(x) = x$.

3. Describe the fundamental groups and the universal covers of the following spaces:
   (a) The torus $T^2 = S^1 \times S^1$.
   (b) The punctured torus $X = T^2 - \{p\}$ where $p \in T^2$ is a single point.
   (c) The real projective plane $\mathbb{RP}^2$.
   (d) The punctured projective plane $\mathbb{RP}^2 - \{p\}$ where $p \in \mathbb{RP}^2$ is a single point.

4. Let $T^2 := S^1 \times S^1$ be the 2-torus. Let $X = S^1 \times \{1\} \cup \{1\} \times S^1 \subset T^2$. And let $i : X \to T^2$ denote the inclusion map.
   (a) Prove that $i_* : H_1(X) \to H_1(T^2)$ is an isomorphism.
   (b) Prove that $i_* : \pi_1(X) \to \pi_1(T^2)$ is not an isomorphism. What is the kernel of $i_*$?

5. (a) Let $f, g : S^n \to S^n$ be continuous maps such that $f(x) \neq g(x)$ for all $x \in S^n$. Show that $f \simeq a \circ g$, where $a$ is the antipodal map.
   (b) Prove that any continuous map $f : S^{2n} \to S^{2n}$ either has a fixed point or there is a point $x$ with $f(x) = -x$.
   (c) Prove that any continuous map $f : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point.

6. Let $f : X \to Y$. Define the mapping cylinder $M_f$ by $M_f := (X \times I) \cup Y/ \sim$ where $(x, 1) \sim f(x)$ for all $x \in X$. Define the mapping cone $C_f$ by $C_f := M_f/ \sim$ where $(x, 0) \sim (x', 0)$ for all $x, x' \in X$.
   (a) Show that there is a long exact homology sequence
      \[ \cdots \to H_n(X) \to H_n(Y) \to H_n(C_f) \to H_{n-1}(X) \to \cdots \]
   (b) Let $f : S^1 \to S^1$ be the function $f(x) = x^2$. Compute the homology groups $H_*(C_f)$.
   (c) Identify the homotopy type of the space $C_f$ for the $f$ of the previous part. (That is, what is the “well known” space which is homotopy equivalent to $C_f$.)

Good Luck!